Chapter 7

Topics in Module Theory

This chapter will be concerned with collecting a number of results and constructions concerning modules over (primarily) noncommutative rings that will be needed to study group representation theory in Chapter 8.

7.1 Simple and Semisimple Rings and Modules

In this section we investigate the question of decomposing modules into "simpler" modules.

(1.1) Definition. *If R is a ring (not necessarily commutative) and $M \neq \langle 0 \rangle$ is a nonzero R-module, then we say that M is a* **simple** *or* **irreducible** R**-module** *if $\langle 0 \rangle$ and M are the only submodules of M.*

(1.2) Proposition. *If an R-module M is simple, then it is cyclic.*

Proof. Let x be a nonzero element of M and let $N = \langle x \rangle$ be the cyclic submodule generated by x. Since M is simple and $N \neq \langle 0 \rangle$, it follows that $M = N$. $\square$

(1.3) Proposition. *If R is a ring, then a cyclic R-module $M = \langle m \rangle$ is simple if and only if* $\mathrm{Ann}(m)$ *is a maximal left ideal.*

Proof. By Proposition 3.2.15, $M \cong R/\mathrm{Ann}(m)$, so the correspondence theorem (Theorem 3.2.7) shows that M has no submodules other than M and $\langle 0 \rangle$ if and only if R has no submodules (i.e., left ideals) containing $\mathrm{Ann}(m)$ other than R and $\mathrm{Ann}(m)$. But this is precisely the condition for $\mathrm{Ann}(m)$ to be a maximal left ideal. $\square$

(1.4) Examples.

(1) An abelian group A is a simple $\mathbf{Z}$-module if and only if A is a cyclic group of prime order.

(2) The hypothesis in Proposition 1.3 that M be cyclic is necessary. The $\mathbf{Z}$-module $A = \mathbf{Z}_2^2$ has annihilator $2\mathbf{Z}$ but the module A is not simple.

(3) Consider the vector space F^2 (where F is any field) as an $F[x]$-module via the linear transformation $T(u_1, u_2) = (u_2, 0)$. Then F^2 is a cyclic $F[X]$-module, but it is not a simple $F[X]$-module. Indeed,

$$F^2 = F[X] \cdot (0, 1)$$

but $N = \{(u, 0) : u \in F\}$ is an $F[X]$-submodule of F^2. Thus the converse of Proposition 1.2 is not true.

(4) Let $V = \mathbf{R}^2$ and consider the linear transformation $T : V \to V$ defined by $T(u, v) = (-v, u)$. Then the $\mathbf{R}[X]$-module V_T is simple. To see this let $w = (u_1, v_1) \neq 0 \in V$ and let N be the $\mathbf{R}[X]$-submodule of V_T generated by w. Then $w \in N$ and $Xw = T(w) = (-v_1, u_1) \in N$. Since any $(x, y) \in V$ can be written as $(x, y) = \alpha w + \beta X w$ where $\alpha = (xu_1 + yv_1)/(u_1^2 + v_1^2)$ and $\beta = (yu_1 - xv_1)/(u_1^2 + v_1^2)$, it follows that $N = V_T$ and hence V_T is simple.

(5) Now let $W = \mathbf{C}^2$ and consider the linear transformation $T : W \to W$ defined by $T(u, v) = (-v, u)$. Note that T is defined by the same formula used in Example 1.4 (4). However, in this case the $\mathbf{C}[X]$-module W_T is not simple. Indeed, the $\mathbf{C}$-subspace $\mathbf{C} \cdot (i, 1)$ is a T-invariant subspace of W, and hence, it is a $\mathbf{C}[X]$-submodule of W_T different from W and from $\langle 0 \rangle$.

The following lemma is very easy, but it turns out to be extremely useful:

(1.5) Proposition. (Schur's lemma)

(1) *Let M be a simple R-module. Then the ring* $\mathrm{End}_R(M)$ *is a division ring.*

(2) *If M and N are simple R-modules, then* $\mathrm{Hom}_R(M, N) \neq \langle 0 \rangle$ *if and only if M and N are isomorphic.*

Proof. (1) Let $f \neq 0 \in \mathrm{End}_R(M)$. Then $\mathrm{Im}(f)$ is a nonzero submodule of M and $\mathrm{Ker}(f)$ is a submodule of M different from M. Since M is simple, it follows that $\mathrm{Im}(f) = M$ and $\mathrm{Ker}(f) = \langle 0 \rangle$, so f is an R-module isomorphism and hence is invertible as an element of the ring $\mathrm{End}_R(M)$.

(2) The same argument as in (1) shows that any nonzero homomorphism $f : M \to N$ is an isomorphism. $\square$

We have a second concept of decomposition of modules into simpler pieces, with simple modules again being the building blocks.

(1.6) Definition. *If R is a ring (not necessarily commutative), then an R-module M is said to be* **indecomposable** *if it has no proper nontrivial com-*

plemented submodule M_1, *i.e., if* $M = M_1 \oplus M_2$ *implies that* $M_1 = \langle 0 \rangle$ *or* $M_1 = M$.

If M is a simple R-module, then M is also indecomposable, but the converse is false. For example, **Z** is an indecomposable **Z**-module, but **Z** is not a simple **Z**-module; note that **Z** contains the proper submodule 2**Z**.

One of the major classes of modules we wish to study is the following:

(1.7) Definition. *An* R*-module* M *is said to be* **semisimple** *if it is a direct sum of simple* R*-modules.*

The idea of semisimple modules is to study modules by decomposing them into a direct sum of simple submodules. In our study of groups there was also another way to construct groups from simpler groups, namely, the extension of one group by another, of which a special case was the semidirect product. Recall from Definition 1.6.6 that a group G is an extension of a group N by a group H if there is an exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1.$$

If this exact sequence is a split exact sequence, then G is a semidirect product of N and H. In the case of abelian groups, semidirect and direct products coincide, but extension of N by H is still a distinct concept.

If G is an abelian group and N is a subgroup, then the exact sequence

$$\langle 0 \rangle \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow \langle 0 \rangle$$

is completely determined by the chain of subgroups $\langle 0 \rangle \subseteq N \subseteq G$. By allowing longer chains of subgroups, we can consider a group as obtained by multiple extensions. We will consider this concept within the class of R-modules.

(1.8) Definition.

(1) *If* R *is a ring (not necessarily commutative) and* M *is an* R*-module, then a* **chain of submodules** *of* M *is a sequence* $\{M_i\}_{i=0}^n$ *of submodules of* M *such that*

$$\langle 0 \rangle = M_0 \subsetneqq M_1 \subsetneqq M_2 \subsetneqq \cdots \subsetneqq M_n = M. \tag{1.1}$$

The **length** *of the chain is* n.

(2) *We say that a chain* $\{N_j\}_{j=0}^m$ *is a* **refinement** *of the chain* $\{M_i\}_{i=0}^n$ *if each* N_j *is equal to* M_i *for some* i. *Refinement of chains defines a partial order on the set* $\mathcal{C}$ *of all chains of submodules of* M.

(3) *A maximal element of* $\mathcal{C}$ *(if it exists) is called a* **composition series** *of* M.

(1.9) *Remarks.*

(1) Note that the chain (1.1) is a composition series if and only if each of the modules M_i/M_{i-1} $(1 \le i \le n)$ is a simple module.

(2) Our primary interest will be in decomposing a module as a direct sum of simple modules. Note that if $M = \oplus_{i=1}^n M_i$ where M_i is a simple R-module, then M has a composition series

$$\langle 0 \rangle \subsetneqq M_1 \subsetneqq M_1 \oplus M_2 \subsetneqq \cdots \subsetneqq \bigoplus_{i=1}^{n} M_i = M.$$

On the other hand, if $M = \oplus_{i=1}^\infty M_i$, then M does not have a composition series. In a moment (Example 1.10 (2)) we shall see an example of a module that is not semisimple but does have a composition series. Thus, while these two properties—semisimplicity and having a composition series—are related, neither implies the other. However, our main interest in composition series is as a tool in deriving results about semisimple modules.

(1.10) Examples.

(1) Let D be a division ring and let M be a D-module with a basis $\{x_1, \ldots, x_m\}$. Let $M_0 = \langle 0 \rangle$ and for $1 \le i \le n$ let $M_i = \langle x_1, \ldots, x_i \rangle$. Then $\{M_i\}_{i=0}^n$ is a chain of submodules of length n, and since

$$\begin{aligned} M_i/M_{i-1} &= \langle x_1, \ldots, x_i \rangle / \langle x_1, \ldots, x_{i-1} \rangle \\ &\cong Dx_i \\ &\cong D, \end{aligned}$$

we conclude that this chain is a composition series because D is a simple D-module.

(2) If p is a prime, the chain

$$\langle 0 \rangle \subsetneqq p\mathbf{Z}_{p^2} \subsetneqq \mathbf{Z}_{p^2}$$

is a composition series for the $\mathbf{Z}$-module $\mathbf{Z}_{p^2}$. Note that $\mathbf{Z}_{p^2}$ is not semisimple as a $\mathbf{Z}$-module since it has no proper complemented submodules.

(3) The $\mathbf{Z}$-module $\mathbf{Z}$ does not have a composition series. Indeed, if $\{I_i\}_{i=0}^n$ is any chain of submodules of length n, then writing $I_1 = \langle a_1 \rangle$, we can properly refine the chain by putting the ideal $\langle 2a_1 \rangle$ between I_1 and $I_0 = \langle 0 \rangle$.

(4) If R is a PID, then essentially the same argument as Example 1.10 (3) shows that R does not have a composition series as an R-module.

(1.11) Definition. *Let M be an R-module. If M has a composition series let $\ell(M)$ denote the minimum length of a composition series for M. If M does not have a composition series, let $\ell(M) = \infty$. $\ell(M)$ is called the* **length** *of the R-module M. If $\ell(M) < \infty$, we say that M has* **finite length**.

Note that isomorphic R-modules have the same length, since if $f : M \to N$ is an R-module isomorphism, the image under f of a composition series for M is a composition series for N.

(1.12) Lemma. *Let M be an R-module of finite length and let N be a proper submodule (i.e., $N \neq M$). Then $\ell(N) < \ell(M)$.*

Proof. Let

$$\langle 0 \rangle = M_0 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M_n = M \tag{1.2}$$

be a composition series of M of length $n = \ell(M)$ and let $N_i = N \cap M_i \subseteq N$. Let $\phi : N_i \to M_i/M_{i-1}$ be the inclusion map $N_i \to M_i$ followed by the projection map $M_i \to M_i/M_{i-1}$. Since $\operatorname{Ker}(\phi) = N_{i-1}$, it follows from the first isomorphism theorem that N_i/N_{i-1} is isomorphic to a submodule of M_i/M_{i-1}. But (1.2) is a composition series, so M_i/M_{i-1} is a simple R-module. Hence $N_i = N_{i-1}$ or $N_i/N_{i-1} = M_i/M_{i-1}$ for $i = 1, 2, \ldots, n$. By deleting the repeated terms of the sequence $\{N_i\}_{i=0}^n$ we obtain a composition series for the module N of length $\leq n = \ell(M)$. Suppose that this composition series for N has length n. Then we must have $N_i/N_{i-1} = M_i/M_{i-1}$ for all $i = 1, 2, \ldots, n$. Thus $N_1 = M_1$, $N_2 = M_2$, $\ldots$, $N_n = M_n$, i.e., $N = M$. Since we have assumed that N is a proper submodule, we conclude that the chain $\{N_i\}_{i=0}^n$ has repeated terms, and hence, after deleting repeated terms we find that N has a composition series of length $< \ell(M)$, that is, $\ell(N) < \ell(M)$. □

(1.13) Proposition. *Let M be an R-module of finite length. Then every composition series of M has length $n = \ell(M)$. Moreover, every chain of submodules can be refined to a composition series.*

Proof. We first show that any chain of submodules of M has length $\leq \ell(M)$. Let

$$\langle 0 \rangle = M_0 \subsetneqq M_1 \subsetneqq \cdots \subsetneqq M_k = M$$

be a chain of submodules of M of length k. By Lemma 1.12,

$$0 = \ell(M_0) < \ell(M_1) < \cdots < \ell(M_k) = \ell(M).$$

Thus, $k \leq \ell(M)$.

