This exercise set will constitute 15% of the final exam. The remaining part of the final exam will be an inclass exam covering Chapters 13 and 14 of Dummit and Foote and Chapter 7 of Adkns-Weintraub. The final exam is May 4 (Friday) from 10:00 AM – Noon. This assignment should be turned in no later than the time of the final exam.

1. Let $G = C_2 = \langle a : a^2 = 1 \rangle$, and let $V = F^2$ (where F is a field). For $(\alpha, \beta) \in V$, define the action of G on V by $1(\alpha, \beta) = (\alpha, \beta)$ and $a(\alpha, \beta) = (\beta, \alpha)$, and extend by linearity to make V into an FG-module. Find all of the FG-submodules of V.

2. If $G = C_2 \times C_2 = \langle a, b : a^2 = b^2 = 1, ab = ba \rangle$, write the real group ring $\mathbb{R}G$ as a direct sum of $\mathbb{R}G$-submodules, each of which is 1-dimensional over $\mathbb{R}$.

3. Let $G = D_{12} = \langle a, b : a^6 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Define matrices A, B, C, D over $\mathbb{C}$ by

$$A = \begin{bmatrix} e^{i\pi/3} & 0 \\ 0 & e^{-i\pi/3} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

   (a) Verify that each of the functions $\rho_k : G \to \mathrm{GL}(2, \mathbb{C})$ ($k = 1, 2, 3, 4$), given by (i) $\rho_1(a^r b^s) = A^r B^s$, (ii) $\rho_2(a^r b^s) = A^{3r}(-B)^s$, (iii) $\rho_3(a^r b^s) = (-A)^r B^s$, (iv) $\rho_4(a^r b^s) = C^r D^s$
   for $0 \le r \le 5$, $0 \le s \le 1$, is a representation of G.

   (b) Which of the representations ρ_k are faithful?

   (c) Which of these representations are equivalent?

   (d) Which are irreducible?

4. Find the missing row in the following character table:

Order of conjugacy class	(1)	(3)	(6)	(6)	(8)
Conjugacy class	$Cl(1)$	$Cl(a)$	$Cl(b)$	$Cl(c)$	$Cl(d)$
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	3	-1	1	-1	0
χ_4	3	-1	-1	1	0
χ_5					

5. The character table of S_3 is

Conjugacy class	$Cl(1)$	$Cl((1\,2))$	$Cl((1\,2\,3))$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

   Let ϕ be a character such that $\phi(1) = 5$, $\phi((1\,2)) = 1$, $\phi((1\,2\,3)) = 2$.

   (a) Compute the inner products $\langle \phi, \chi_1 \rangle$, $\langle \phi, \chi_2 \rangle$, and $\langle \phi, \chi_3 \rangle$.

   (b) Write the character ϕ as a linear combination of χ_1, χ_2, χ_3.