Now consider a composition series of M of length k. By the definition of composition series, $k \geq \ell(M)$ and we just proved that $k \leq \ell(M)$. Thus, $k = \ell(M)$. If a chain has length $\ell(M)$, then it must be maximal and, hence, is a composition series. If the chain has length $< \ell(M)$, then it is not a composition series and hence it may be refined until its length is $\ell(M)$, at which time it will be a composition series. $\square$

According to Example 1.10 (1), if D is a division ring and M is a D-module, then a basis $S = \{x_1, \dots, x_n\}$ with n elements determines a composition series of M of length n. Since all composition series of M must have the same length, we conclude that any two finite bases of M must have the same length n. Moreover, if M had also an infinite basis T, then M would have a linearly independent set consisting of more than n elements. Call this set $\{y_1, \dots, y_k\}$ with $k > n$. Then

$$\langle 0 \rangle \subsetneqq \langle y_1 \rangle \subsetneqq \langle y_1, y_2 \rangle \subsetneqq \cdots \subsetneqq \langle y_1, \dots, y_k \rangle \subsetneqq M$$

is a chain of length $> n$, which contradicts Proposition 1.13. Thus, every basis of M is finite and has n elements. We have arrived at the following result.

(1.14) Proposition. *Let D be a division ring and let M be a D-module with a finite basis. Then every basis of M is finite and all bases have the same number of elements.*

Proof. $\square$

An (almost) equivalent way to state the same result is the following. It can be made equivalent by the convention that D^∞ refers to any infinite direct sum of copies of D, without regard to the cardinality of the index set.

(1.15) Corollary. *If D is a division ring and $D^m \cong D^n$ then $m = n$.*

Proof. $\square$

We conclude our treatment of composition series with the following result, which is frequently useful in constructing induction arguments.

(1.16) Proposition. *Let $0 \longrightarrow K \xrightarrow{\phi} M \xrightarrow{\psi} L \longrightarrow 0$ be a short exact sequence of R-modules. If K and L are of finite length then so is M, and*

$$\ell(M) = \ell(K) + \ell(L).$$

Proof. Let

$$\langle 0 \rangle = K_0 \subsetneqq K_1 \subsetneqq \cdots \subsetneqq K_n = K$$

be a composition series of K, and let

$$\langle 0 \rangle = L_0 \subsetneqq L_1 \subsetneqq \cdots \subsetneqq L_m = L$$

be a composition series for L. For $0 \leq i \leq n$, let $M_i = \phi(K_i)$, and for $n+1 \leq i \leq n+m$, let $M_i = \psi^{-1}(L_{i-n})$. Then $\{M_i\}_{i=0}^{n+m}$ is a chain of submodules of M and

$$M_i/M_{i-1} \cong \begin{cases} K_i/K_{i-1} & \text{for } 1 \leq i \leq n \\ L_{i-n}/L_{i-n-1} & \text{for } n+1 \leq i \leq n+m \end{cases}$$

so that $\{M_i\}_{i=0}^{n+m}$ is a composition series of M. Thus, $\ell(M) = n+m$. $\square$

(1.17) Example. Let R be a PID and let M be a finitely generated torsion R-module. We may write M as a finite direct sum of primary cyclic torsion modules:

$$M \cong \bigoplus_{i=1}^{k} R/\langle p_i^{e_i}\rangle.$$

Then it is an easy exercise to check that M is of finite length and

$$\ell(M) = \sum_{i=1}^{k} e_i.$$

We now return to our consideration of semisimple modules. For this purpose we introduce the following convenient notation.

If M is an R-module and s is a positive integer, then sM will denote the direct sum $M \oplus \cdots \oplus M$ (s summands). More generally, if Γ is any index set then ΓM will denote the R-module $\Gamma M = \oplus_{\gamma\in\Gamma} M_\gamma$ where $M_\gamma = M$ for all $\gamma \in \Gamma$. Of course, if $|\Gamma| = s < \infty$ then $\Gamma M = sM$, and we will prefer the latter notation.

This notation is convenient for describing semisimple modules as direct sums of simple R-modules. If M is a semisimple R-module, then

$$M \cong \bigoplus_{i\in I} M_i \tag{1.3}$$

where M_i is simple for each $i \in I$. If we collect all the simple modules in Equation (1.3) that are isomorphic, then we obtain

$$M \cong \bigoplus_{\alpha\in A} (\Gamma_\alpha M_\alpha) \tag{1.4}$$

where $\{M_\alpha\}_{\alpha\in A}$ is a set of pairwise distinct (i.e., $M_\alpha \ncong M_\beta$ if $\alpha \neq \beta$) simple modules. Equation (1.4) is said to be a **simple factorization** of the semisimple module M. Notice that this is analogous to the prime factorization of elements in a PID. This analogy is made even more compelling by the following uniqueness result for the simple factorization.

(1.18) Theorem. *Suppose that M and N are semisimple R-modules with simple factorizations*

$$M \cong \bigoplus_{\alpha \in A} (\Gamma_\alpha M_\alpha) \tag{1.5}$$

and

$$N \cong \bigoplus_{\beta \in B} (\Lambda_\beta N_\beta) \tag{1.6}$$

where $\{M_\alpha\}_{\alpha \in A}$ *and* $\{N_\beta\}_{\beta \in B}$ *are the distinct simple factors of* M *and* N*, respectively. If* M *is isomorphic to* N*, then there is a bijection* $\psi : A \to B$ *such that* $M_\alpha \cong N_{\psi(\alpha)}$ *for all* $\alpha \in A$*. Moreover,* $|\Gamma_\alpha| < \infty$ *if and only if* $|\Lambda_{\psi(\alpha)}| < \infty$ *and in this case* $|\Gamma_\alpha| = |\Lambda_{\psi(\alpha)}|$.

Proof. Let $\phi : M \to N$ be an isomorphism and let $\alpha \in A$ be given. We may write $M \cong M_\alpha \oplus M'$ with $M' = \oplus_{\gamma \in A \setminus \{\alpha\}} (\Gamma_\gamma M_\gamma) \oplus \Gamma'_\alpha M_\alpha$ where Γ'_α is Γ_α with one element deleted. Then by Proposition 3.3.15,

$$\begin{aligned} \operatorname{Hom}_R(M, N) &\cong \operatorname{Hom}_R(M_\alpha, N) \oplus \operatorname{Hom}_R(M', N) \\ &\cong \Big(\bigoplus_{\beta \in B} \Lambda_\beta \operatorname{Hom}_R(M_\alpha, N_\beta)\Big) \oplus \operatorname{Hom}_R(M', N). \end{aligned} \tag{1.7}$$

By Schur's lemma, $\operatorname{Hom}_R(M_\alpha, N_\beta) = \langle 0 \rangle$ unless $M_\alpha \cong N_\beta$. Therefore, in Equation (1.7) we will have $\operatorname{Hom}_R(M_\alpha, N) = 0$ or $\operatorname{Hom}_R(M_\alpha, N) \cong \Lambda_\beta \operatorname{Hom}_R(M_\alpha, N_\beta)$ for a unique $\beta \in B$. The first alternative cannot occur since the isomorphism $\phi : M \to N$ is identified with $(\phi \circ \iota_1, \phi \circ \iota_2)$ where $\iota_1 : M_\alpha \to M$ is the canonical injection (and $\iota_2 : M' \to M$ is the injection). If $\operatorname{Hom}_R(M_\alpha, N) = 0$ then $\phi \circ \iota_1 = 0$, which means that $\phi|_{M_\alpha} = 0$. This is impossible since ϕ is injective. Thus the second case occurs and we define $\phi(\alpha) = \beta$ where $\operatorname{Hom}_R(M_\alpha, N_\beta) \neq \langle 0 \rangle$. Thus we have defined a function $\phi : A \to B$, which is one-to-one by Schur's lemma. It remains to check that ϕ is surjective. But given $\beta \in B$, we may write $N \cong N_\beta \oplus N'$. Then

$$\operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_R(M, N_\beta) \oplus \operatorname{Hom}_R(M, N')$$

and

$$\operatorname{Hom}_R(M, N_\beta) \cong \prod_{\alpha \in A} \Big(\prod_{\Gamma_\alpha} \operatorname{Hom}_R(M_\alpha, N_\beta)\Big).$$

Since ϕ is surjective, we must have $\operatorname{Hom}_R(M, N_\beta) \neq \langle 0 \rangle$, and thus, Schur's lemma implies that

$$\operatorname{Hom}_R(M, N_\beta) \cong \prod_{\Gamma_\alpha} \operatorname{Hom}(M_\alpha, N_\beta)$$

for a unique $\alpha \in A$. Then $\psi(\alpha) = \beta$, so ψ is surjective.

According to Proposition 3.3.15 and Schur's lemma,

$$\begin{aligned} \operatorname{Hom}_R(M, N) &\cong \prod_{\alpha \in A} \Big(\bigoplus_{\beta \in B} \operatorname{Hom}_R(\Gamma_\alpha M_\alpha, \Lambda_\beta N_\beta)\Big) \\ &\cong \prod_{\alpha \in A} \operatorname{Hom}_R(\Gamma_\alpha M_\alpha, \Lambda_{\psi(\alpha)} N_{\psi(\alpha)}). \end{aligned}$$

Therefore, $\phi \in \mathrm{Hom}_R(M, N)$ is an isomorphism if and only if

$$\phi_\alpha = \phi|_{\Gamma_\alpha M_\alpha} : \Gamma_\alpha M_\alpha \to \Lambda_{\psi(\alpha)} N_{\psi(\alpha)}$$

is an isomorphism for all $\alpha \in A$. But by the definition of ψ and Schur's lemma, M_α is isomorphic to $N_{\psi(\alpha)}$. Also, $\Gamma_\alpha M_\alpha$ has length $|\Gamma_\alpha|$, and $\Lambda_{\psi(\alpha)} N_{\psi(\alpha)}$ has length $|\Lambda_{\psi(\alpha)}|$, and since isomorphic modules have the same length, $|\Gamma_{\alpha)}| = |\Lambda_{\psi(\alpha)}|$, completing the proof. □

(1.19) Corollary. *Let M be a semisimple R-module and suppose that M has two simple factorizations*

$$M \cong \bigoplus_{\alpha \in A} (\Gamma_\alpha M_\alpha) \cong \bigoplus_{\beta \in B} (\Lambda_\beta N_\beta)$$

with distinct simple factors $\{M_\alpha\}_{\alpha \in A}$ and $\{N_\beta\}_{\beta \in B}$. Then there is a bijection $\psi : A \to B$ such that $M_\alpha \cong N_{\psi(\alpha)}$ for all $\alpha \in A$. Moreover, $|\Gamma_\alpha| < \infty$ if and only if $|\Lambda_{\psi(\alpha)}| < \infty$ and in this case $|\Gamma_\alpha| = |\Lambda_{\psi(\alpha)}|$.

Proof. Take $\phi = 1_M$ in Theorem 1.18. □

(1.20) *Remarks.*

(1) While it is true in Corollary 1.19 that $M_\alpha \cong N_{\psi(\alpha)}$ (isomorphism as R-modules), it is not necessarily true that $M_\alpha = N_{\psi(\alpha)}$. For example, let $R = F$ be a field and let M be a vector space over F of dimension s. Then for *any* choice of basis $\{m_1, \ldots, m_s\}$ of M, we obtain a direct sum decomposition

$$M \cong Rm_1 \oplus \cdots \oplus Rm_s.$$

(2) In Theorem 1.18 we have been content to distinguish between finite and infinite index sets Γ_α, but we are not distinguishing between infinite sets of different cardinality. Using the theory of cardinal arithmetic, one can refine Theorem 1.18 to conclude that $|\Gamma_\alpha| = |\Lambda_{\psi(\alpha)}|$ for *all* $\alpha \in A$, where $|S|$ denotes the cardinality of the set S.

We will now present some alternative characterizations of semisimple modules. The following notation, which will be used only in this section, will be convenient for this purpose. Let $\{M_i\}_{i \in I}$ be a set of submodules of a module M. Then let

$$M_I = \sum_{i \in I} M_i$$

be the sum of the submodules $\{M_i\}_{i \in I}$.

(1.21) Lemma. *Let M be an R-module that is a sum of simple submodules $\{M_i\}_{i \in I}$, and let N be an arbitrary submodule of M. Then there is a subset $J \subseteq I$ such that*

$$M \cong N \oplus \Big(\bigoplus_{i \in J} M_i \Big).$$

Proof. The proof is an application of Zorn's lemma. Let

$$\mathcal{S} = \Big\{ P \subseteq I : M_P \cong \bigoplus_{i \in P} M_i \text{ and } M_P \cap N = \langle 0 \rangle \Big\}.$$

Partially order $\mathcal{S}$ by inclusion and let $\mathcal{C} = \{P_\alpha\}_{\alpha \in A}$ be an arbitrary chain in $\mathcal{S}$. If $P = \cup_{\alpha \in A} P_\alpha$, we claim that $P \in \mathcal{S}$. Suppose that $P \notin \mathcal{S}$. Since it is clear that $M_P \cap N = \langle 0 \rangle$, we must have that $M_P \not\cong \oplus_{i \in P} M_i$. Then Theorem 3.3.2 shows that there is some $p_0 \in P$, such that $M_{p_0} \cap M_{P'} \neq \langle 0 \rangle$, where $P' = P \setminus \{p_0\}$. Suppose that $0 \neq x \in M_{p_0} \cap M_{P'}$. Then we may write

$$x = x_{p_1} + \cdots + x_{p_k} \tag{1.8}$$

where $x_{p_i} \neq 0 \in M_{p_i}$ for $\{p_1, \ldots, p_k\} \subseteq P'$. Since $\mathcal{C}$ is a chain, there is an index $\alpha \in A$ such that $\{p_0, p_1, \ldots, p_k\} \subseteq P_\alpha$. Equation (1.8) shows that $M_{P_\alpha} \not\cong \oplus_{i \in P_\alpha} M_i$, which contradicts the fact that $P_\alpha \in \mathcal{S}$. Therefore, we must have $P \in \mathcal{S}$, and Zorn's lemma applies to conclude that $\mathcal{S}$ has a maximal element J.

Claim. $M = N + M_J \cong N \oplus \big(\bigoplus_{i \in J} M_i \big)$.

If this were not true, then there would be an index $i_0 \in I$ such that $M_{i_0} \not\subseteq N + M_J$. This implies that $M_{i_0} \not\subseteq N$ and $M_{i_0} \not\subseteq M_J$. Since $M_{i_0} \cap N$ and $M_{i_0} \cap M_J$ are proper submodules of M_{i_0}, it follows that $M_{i_0} \cap N = \langle 0 \rangle$ and $M_{i_0} \cap M_J = \langle 0 \rangle$ because M_{i_0} is a simple R-module. Therefore, $\{i_0\} \cup J \in \mathcal{S}$, contradicting the maximality of J. Hence, the claim is proved. □

(1.22) Corollary. *If an R-module M is a sum of simple submodules, then M is semisimple.*

Proof. Take $N = \langle 0 \rangle$ in Theorem 1.21. □

(1.23) Theorem. *If M is an R-module, then the following are equivalent:*

(1) *M is a semisimple module.*
(2) *Every submodule of M is complemented.*
(3) *Every submodule of M is a sum of simple R-modules.*

Proof. (1) $\Rightarrow$ (2) follows from Lemma 1.21, and (3) $\Rightarrow$ (1) is immediate from Corollary 1.22. It remains to prove (2) $\Rightarrow$ (3).

Let M_1 be a submodule of M. First we observe that every submodule of M_1 is complemented in M_1. To see this, suppose that N is any submodule of M_1. Then N is complemented in M, so there is a submodule N' of M such that $N \oplus N' \cong M$. But then $N + (N' \cap M_1) = M_1$ so that $N \oplus (N' \cap M_1) \cong M_1$, and hence N is complemented in M_1.

Next we claim that every nonzero submodule M_2 of M contains a nonzero simple submodule. Let $m \in M_2$, $m \neq 0$. Then $Rm \subseteq M_2$ and, furthermore, $R/\operatorname{Ann}(m) \cong Rm$ where $\operatorname{Ann}(m) = \{a \in R : am = 0\}$ is a left ideal of R. A simple Zorn's lemma argument (see the proof of Theorem 2.2.16) shows that there is a maximal left ideal I of R containing $\operatorname{Ann}(m)$. Then Im is a maximal submodule of Rm by the correspondence theorem. By the previous paragraph, Im is a complemented submodule of Rm, so there is a submodule N of Rm with $N \oplus Im \cong Rm$, and since Im is a maximal submodule of Rm, it follows that the submodule N is simple. Therefore, we have produced a simple submodule of M_2.

Now consider an arbitrary submodule N of M, and let $N_1 \subseteq N$ be the sum of all the simple submodules of N. We claim that $N_1 = N$. N_1 is complemented in N, so we may write $N \cong N_1 \oplus N_2$. If $N_2 \neq \langle 0 \rangle$ then N_2 has a nonzero simple submodule N', and since $N' \subseteq N$, it follows that $N' \subseteq N_1$. But $N_1 \cap N_2 = \langle 0 \rangle$. This contradiction shows that $N_2 = \langle 0 \rangle$, i.e., $N = N_1$, and the proof is complete. □

(1.24) Corollary. *Sums, submodules, and quotient modules of semisimple modules are semisimple.*

Proof. Sums: This follows immediately from Corollary 1.22.

Submodules: Any submodule of a semisimple module satisfies condition (3) of Theorem 2.23.

Quotient modules: If M is a semisimple module, $N \subseteq M$ is a submodule, and $Q = M/N$, then N has a complement N' in M, i.e., $M \cong N \oplus N'$. But then $Q \cong N'$, so Q is isomorphic to a submodule of M, and hence, is semisimple. □

(1.25) Corollary. *Let M be a semisimple R-module and let $N \subseteq M$ be a submodule. Then N is irreducible (simple) if and only if N is indecomposable.*

Proof. Since every irreducible module is indecomposable, we need to show that if N is not irreducible, then N is not indecomposable. Let N_1 be a nontrivial proper submodule of N. Then N is semisimple by Corollary 1.24, so N_1 has a complement by Theorem 1.23, and N is not indecomposable. □

(1.26) *Remark.* The fact that every submodule of a semisimple R-module M is complemented is equivalent (by Theorem 3.3.9) to the statment that whenever M is a semisimple R-module, every short exact sequence

$$0 \longrightarrow N \longrightarrow M \longrightarrow K \longrightarrow 0$$

of R-modules splits.

(1.27) Definition. *A ring R is called* **semisimple** *if R is semisimple as a left R-module.*

Remark. The proper terminology should be "left semisimple," with an analogous definition of "right semisimple," but we shall see below that the two notions coincide.

(1.28) Theorem. *The following are equivalent for a ring R:*

(1) *R is a semisimple ring.*
(2) *Every R-module is semisimple.*
(3) *Every R-module is projective.*

Proof. (1) $\Rightarrow$ (2). Let M be an R-module. By Proposition 3.4.14, M has a free presentation

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

so that M is a quotient of the free R-module F. Since F is a direct sum of copies of R and R is assumed to be semisimple, it follows that F is semisimple, and hence M is also (Corollary 1.24).

(2) $\Rightarrow$ (3). Assume that every R-module is semisimple, and let P be an arbitrary R-module. Suppose that

$$0 \longrightarrow K \longrightarrow M \longrightarrow P \longrightarrow 0 \tag{1.9}$$

is a short exact sequence. Since M is R-module, our assumption is that it is semisimple and then Remark 1.26 implies that sequence (1.9) is split exact. Since (1.9) is an arbitrary short exact sequence with P on the right, it follows from Theorem 3.5.1 that P is projective.

(3) $\Rightarrow$ (1). Let M be an arbitrary submodule of R (i.e., an arbitrary left ideal). Then we have a short exact sequence

$$0 \longrightarrow M \longrightarrow R \longrightarrow R/M \longrightarrow 0.$$

Since all R-modules are assumed projective, we have that R/M is projective, and hence (by Theorem 3.5.1) this sequence splits. Therefore, $R \cong M \oplus N$ for some submodule $N \subseteq R$, which is isomorphic (as an R-module) to R/M. Then by Theorem 1.23, R is semisimple. $\square$

(1.29) Corollary. *Let R be a semisimple ring and let M be an R-module. Then M is irreducible (simple) if and only if M is indecomposable.*

Proof. $\square$

(1.30) Theorem. *Let R be a semisimple ring. Then every simple R-module is isomorphic to a submodule of R.*

Proof. Let N be a simple R-module, and let $R = \oplus_{i \in I} M_i$ be a simple factorization of the semisimple R-module R. We must show that at least one of the simple R-modules M_i is isomorphic to N. If this is not the case, then

$$\mathrm{Hom}_R(R,\, N) \cong \mathrm{Hom}_R\Big(\bigoplus_{i\in I} M_i,\, N\Big) \cong \prod_{i\in I} \mathrm{Hom}_R(M_i,\, N) = \langle 0\rangle$$

where the last equality is because $\mathrm{Hom}_R(M_i,\, N) = \langle 0\rangle$ if M_i is not isomorphic to N (Schur's lemma). But $\mathrm{Hom}_R(R,\, N) \cong N \neq \langle 0\rangle$, and this contradiction shows that we must have N isomorphic to one of the simple submodules M_i of R. □

(1.31) Corollary. *Let R be a semisimple ring.*

(1) *There are only finitely many isomorphism classes of simple R-modules.*
(2) *If $\{M_\alpha\}_{\alpha\in A}$ is the set of isomorphism classes of simple R-modules and*

$$R \cong \bigoplus_{\alpha\in A} \Big(\bigoplus \Gamma_\alpha M_\alpha\Big),$$

then each Γ_α is finite.

Proof. Since R is semisimple, we may write

$$R = \bigoplus_{\beta\in B} N_\beta$$

where each N_β is simple. We will show that B is finite, and then both finiteness statements in the corollary are immediate from Theorem 1.30.

Consider the identity element $1 \in R$. By the definition of direct sum, we have

$$1 = \sum_{\beta\in B} r_\beta n_\beta$$

for some elements $r_\beta \in R$, $n_\beta \in N_\beta$, with all but finitely many r_β equal to zero. Of course, each N_β is a left R-submodule of R, i.e., a left ideal.

Now suppose that B is infinite. Then there is a $\beta_0 \in B$ for which $r_{\beta_0} = 0$. Let n be any nonzero element of N_{β_0}. Then

$$n = n\cdot 1 = n\Big(\sum_{\beta\in B} r_\beta n_\beta\Big) = \sum_{\beta\in B\setminus\{\beta_0\}} (nr_\beta)n_\beta,$$

so

$$n \in \bigoplus_{\beta\in B\setminus\{\beta_0\}} N_\beta.$$

Thus,

$$n \in N_{\beta_0} \bigcap \Big(\bigoplus_{\beta\in B\setminus\{\beta_0\}} N_\beta\Big) = \{0\},$$

by the definition of direct sum again, which is a contradiction. Hence, B is finite. □

We now come to the basic structure theorem for semisimple rings.

(1.32) Theorem. (Wedderburn) *Let R be a semisimple ring. Then there is a finite collection of integers $n_1, \ldots, n_k$, and division rings $D_1, \ldots, D_k$ such that*

$$R \cong \bigoplus_{i=1}^{k} \mathrm{End}_{D_i}(D_i^{n_i}).$$

Proof. By Corollary 1.31, we may write

$$R \cong \bigoplus_{i=1}^{k} n_i M_i$$

where $\{M_i\}_{i=1}^{k}$ are the distinct simple R-modules and $n_1, \ldots, n_k$ are positive integers. Then

$$\begin{aligned} R &\cong \mathrm{End}_R(R) \\ &\cong \mathrm{Hom}_R\Big(\bigoplus_{i=1}^{k} n_i M_i, \bigoplus_{i=1}^{k} n_i M_i\Big) \\ &\cong \bigoplus_{i=1}^{k} \mathrm{Hom}_R(n_i M_i, n_i M_i) \\ &\cong \bigoplus_{i=1}^{k} \mathrm{End}_R(n_i M_i), \end{aligned}$$

by Schur's lemma. Also, by Schur's lemma, $\mathrm{End}_R(M_i)$ is a division algebra, which we denote by D_i, for each $i = 1, \ldots, k$. Then it is easy to check (compare the proof of Theorem 1.18) that

$$\mathrm{End}_R(n_i M_i) \cong \mathrm{End}_{D_i}(D_i^{n_i}),$$

completing the proof. □

Remark. Note that by Corollary 4.3.9, $\mathrm{End}_D(D^n)$ is isomorphic to $M_n(D^{\mathrm{op}})$. Thus, Wedderburn's theorem is often stated as, *Every semisimple ring is isomorphic to a finite direct sum of matrix rings over division rings.*

(1.33) Lemma. *Let D be a division ring and n a positive integer. Then $R = \mathrm{End}_D(D^n)$ is semisimple as a left R-module and also as a right R-module. Furthermore, R is semisimple as a left D-module and as a right D-module.*

Proof. Write $D^n = D_1 \oplus D_2 \oplus \cdots \oplus D_n$ where $D_i = D$. Let

$$M_i = \Big\{ f \in \mathrm{End}_D(D^n) : \mathrm{Ker}(f) \supseteq \bigoplus_{k \neq i} D_k \Big\},$$
$$N_j = \{ f \in \mathrm{End}_D(D^n) : \mathrm{Im}(f) \subseteq D_j \},$$

and let

$$P_{ij} = M_i \cap N_j.$$

Note that $P_{ij} \cong D$. Then

$$\mathrm{End}_D(D^n) \cong M_1 \oplus \cdots \oplus M_n$$

as a left R-module, and

$$\mathrm{End}_D(D^n) \cong N_1 \oplus \cdots \oplus N_n$$

as a right R-module. We leave it to the reader to check that each M_i (resp., N_j) is a simple left (resp., right) R-module. Also,

$$\mathrm{End}_D(D^n) \cong \bigoplus P_{ij}$$

as a left (resp., right) D-module, and each P_{ij} is certainly simple (on either side). □

(1.34) Corollary. *A ring R is semisimple as a left R-module if and only if it is semisimple as a right R-module.*

Proof. This follows immediately from Theorem 1.32 and Lemma 1.33. □

Observe that R is a simple left R-module (resp., right R-module) if and only if R has no nontrivial proper left (resp., right) ideals, which is the case if and only if R is a division algebra. Thus, to define simplicity of R in this way would bring nothing new. Instead we make the following definition:

(1.35) Definition. *A ring R with identity is* **simple** *if it has no nontrivial proper (two-sided) ideals.*

Remark. In the language of the next section, this definition becomes *"A ring R with identity is simple if it is simple as an (R, R)-bimodule."*

(1.36) Corollary. *Let D be a division ring and n a positive integer. Then $\mathrm{End}_D(D^n)$ is a simple ring that is semisimple as a left $\mathrm{End}_D(D^n)$-module.*

Conversely, if R is a simple ring that is semisimple as a left R-module, or, equivalently, as a right R-module, then

$$R \cong \mathrm{End}_D(D^n)$$

for some division ring D and positive integer n.

Proof. We leave it to the reader to check that $\mathrm{End}_D(D^n)$ is simple (compare Theorem 2.2.26 and Corollary 2.2.27), and then the first part of the corollary follows from Lemma 1.33. Conversely, if R is semisimple we have the decomposition given by Wedderburn's theorem (Theorem 1.32), and then the condition of simplicity forces $k = 1$. □

Our main interest in semisimple rings and modules is in connection with our investigation of group representation theory, but it is also of interest to reconsider modules over a PID from this point of view. Thus let R be a PID. We wish to give a criterion for R-modules to be semisimple. The following easy lemma is left as an exercise.

(1.37) Lemma. *Let R be an integral domain. Then R is a semisimple ring if and only if R is a field. If R is a field, R is simple.*

Proof. Exercise. □

From this lemma and Theorem 1.28, we see that if R is a field, then every R-module (i.e., vector space) is semisimple and there is nothing more to say. For the remainder of this section, we will assume that R is a PID that is not a field.

Let M be a finitely generated R-module. Then by Corollary 3.6.9, we have that $M \cong F \oplus M_\tau$, where F is free (of finite rank) and M_τ is the torsion submodule of M. If $F \neq \langle 0 \rangle$ then Lemma 1.37 shows that M is not semisimple. It remains to consider the case where $M = M_\tau$, i.e., where M is a finitely generated torsion module. Recall from Theorem 3.7.13 that each such M is a direct sum of primary cyclic R-modules.

(1.38) Proposition. *Let M be a primary cyclic R-module (where R is a* PID *is not a field) and assume that* $\mathrm{Ann}(M) = \langle p^e \rangle$ *where $p \in R$ is a prime. If $e = 1$ then M is simple. If $e > 1$, then M is not semisimple.*

Proof. First suppose that $e = 1$, so that $M \cong R/\langle p \rangle$. Then M is a simple R-module because $\langle p \rangle$ is a prime ideal in the PID R, and hence, it is a maximal ideal.

Next suppose that $e > 1$. Then

$$\langle 0 \rangle \neq p^{e-1}M \subsetneqq M,$$

and $p^{e-1}M$ is a proper submodule of M, which is not complemented; hence, M is not semisimple by Theorem 1.28. □

(1.39) Theorem. *Let M be a finitely generated torsion R-module (where R is a* PID *that is not a field). Then M is semisimple if and only if* $\mathrm{me}(M)$

(see Definition 3.7.8) is a product of distinct prime factors. M is a simple R-module if and only if

$$\mathrm{me}(M) = \mathrm{co}(M) = \langle p \rangle$$

where $p \in R$ is a prime.

Proof. First suppose that M is cyclic, and $\mathrm{me}(M) = \langle p_1^{e_1} \dots p_k^{e_k} \rangle$. Then the primary decomposition of M is given by

$$M \cong (R/\langle p_1^{e_1} \rangle) \oplus \cdots \oplus (R/\langle p_k^{e_k} \rangle) \,,$$

and M is semisimple if and only if each of the summands is semisimple, which by Proposition 1.38, is true if and only if

$$e_1 = e_2 = \cdots = e_k = 1.$$

Now let M be general. Then by Theorem 3.7.1, there is a cyclic decomposition

$$M \cong Rw_1 \oplus \cdots \oplus Rw_n$$

such that $\mathrm{Ann}(w_i) = \langle s_i \rangle$ and $s_i \mid s_{i+1}$ for $1 \le i \le n-1$. Then M is semisimple if and only if each of the cyclic submodules Rw_i is semisimple, which occurs (by the previous paragraph) if and only if s_i is a product of distinct prime factors. Since $s_i \mid s_{i+1}$, this occurs if and only if $s_n = \mathrm{me}(M)$ is a product of distinct prime factors. The second assertion is then easy to verify. □

(1.40) *Remark.* In the two special cases of finite abelian groups and linear transformations that we considered in some detail in Chapters 3 and 4, Theorem 1.39 takes the following form:

(1) A finite abelian group is semisimple if and only if it is the direct product of cyclic groups of prime order, and it is simple if and only if it is cyclic of prime order.

(2) Let V be a finite-dimensional vector space over a field F and let $T : V \to V$ be a linear transformation. Then V_T is a semisimple $F[X]$-module if and only if the minimal polynomial $m_T(X)$ of T is a product of distinct irreducible factors and is simple if and only if its characteristic polynomial $c_T(X)$ is equal to its minimal polynomial $m_T(X)$, this polynomial being irreducible (see Lemma 4.4.11.) If F is algebraically closed (so that the only irreducible polynomials are linear ones) then V_T is semisimple if and only if T is diagonalizable and simple if and only if V is one-dimensional (see Corollary 4.4.32).

7.2 Multilinear Algebra

We have three goals in this section: to introduce the notion of a bimodule, to further our investigation of "Hom," and to introduce and investigate tensor products. The level of generality of the material presented in this section is dictated by the applications to the theory of group representations. For this reason, most of the results will be concerned with modules over rings that are not commutative; frequently there will be more than one module structure on the same abelian group, and many of the results are concerned with the interaction of these various module structures. We start with the concept of bimodule.

(2.1) Definition. *Let R and S be rings. An abelian group M is an* (R, S)**-bimodule** *if M is both a left R-module and a right S-module, and the compatibility condition*

$$r(ms) = (rm)s \tag{2.1}$$

is satisfied for every $r \in R$, $m \in M$, and $s \in S$.

(2.2) Examples.

(1) Every left R-module is an $(R, \mathbf{Z})$-bimodule, and every right S-module is a $(\mathbf{Z}, S)$-bimodule.

(2) If R is a commutative ring, then every left or right R-module is an (R, R)-bimodule in a natural way. Indeed, if M is a left R-module, then according to Remark 3.1.2 (1), M is also a right R-module by means of the operation $mr = rm$. Then Equation (2.1) is

$$r(ms) = r(sm) = (rs)m = (sr)m = s(rm) = (rm)s.$$

(3) If T is a ring and R and S are subrings of T (possibly with $R = S = T$), then T is an (R, S)-bimodule. Note that Equation (2.1) is simply the associative law in T.

(4) If M and N are left R-modules, then the abelian group $\mathrm{Hom}_R(M, N)$ has the structure of an $(\mathrm{End}_R(N), \mathrm{End}_R(M))$-bimodule, as follows. If $f \in \mathrm{Hom}_R(M, N)$, $\phi \in \mathrm{End}_R(M)$, and $\psi \in \mathrm{End}_R(N)$, then define $f\phi = f \circ \phi$ and $\psi f = \psi \circ f$. These definitions provide a left $\mathrm{End}_R(N)$-module and a right $\mathrm{End}_R(M)$-module structure on $\mathrm{Hom}_R(M, N)$, and Equation (2.1) follows from the associativity of composition of functions.

(5) Recall that a ring T is an R-algebra, if T is an R-module and the R-module structure on T and the ring structure of T are compatible, i.e., $r(t_1t_2) = (rt_1)t_2 = t_1(rt_2)$ for all $r \in R$ and $t_1, t_2 \in T$. If T happens to be an (R, S)-bimodule, such that $r(t_1t_2) = (rt_1)t_2 = t_1(rt_2)$ and $(t_1t_2)s = t_1(t_2s) = (t_1s)t_2$ for all $r \in R$, $s \in S$, and $t_1, t_2 \in T$, then we

say that T is an (R, S)**-bialgebra**. For example, if R and S are subrings of a commutative ring T, then T is an (R, S)-bialgebra.

Suppose that M is an (R, S)-bimodule and $N \subseteq M$ is a subgroup of the additive abelian group of M. Then N is said to be an (R, S)**-bisubmodule** of M if N is both a left R-submodule and a right S-submodule of M. If M_1 and M_2 are (R, S)-bimodules, then a function $f : M_1 \to M_2$ is an (R, S)**-bimodule homomorphism** if it is both a left R-module homomorphism and a right S-module homomorphism. The set of (R, S)-bimodule homomorphisms will be denoted $\mathrm{Hom}_{(R,S)}(M_1, M_2)$. Since bimodule homomorphisms can be added, this has the structure of an abelian group, but, a priori, nothing more. If $f : M_1 \to M_2$ is an (R, S)-bimodule homomorphism, then it is a simple exercise to check that $\mathrm{Ker}(f) \subseteq M_1$ and $\mathrm{Im}(f) \subseteq M_2$ are (R, S)-bisubmodules.

Furthermore, if $N \subseteq M$ is an (R, S)-bisubmodule, then the quotient abelian group is easily seen to have the structure of an (R, S)-bimodule. We leave it as an exercise for the reader to formulate and verify the noether isomorphism theorems (see Theorems 3.2.3 to 3.2.6) in the context of (R, S)-bimodules. It is worth pointing out that if M is an (R, S)-bimodule, then there are three distinct concepts of submodule of M, namely, R-submodule, S-submodule, and (R, S)-bisubmodule. Thus, if $X \subseteq M$, then one has three concepts of submodule of M generated by the set X. To appreciate the difference, suppose that $X = \{x\}$ consists of a single element $x \in M$. Then the R-submodule generated by X is the set

$$Rx = \{rx : r \in R\}, \tag{2.2}$$

the S-submodule generated by X is the set

$$xS = \{xs : s \in S\}, \tag{2.3}$$

while the (R, S)-bisubmodule generated by X is the set

$$RxS = \{\sum_{i=1}^{n} r_i x s_i : n \in \mathbf{N} \text{ and } r_i \in R,\ s_i \in S \text{ for } 1 \le i \le n\}. \tag{2.4}$$

(2.3) Examples.

(1) If R is a ring, then a left R-submodule of R is a left ideal, a right R-submodule is a right ideal, and an (R, R)-bisubmodule of R is a (two-sided) ideal.

(2) As a specific example, let $R = M_2(\mathbf{Q})$ and let $x = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then the left R-submodule of R generated by $\{x\}$ is

$$Rx = \left\{ \begin{bmatrix} a & 0 \\ b & 0 \end{bmatrix} : a,\, b \in \mathbf{Q} \right\},$$

the right R-submodule of R generated by $\{x\}$ is

$$xR = \left\{ \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} : a,\, b \in \mathbf{Q} \right\},$$

while the (R, R)-bisubmodule of R generated by $\{x\}$ is R itself (see Theorem 2.2.26).

When considering bimodules, there are (at least) three distinct types of homomorphisms that can be considered. In order to keep them straight, we will adopt the following notational conventions. If M and N are left R-modules (in particular, both could be (R, S)-bimodules, or one could be an (R, S)-bimodule and the other a (R, T)-bimodule), then $\mathrm{Hom}_R(M,\, N)$ will denote the set of (left) R-module homomorphisms from M to N. If M and N are right S-modules, then $\mathrm{Hom}_{-S}(M,\, N)$ will denote the set of all (right) S-module homomorphisms. If M and N are (R, S)-bimodules, then $\mathrm{Hom}_{(R,S)}(M,\, N)$ will denote the set of all (R, S)-bimodule homomorphisms from M to N. With no additional hypotheses, the only algebraic structure that can be placed upon these sets of homomorphisms is that of abelian groups, i.e., addition of homomorphisms is a homomorphism in each situation described. The first thing to be considered is what additional structure is available.

(2.4) Proposition. *Suppose that M is an (R, S)-bimodule and N is an (R, T)-bimodule. Then $\mathrm{Hom}_R(M,\, N)$ can be given the structure of an (S, T)-bimodule.*

Proof. We must define compatible left S-module and right T-module structures on $\mathrm{Hom}_R(M,\, N)$. Thus, let $f \in \mathrm{Hom}_R(M,\, N)$, $s \in S$, and $t \in T$. Define sf and ft as follows:

$$sf(m) = f(ms) \quad \text{for all } m \in M \tag{2.5}$$

and

$$ft(m) = f(m)t \quad \text{for all } m \in M. \tag{2.6}$$

We must show that Equation (2.5) defines a left S-module structure on $\mathrm{Hom}_R(M,\, N)$ and that Equation (2.6) defines a right T-module structure on $\mathrm{Hom}_R(M,\, N)$, and we must verify the compatibility condition $s(ft) = (sf)t$.

We first verify that sft is an R-module homomorphism. To see this, suppose that $r_1,\, r_2 \in R$, $m_1,\, m_2 \in M$ and note that

$$\begin{aligned} sft(r_1m_1 + r_2m_2) &= f((r_1m_1 + r_2m_2)s)t \\ &= f((r_1m_1)s + (r_2m_2)s)t \\ &= f(r_1(m_1s) + r_2(m_2s))t \\ &= (r_1f(m_1s) + r_2f(m_2s))\,t \end{aligned}$$

$$
\begin{aligned}
&= (r_1 f(m_1 s))t + (r_2 f(m_2 s))t \\
&= r_1(f(m_1 s)t) + r_2(f(m_2 s)t) \\
&= r_1(sft)(m_1) + r_2(sft)(m_2),
\end{aligned}
$$

where the third equality follows from the (R, S)-bimodule structure on M, while the next to last equality is a consequence of the (R, T)-bimodule structure on N. Thus, sft is an R-module homomorphism for all $s \in S$, $t \in T$, and $f \in \mathrm{Hom}_R(M,\, N)$.

Now observe that, if s_1, $s_2 \in S$ and $m \in M$, then

$$
\begin{aligned}
(s_1(s_2 f))\,(m) &= (s_2 f)(m s_1) \\
&= f((m s_1) s_2) \\
&= f(m(s_1 s_2)) \\
&= ((s_1 s_2) f)\,(m)
\end{aligned}
$$

so that $\mathrm{Hom}_R(M,\, N)$ satisfies axiom $(\mathrm{c_l})$ of Definition 3.1.1. The other axioms are automatic, so $\mathrm{Hom}_R(M,\, N)$ is a left S-module. Similarly, if t_1, $t_2 \in T$ and $m \in M$, then

$$
\begin{aligned}
((f t_1) t_2)\,(m) &= ((f t_1)(m))\, t_2 \\
&= (f(m) t_1)\, t_2 \\
&= f(m)(t_1 t_2) \\
&= (f(t_1 t_2))\,(m).
\end{aligned}
$$

Thus, $\mathrm{Hom}_R(M,\, N)$ is a right T-module by Definition 3.1.1 (2). We have only checked axiom $(\mathrm{c_r})$, the others being automatic.

It remains to check the compatibility of the left S-module and right T-module structures. But, if $s \in S$, $t \in T$, $f \in \mathrm{Hom}_R(M,\, N)$, and $m \in M$, then

$$
((sf)t)\,(m) = (sf)(m)t = f(ms)t = (ft)(ms) = s(ft)(m).
$$

Thus, $(sf)t = s(ft)$ and $\mathrm{Hom}_R(M,\, N)$ is an (S, T)-bimodule, which completes the proof of the proposition. □

Proved in exactly the same way is the following result concerning the bimodule structure on the set of right R-module homomorphisms.

(2.5) Proposition. *Suppose that M is an (S, R)-bimodule and N is a (T, R)-bimodule. Then* $\mathrm{Hom}_{-R}(M,\, N)$ *has the structure of a (T, S)-bimodule, via the module operations*

$$
(tf)(m) = t(f(m)) \quad \textit{and} \quad (fs)(m) = f(sm)
$$

where $s \in S$, $t \in T$, $f \in \mathrm{Hom}_{-R}(M,\, N)$, and $m \in M$.

Proof. Exercise. □

Some familiar results are corollaries of these propositions. (Also see Example 3.1.5 (10).)

(2.6) Corollary.

(1) *If M is a left R-module, then $M^* = \operatorname{Hom}_R(M, R)$ is a right R-module.*
(2) *If M and N are (R, R)-bimodules, then $\operatorname{Hom}_R(M, N)$ is an (R, R)-bimodule, and $\operatorname{End}_R(M)$ is an (R, R)-bialgebra. In particular, this is the case when the ring R is commutative.*

Proof. Exercise. □

Remark. If M and N are both (R, S)-bimodules, then the set of bimodule homomorphisms $\operatorname{Hom}_{(R,S)}(M, N)$ has only the structure of an abelian group.

Theorem 3.3.10 generalizes to the following result in the context of bimodules. The proof is identical, and hence it will be omitted.

(2.7) Theorem. *Let*

$$0 \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \tag{2.7}$$

be a sequence of (R, S)-bimodules and (R, S)-bimodule homomorphisms. Then the sequence (2.7) *is exact if and only if the sequence*

$$0 \longrightarrow \operatorname{Hom}_R(N, M_1) \xrightarrow{\phi_*} \operatorname{Hom}_R(N, M) \xrightarrow{\psi_*} \operatorname{Hom}_R(N, M_2) \tag{2.8}$$

is an exact sequence of (T, S)-bimodules for all (R, T)-bimodules N.

If

$$M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \longrightarrow 0 \tag{2.9}$$

is a sequence of (R, S)-bimodules and (R, S)-bimodule homomorphisms, then the sequence (2.9) *is exact if and only if the sequence*

$$0 \longrightarrow \operatorname{Hom}_R(M_2, N) \xrightarrow{\psi^*} \operatorname{Hom}_R(M, N) \xrightarrow{\phi^*} \operatorname{Hom}_R(M_1, N) \tag{2.10}$$

is an exact sequence of (S, T)-bimodules for all (R, T)-bimodules N.

Proof. □

Similarly, the proof of the following result is identical to the proof of Theorem 3.3.12.

(2.8) Theorem. *Let N be a fixed (R, T)-bimodule. If*

$$0 \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \longrightarrow 0 \tag{2.11}$$

is a split short exact sequence of (R,S)*-bimodules, then*

$$0 \longrightarrow \operatorname{Hom}_R(N,\, M_1) \xrightarrow{\phi_*} \operatorname{Hom}_R(N,\, M) \xrightarrow{\psi_*} \operatorname{Hom}_R(N,\, M_2) \longrightarrow 0 \tag{2.12}$$

is a split short exact sequence of (T,S)*-bimodules, and*

$$0 \longrightarrow \operatorname{Hom}_R(M_2,\, N) \xrightarrow{\psi^*} \operatorname{Hom}_R(M,\, N) \xrightarrow{\phi^*} \operatorname{Hom}_R(M_1,\, N) \longrightarrow 0 \tag{2.13}$$

is a split short exact sequence of (S,T)*-bimodules.*

Proof. □

This concludes our brief introduction to bimodules and module structures on spaces of homomorphisms; we turn our attention now to the concept of tensor product of modules. As we shall see, Hom and tensor products are closely related, but unfortunately, there is no particularly easy definition of tensor products. On the positive side, the use of the tensor product in practice does not usually require an application of the definition, but rather fundamental properties (easier than the definition) are used.

Let M be an (R,S)-bimodule and let N be an (S,T)-bimodule. Let F be the free abelian group on the index set $M \times N$ (Remark 3.4.5). Recall that this means that $F = \oplus_{(m,n)\in M\times N}\mathbf{Z}_{(m,n)}$ where $\mathbf{Z}_{(m,n)} = \mathbf{Z}$ for all $(m,\, n) \in M \times N$, and that a basis of F is given by $S = \{e_{(m,n)}\}_{(m,n)\in M\times N}$ where $e_{(m,n)} = (\delta_{mk}\delta_{n\ell})_{(k,\ell)\in M\times N}$, that is, $e_{(m,n)} = 1$ in the component of F corresponding to the element $(m,n) \in M \times N$ and $e_{(m,n)} = 0$ in all other components. As is conventional, we will identify the basis element $e_{(m,n)}$ with the element $(m,n) \in M \times N$. Thus a typical element of F is a linear combination

$$\sum_{(m,n)\in M\times N} c_{(m,n)}(m,n)$$

where $c_{(m,n)} \in \mathbf{Z}$ and all but finitely many of the integers $c_{(m,n)}$ are 0. Note that F can be given the structure of an (R,T)-bimodule via the multiplication

$$r\left(\sum_{i=1}^{k} c_i(m_i,\, n_i)\right)t = \sum_{i=1}^{k} c_i(rm_i,\, n_it) \tag{2.14}$$

where $r \in R$, $t \in T$, and $c_1,\, \ldots\, , c_k \in \mathbf{Z}$.

Let $K \subseteq F$ be the subgroup of F generated by the subset $H_1 \cup H_2 \cup H_3$ where the three subsets H_1, H_2, and H_3 are defined by

$$\begin{aligned}
H_1 &= \{(m_1 + m_2,\, n) - (m_1,\, n) - (m_2,\, n) : m_1,\, m_2 \in M,\, n \in N\}\\
H_2 &= \{(m,\, n_1 + n_2) - (m,\, n_1) - (m,\, n_2) : m \in M,\, n_1,\, n_2 \in N\}\\
H_3 &= \{(ms,\, n) - (m,\, sn) : m \in M,\, n \in N,\, s \in S\}.
\end{aligned}$$

Note that K is an (R, T)-submodule of F using the bimodule structure given by Equation (2.14).

With these preliminaries out of the way, we can define the tensor product of M and N.

(2.9) Definition. *With the notation introduced above, the* **tensor product** *of the (R, S)-bimodule M and the (S, T)-bimodule N, denoted $M \otimes_S N$, is the quotient (R, T)-bimodule*

$$M \otimes_S N = F/K.$$

If $\pi : F \to F/K$ is the canonical projection map, then we let $m \otimes_S n = \pi((m, n))$ for each $(m, n) \in M \times N \subseteq F$. When S is clear from the context we will frequently write $m \otimes n$ in place of $m \otimes_S n$.

Note that the set

$$\{m \otimes_S n : (m, n) \in M \times N\} \tag{2.15}$$

generates $M \otimes_S N$ as an (R, T)-bimodule, but it is important to recognize that $M \otimes_S N$ is not (in general) equal to the set in (2.15). Also important to recognize is the fact that $m \otimes_S n = (m, n) + K$ is an equivalence class, so that $m \otimes n = m' \otimes n'$ does not necessarily imply that $m = m'$ and $n = n'$. As motivation for this rather complicated definition, we have the following proposition. The proof is left as an exercise.

(2.10) Proposition. *Let M be an (R, S)-bimodule, N an (S, T)-bimodule, and let m, $m_i \in M$, n, $n_i \in N$, and $s \in S$. Then the following identities hold in $M \otimes_S N$.*

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n \tag{2.16}$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2 \tag{2.17}$$

$$ms \otimes n = m \otimes sn. \tag{2.18}$$

Proof. Exercise. □

Indeed, the tensor product $M \otimes_S N$ is obtained from the cartesian product $M \times N$ by "forcing" the relations (2.16)–(2.18), but no others, to hold. This idea is formalized in Theorem 2.12, the statement of which requires the following definition.

(2.11) Definition. *Let M be an (R, S)-bimodule, N an (S, T)-bimodule, and let $M \times N$ be the cartesian product of M and N as sets. Let Q be any (R, T)-bimodule. A map $g : M \times N \to Q$ is said to be* **S-middle linear** *if it satisfies the following properties (where $r \in R$, $s \in S$, $t \in T$, m, $m_i \in M$ and $n, n_i \in N$):*

(1) $g(rm, nt) = rg(m, n)t$,
(2) $g(m_1 + m_2, n) = g(m_1, n) + g(m_2, n)$,
(3) $g(m, n_1 + n_2) = g(m, n_1) + g(m, n_2)$, *and*
(4) $g(ms, n) = g(m, sn)$.

Note that conditions (1), (2), and (3) simply state that for each $m \in M$ the function $g_m : N \to Q$ defined by $g_m(n) = g(m, n)$ is in $\mathrm{Hom}_{-T}(N, Q)$ and for each $n \in N$ the function $g^n : M \to Q$ defined by $g^n(m) = g(m, n)$ is in $\mathrm{Hom}_R(M, Q)$. Condition (4) is compatibility with the S-module structures on M and N.

If $\pi : F \to M \otimes_S N = F/K$ is the canonical projection map and $\iota : M \times N \to F$ is the inclusion map that sends (m, n) to the basis element $(m, n) \in F$, then we obtain a map $\theta : M \times N \to M \otimes N$. According to Proposition 2.10, the function θ is S-middle linear. The content of the following theorem is that *every* S-middle linear map "factors" through θ. This can, in fact, be taken as the fundamental defining property of the tensor product.

(2.12) Theorem. *Let M be an (R, S)-bimodule, N an (S, T)-bimodule, Q an (R, T)-bimodule, and $g : M \times N \to Q$ an S-middle linear map. Then there exists a unique (R, T)-bimodule homomorphism $\widetilde{g} : M \otimes_S N \to Q$ with $g = \widetilde{g} \circ \theta$. Furthermore, this property characterizes $M \otimes_S N$ up to isomorphism.*

Proof. If F denotes the free $\mathbf{Z}$-module on the index set $M \times N$, which is used to define the tensor product $M \otimes_S N$, then Equation (2.14) gives an (R, T)-bimodule structure on F. Since F is a free $\mathbf{Z}$-module with basis $M \times N$ and $g : M \times N \to Q$ is a function, Proposition 3.4.9 shows that there is a unique $\mathbf{Z}$-module homomorphism $g' : F \to Q$ such that $g' \circ \iota = g$ where $\iota : M \times N \to F$ is the inclusion map. The definition of the (R, T)-bimodule structure on F and the fact that g is S-middle linear implies that g' is in fact an (R, T)-bimodule homomorphism. Let $K' = \mathrm{Ker}(g')$, so the first isomorphism theorem provides an injective (R, T)-bimodule homomorphism $g'' : F/K' \to Q$ such that $g' = \pi' \circ g''$ where $\pi' : F \to F/K'$ is the canonical projection map. Recall that $K \subset F$ is the subgroup of F generated by the sets H_1, H_2, and H_3 defined prior to Definition 2.9. Since g is an S-middle linear map, it follows that $K \subseteq \mathrm{Ker}(g') = K'$, so there is a map $\pi_2 : F/K \to F/K'$ such that $\pi_2 \circ \pi = \pi'$.

Thus, $g : M \times N \to Q$ can be factored as follows:

$$M \times N \xrightarrow{\iota} F \xrightarrow{\pi} F/K \xrightarrow{\pi_2} F/K' \xrightarrow{g''} Q. \tag{2.19}$$

Recalling that $F/K = M \otimes_S N$, we define $\widetilde{g} = g'' \circ \pi_2$. Since $\theta = \pi \circ \iota$, Equation (2.19) shows that $g = \widetilde{g} \circ \theta$.

It remains to consider uniqueness of $\widetilde{g}$. But $M \otimes_S N$ is generated by the set $\{m \otimes_S n = \theta(m, n) : m \in M,\ n \in N\}$, and any function $\widetilde{g}$ such that $\widetilde{g} \circ \theta = g$ satisfies $\widetilde{g}(m \otimes n) = \widetilde{g}(\theta(m, n)) = g(m, n)$, so $\widetilde{g}$ is uniquely specified on a generating set and, hence, is uniquely determined.

Now suppose that we have (R, T)-bimodules P_i and S-middle linear maps $\theta_i : M \times N \to P_i$ such that, for any (R, T)-bimodule Q and any S-middle linear map $g : M \times N \to Q$, there exist unique (R, T)-bimodule homomorphisms $\widetilde{g}_i : P_i \to Q$ with $g = \widetilde{g}_i \circ \theta_i$ for $i = 1, 2$. We will show that P_1 and P_2 are isomorphic, and indeed that there is a unique (R, T)-bimodule isomorphism $\phi : P_1 \to P_2$ with the property that $\theta_2 = \phi \circ \theta_1$.

Let $Q = P_2$ and $g = \theta_2$. Then by the above property of P_1 there is a unique (R, T)-bimodule homomorphism $\phi : P_1 \to P_2$ with $\theta_2 = \phi \circ \theta_1$. We need only show that ϕ is an isomorphism. To this end, let $Q = P_1$ and $g = \theta_1$ to obtain $\psi : P_2 \to P_1$ with $\theta_1 = \psi \circ \theta_2$. Then

$$\theta_1 = \psi \circ \theta_2 = \psi \circ (\phi \circ \theta_1) = (\psi \circ \phi) \circ \theta_1.$$

Now apply the above property of P_1 again with $Q = P_1$ and $g = \theta_1$. Then there is a unique $\widetilde{g}$ with $g = \widetilde{g} \circ \theta_1$, i.e., a unique $\widetilde{g}$ with $\theta_1 = \widetilde{g} \circ \theta_1$. Obviously, $\widetilde{g} = 1_{P_1}$ satisfies this condition but so does $\widetilde{g} = \psi \circ \phi$, so we conclude that $\psi \circ \phi = 1_{P_1}$.

Similarly, $\phi \circ \psi = 1_{P_2}$, so $\psi = \phi^{-1}$, and we are done. □

(2.13) *Remarks.*

(1) If M is a right R-module and N is a left R-module, then $M \otimes_R N$ is an abelian group.

(2) If M and N are both (R, R)-bimodules, then $M \otimes_R N$ is an (R, R)-bimodule. A particular (important) case of this occurs when R is a commutative ring. In this case every left R-module is automatically a right R-module, and vice-versa. Thus, over a commutative ring R, it is meaningful to speak of the tensor product of R-modules, without explicit attention to the subtleties of bimodule structures.

(3) Suppose that M is a left R-module and S is a ring that contains R as a subring. Then we can form the tensor product $S \otimes_R M$ which has the structure of an $(S, \mathbf{Z})$-bimodule, i.e, $S \otimes_R M$ is a left S-module. This construction is called **change of rings** and it is useful when one would like to be able to multiply elements of M by scalars from a bigger ring. For example, if V is any vector space over $\mathbf{R}$, then $\mathbf{C} \otimes_{\mathbf{R}} V$ is a vector space over the complex numbers. This construction has been implicitly used in the proof of Theorem 4.6.23.

(4) If R is a commutative ring, M a free R-module, and ϕ a bilinear form on M, then $\phi : M \times M \to R$ is certainly middle linear, and so ϕ induces an R-module homomorphism

$$\widetilde{\phi} : M \otimes_R M \to R.$$

(2.14) Corollary.

(1) *Let M and M' be (R, S)-bimodules, let N and N' be (S, T)-bimodules, and suppose that $f : M \to M'$ and $g : N \to N'$ are bimodule homomorphisms. Then there is a unique (R, T)-bimodule homomorphism*

$$f \otimes g = f \otimes_S g : M \otimes_S N \longrightarrow M' \otimes_S N' \tag{2.20}$$

satisfying $(f \otimes g)(m \otimes n) = f(m) \otimes g(n)$ for all $m \in M$, $n \in N$.

(2) *If M'' is another (R, S)-bimodule, N'' is an (S, T)-bimodule, and $f'' : M' \to M''$, $g'' : N' \to N''$ are bimodule homomorphisms, then letting $f \otimes g : M \otimes N \to M' \otimes N'$ and $f' \otimes g' : M' \otimes N' \to M'' \otimes N''$ be defined as in part (1), we have*

$$(f' \otimes g')(f \otimes g) = (f'f) \otimes (g'g) : M \otimes N \longrightarrow M'' \otimes N''.$$

Proof. (1) Let F be the free abelian group on $M \times N$ used in the definition of $M \otimes_S N$, and let $h : F \to M' \otimes_S N'$ be the unique $\mathbf{Z}$-module homomorphism such that $h(m, n) = f(m) \otimes_S g(n)$. Since f and g are bimodule homomorphisms, it is easy to check that h is an S-middle linear map, so by Theorem 2.12, there is a unique bimodule homomorphism $\widetilde{h} : M \otimes N \to M' \otimes N'$ such that $h = \widetilde{h} \circ \theta$ where $\theta : M \times N \to M \otimes N$ is the canonical map. Let $f \otimes g = \widetilde{h}$. Then

$$(f \otimes g)(m \otimes n) = \widetilde{h}(m \otimes n) = h \circ \theta(m, n) = h(m, n) = f(m) \otimes g(n)$$

as claimed.

(2) is a routine calculation, which is left as an exercise. □

We will now consider some of the standard canonical isomorphisms relating various tensor product modules. The verifications are, for the most part, straightforward applications of Theorem 2.12. A few representative calculations will be presented; the others are left as exercises.

(2.15) Proposition. *Let M be an (R, S)-bimodule. Then there are (R, S)-bimodule isomorphisms*

$$R \otimes_R M \cong M \quad \text{and} \quad M \otimes_S S \cong M.$$

Proof. We check the first isomorphism; the second is similar. Let $f : R \times M \to M$ be defined by $f(r, m) = rm$. It is easy to check that f is an R-middle linear map, and thus Theorem 2.12 gives an (R, S)-bimodule homomorphism $\widetilde{f} : R \otimes_R M \to M$ such that $\widetilde{f}(r \otimes m) = rm$. Define $g : M \to R \otimes_R M$ by $g(m) = 1 \otimes m$. Then g is an (R, S)-bimodule homomorphism, and it is immediate that $\widetilde{f}$ and g are inverses of each other. □

(2.16) Proposition. *Let M and N be (R,R)-bimodules. Then*

$$M \otimes_R N \cong N \otimes_R M.$$

Proof. The isomorphism is given (via an application of Theorem 2.12) by $m \otimes n \mapsto n \otimes m$. □

(2.17) Proposition. *Let M be an (R,S)-bimodule, N an (S,T)-bimodule, and P a (T,U)-bimodule. Then there is an isomorphism of (R,U)-bimodules*

$$(M \otimes_S N) \otimes_T P \cong M \otimes_S (N \otimes_T P).$$

Proof. Fix an element $p \in P$ and define a function

$$f_p : M \times N \to M \otimes_S (N \otimes_T P)$$

by

$$f_p(m,\, n) = m \otimes_S (n \otimes_T p).$$

f_p is easily checked to be S-middle linear, so Theorem 2.12 applies to give an (R,T)-bimodule homomorphism $\widetilde{f}_p : M \otimes_S N \to M \otimes_S (N \otimes_T P)$. Then we have a map $f : (M \otimes_S N) \times P \to M \otimes_S (N \otimes_T P)$ defined by

$$f((m \otimes n),\, p) = \widetilde{f}_p(m \otimes n) = m \otimes (n \otimes p).$$

But f is T-middle linear, and hence there is a map of (R,U)-bimodules

$$\widetilde{f} : (M \otimes_S N) \otimes_T P \longrightarrow M \otimes_S (N \otimes_T P)$$

satisfying $\widetilde{f}((m \otimes n) \otimes p) = m \otimes (n \otimes p)$. Similarly, there is an (R,U)-bimodule homomorphism

$$\widetilde{g} : M \otimes_S (N \otimes_T P) \longrightarrow (M \otimes_S N) \otimes_T P$$

satisfying $\widetilde{g}(m \otimes (n \otimes p)) = (m \otimes n) \otimes p$. Clearly, $\widetilde{g}\widetilde{f}$ (respectively $\widetilde{f}\widetilde{g}$) is the identity on elements of the form $(m \otimes n) \otimes p$ (respectively, $m \otimes (n \otimes p)$), and since these elements generate the respective tensor products, we conclude that $\widetilde{f}$ and $\widetilde{g}$ are isomorphisms. □

(2.18) Proposition. *Let $M = \oplus_{i \in I} M_i$ be a direct sum of (R,S)-bimodules, and let $N = \oplus_{j \in J} N_j$ be a direct sum of (S,T)-bimodules. Then there is an isomorphism*

$$M \otimes_S N \cong \bigoplus_{i \in I} \bigoplus_{j \in J} (M_i \otimes_S N_j)$$

of (R,T)-bimodules.

Proof. Exercise. □

(2.19) *Remark.* When one is taking Hom and tensor product of various bimodules, it can be somewhat difficult to keep track of precisely what type of module structure is present on the given Hom or tensor product. The following is a useful mnemonic device for keeping track of the various module structures when forming Hom and tensor products. We shall write ${}_RM_S$ to indicate that M is an (R,S)-bimodule. When we form the tensor product of an (R,S)-bimodule and an (S,T)-bimodule, then the resulting module has the structure of an (R,T)-bimodule (Definition 2.9). This can be indicated mnemonically by

$$(2.21) \qquad {}_RM_S \otimes_S {}_SN_T = {}_RP_T.$$

Note that the two subscripts "S" on the bimodules appear adjacent to the subscript "S" on the tensor product sign, and after forming the tensor product they all disappear leaving the outside subscripts to denote the bimodule type of the answer (= tensor product).

A similar situation holds for Hom, but with one important difference. Recall from Proposition 2.4 that if M is an (R,S)-bimodule and N is an (R,T)-bimodule, then $\mathrm{Hom}_R(M,\, N)$ has the structure of an (S,T)-bimodule. (Recall that $\mathrm{Hom}_R(M,\, N)$ denotes the *left* R-module homomorphisms.) In order to create a simple mnemonic device similar to that of Equation (2.21), we make the following definition. If M and N are left R-modules, then we will write $M \pitchfork_R N$ for $\mathrm{Hom}_R(M,\, N)$. Using $\pitchfork_R$ in place of $\otimes_R$, we obtain the same convention about matching subscripts disappearing, leaving the outer subscripts to give the bimodule type, *provided* that the order of the subscripts of the module on the left of the $\pitchfork_R$ sign are reversed. Thus, Proposition 2.4 is encoded in this context as the statement

$${}_RM_S \quad \text{and} \quad {}_RN_T \Longrightarrow {}_SM_R \pitchfork_R {}_RN_T = {}_SP_T.$$

A similar convention holds for homomorphisms of *right* T-modules. This is illustrated by

$$\mathrm{Hom}_{-T}({}_RM_T,\, {}_SN_T) = {}_SN_T \pitchfork_{-T} {}_TM_R = {}_SP_R,$$

the result being an (S,R)-bimodule (see Proposition 2.5). Note that we must reverse the subscripts on M and interchange the position of M and N.

We shall now investigate the connection between Hom and tensor product. This relationship will allow us to deduce the effect of tensor products on exact sequences, using the known results for Hom (Theorems 2.7 and 2.8 in the current section, which are generalizations of Theorems 3.3.10 and 3.3.12).

(2.20) Theorem. (Adjoint associativity of Hom and tensor product) *Let M_1 and M_2 be (S,R)-bimodules, N a (T,S)-bimodule, and P a (T,U)-bimodule. If $\psi : M_2 \to M_1$ is an (S,R)-bimodule homomorphism, then* *there are* (R, U)*-bimodule isomorphisms*

$$\Phi_i : \operatorname{Hom}_S(M_i,\, \operatorname{Hom}_T(N,\, P)) \longrightarrow \operatorname{Hom}_T(N \otimes_S M_i,\, P)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \operatorname{Hom}_S(M_1,\, \operatorname{Hom}_T(N,\, P)) & \xrightarrow{\psi^*} & \operatorname{Hom}_S(M_2,\, \operatorname{Hom}_T(N,\, P)) \\ \big\downarrow{\scriptstyle \Phi_1} & & \big\downarrow{\scriptstyle \Phi_2} \\ \operatorname{Hom}_T(N \otimes_S M_1,\, P) & \xrightarrow{(1_N \otimes_S \phi)^*} & \operatorname{Hom}_T(N \otimes_S M_2,\, P) \end{array} \tag{2.22}$$

Proof. Define $\Phi_i : \operatorname{Hom}_S(M_i,\, \operatorname{Hom}_T(N,\, P)) \to \operatorname{Hom}_T(N \otimes_S M_i,\, P)$ by

$$\Phi_i(f)(n \otimes m) = (f(m))(n)$$

where $f \in \operatorname{Hom}_S(M_i,\, \operatorname{Hom}_T(N,\, P))$, $m \in M_i$, and $n \in N$. It is easy to check that $\Phi_i(f) \in \operatorname{Hom}_T(N \otimes_S M,\, P)$ and that Φ is a homomorphism of (R, U)-bimodules. The inverse map is given by

$$\left(\Psi_i(g)(m)\right)(n) = g(m \otimes n)$$

where $g \in \operatorname{Hom}_T(N \otimes_S M,\, P)$, $m \in M$, and $n \in N$. To check the commutativity of the diagram, suppose that $f \in \operatorname{Hom}_S(M,\, \operatorname{Hom}_T(N,\, P)$, $n \in N$, and $m_2 \in M_2$. Then

$$\begin{aligned} \left(\left(\Phi_2 \circ \psi^*\right)(f)\right)(n \otimes m_2) &= \left(\Phi_2(f \circ \psi)\right)(n \otimes m_2) \\ &= \left((f \circ \psi)(m_2)\right)(n) \\ &= f(\psi(m_2))(n) \\ &= \left(\Phi_1(f)\right)(n \otimes \psi(m_2)) \\ &= \left(\Phi_1(f)\right)\left((1_n \otimes \psi)(n \otimes m_2)\right) \\ &= (1_n \otimes \psi)^* \left(\Phi_1(f)\right)(n \otimes m_2) \\ &= \big((1_N \otimes \psi)^* \circ \Phi_1(f)\big)(n \otimes m_2). \end{aligned}$$

Thus, $\Phi_2 \circ \psi^* = (1_N \otimes \psi)^* \circ \Phi_1$ and diagram (2.22) is commutative. □

There is an analogous result concerning homomorphisms of right modules. In general we shall not state results explicitly for right modules; they can usually be obtained by obvious modifications of the left module results. However, the present result is somewhat complicated, so it will be stated precisely.

(2.21) Theorem. *Let* M_1 *and* M_2 *be* (R, S)*-bimodules,* N *an* (S, T)*-bimodule, and* P *a* (U, T)*-bimodule. If* $\psi : M_2 \to M_1$ *is an* (R, S)*-bimodule homomorphism, then there are* (U, R)*-bimodule isomorphisms*

$$\Phi_i : \operatorname{Hom}_{-S}(M_i,\, \operatorname{Hom}_{-T}(N,\, P)) \longrightarrow \operatorname{Hom}_{-T}(M_i \otimes_S N,\, P)$$

such that the following diagram commutes:

$$
\begin{tikzcd}
\mathrm{Hom}_{-S}(M_1,\, \mathrm{Hom}_{-T}(N,\, P)) \arrow[r, "\psi^*"] \arrow[d, "\Phi_1"] & \mathrm{Hom}_{-S}(M_2,\, \mathrm{Hom}_{-T}(N,\, P)) \arrow[d, "\Phi_2"] \\
\mathrm{Hom}_{-T}(M_1 \otimes_S N,\, P) \arrow[r, "(1_N \otimes_S \phi)^*"] & \mathrm{Hom}_{-T}(M_2 \otimes_S N,\, P)
\end{tikzcd}
\tag{2.23}
$$

Proof. The proof is the same as that of Theorem 2.20. □

Remark. Note that Theorems 2.20 and 2.21 are already important results in case $M_1 = M_2 = M$ and $\psi = 1_M$.

As a simple application of adjoint associativity, there is the following result.

(2.22) Corollary. *Let M be an (R, S)-bimodule, N an (S, T)-bimodule, and let $P = M \otimes_S N$ (which is an (R, T)-bimodule). If M is projective as a left R-module (resp., as a right S-module) and N is projective as a left S-module (resp., as a right T-module), then P is projective as a left R-module (resp., as a right T-module).*

Proof. To show that P is projective as a left R-module, we must show that, given any surjection $f : A \to B$ of R-modules, the induced map

$$f_* : \mathrm{Hom}_R(P,\, A) \longrightarrow \mathrm{Hom}_R(P,\, B)$$

is also surjective. By hypothesis, M is projective as a left R-module so that

$$f_* : \mathrm{Hom}_R(M,\, A) \longrightarrow \mathrm{Hom}_R(M,\, B)$$

is surjective. Also, N is assumed to be projective as a left S-module, so the map

$$(f_*)_* : \mathrm{Hom}_S(N,\, \mathrm{Hom}_R(M,\, A)) \longrightarrow \mathrm{Hom}_S(N,\, \mathrm{Hom}_R(M,\, A))$$

is also surjective. But, by Theorem 2.20, if $C = A$ or B, then

$$\mathrm{Hom}_S(N, \mathrm{Hom}_R(M,\, C)) \cong \mathrm{Hom}_R(P,\, C).$$

It is simple to check that in fact there is a commutative diagram

$$
\begin{tikzcd}
\mathrm{Hom}_S(N,\, \mathrm{Hom}_R(M,\, A)) \arrow[r, "(f_*)_*"] \arrow[d, "\Phi_1"] & \mathrm{Hom}_S(N,\, \mathrm{Hom}_R(M,\, B)) \arrow[d, "\Phi_2"] \\
\mathrm{Hom}_R(P,\, A) \arrow[r, "f_*"] & \mathrm{Hom}_R(P,\, B)
\end{tikzcd}
$$

and this completes the proof. □

One of the most important consequences of the adjoint associativity property relating Hom and tensor product is the ability to prove theorems concerning the exactness of sequences of tensor product modules by appealing to the theorems on exactness of Hom sequences, namely, Theorems 2.7 and 2.8.

(2.23) Theorem. *Let N be a fixed (R, T)-bimodule. If*

$$M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \longrightarrow 0 \tag{2.24}$$

is an exact sequence of (S, R)-bimodules, then

$$M_1 \otimes_R N \xrightarrow{\phi \otimes 1_N} M \otimes_R N \xrightarrow{\psi \otimes 1_N} M_2 \otimes_R N \longrightarrow 0 \tag{2.25}$$

is an exact sequence of (S, T)-bimodules, while if (2.24) *is an exact sequence of (T, S)-bimodules, then*

$$N \otimes_T M_1 \xrightarrow{1_N \otimes \phi} N \otimes_T M \xrightarrow{1_N \otimes \psi} N \otimes_T M_2 \longrightarrow 0 \tag{2.26}$$

is an exact sequence of (R, S)-bimodules.

Proof. We will prove the exactness of sequence (2.26); exactness of sequence (2.25) is similar and it is left as an exercise. According to Theorem 2.7, in order to check the exactness of sequence (2.26), it is sufficient to check that the induced sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_R(N \otimes_T M_2,\, P) \longrightarrow \mathrm{Hom}_R(N \otimes_T M,\, P) \\ &\longrightarrow \mathrm{Hom}_R(N \otimes_T M_1,\, P) \end{aligned} \tag{2.27}$$

is exact for every (R, U)-bimodule P. But Theorem 2.20 identifies sequence (2.27) with the following sequence, which is induced from sequence (2.24) by the (T, U)-bimodule $\mathrm{Hom}_R(N,\, P)$:

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_T(M_2,\, \mathrm{Hom}_R(N,\, P)) \longrightarrow \mathrm{Hom}_T(M,\, \mathrm{Hom}_R(N,\, P)) \\ &\longrightarrow \mathrm{Hom}_T(M_1,\, \mathrm{Hom}_R(N,\, P)). \end{aligned} \tag{2.28}$$

Since (2.24) is assumed to be exact, Theorem 2.7 shows that sequence (2.28) is exact for any (R, U)-bimodule P. Thus sequence (2.27) is exact for all P, and the proof is complete. □

(2.24) Examples.

(1) Consider the following short exact sequence of $\mathbf{Z}$-modules:

$$0 \longrightarrow \mathbf{Z} \xrightarrow{\phi} \mathbf{Z} \xrightarrow{\psi} \mathbf{Z}_m \longrightarrow 0 \tag{2.29}$$

where $\phi(i) = mi$ and ψ is the canonical projection map. If we take $N = \mathbf{Z}_n$, then exact sequence (2.25) becomes

$$\mathbf{Z} \otimes \mathbf{Z}_n \xrightarrow{\phi\otimes 1} \mathbf{Z} \otimes \mathbf{Z}_n \xrightarrow{\psi\otimes 1} \mathbf{Z}_m \otimes \mathbf{Z}_n \longrightarrow 0. \tag{2.30}$$

By Proposition 2.15, exact sequence (2.30) becomes the exact sequence

$$\mathbf{Z}_n \xrightarrow{\widetilde{\phi}} \mathbf{Z}_n \xrightarrow{\widetilde{\psi}} \mathbf{Z}_m \otimes \mathbf{Z}_n \longrightarrow 0 \tag{2.31}$$

where $(\widetilde{\phi})(i) = mi$. Thus $\mathbf{Z}_m \otimes \mathbf{Z}_n \cong \operatorname{Coker}(\widetilde{\phi})$. Now let $d = \gcd(m, n)$ and write $m = m'd$, $n = n'd$. Then the map $\widetilde{\phi}$ is the composite

$$\mathbf{Z}_n \xrightarrow{\phi_1} \mathbf{Z}_n \xrightarrow{\phi_2} \mathbf{Z}_n$$

where $\phi_1(i) = m'i$ and $\phi_2(i) = di$. Since $\gcd(m', n) = 1$, it follows that ϕ_1 is an isomorphism (Proposition 1.4.11), while $\operatorname{Im}(\phi_2) = d\mathbf{Z}_n$. Hence, $\operatorname{Coker}(\widetilde{\phi}) \cong \mathbf{Z}_n/d\mathbf{Z}_n \cong \mathbf{Z}_d$, i.e.,

$$\mathbf{Z}_m \otimes \mathbf{Z}_n \cong \mathbf{Z}_d.$$

(2) Suppose that M is any finite abelian group. Then

$$M \otimes_{\mathbf{Z}} \mathbf{Q} = \langle 0 \rangle.$$

To see this, consider a typical generator $x \otimes r$ of $M \otimes_{\mathbf{Z}} \mathbf{Q}$, where $x \in M$ and $r \in \mathbf{Q}$. Let $n = |M|$. Then $nx = 0$ and, according to Equation (2.18),

$$x \otimes r = x \otimes n(r/n) = xn \otimes (r/n) = 0 \otimes (r/n) = 0.$$

Since $x \in M$ and $r \in \mathbf{Q}$ are arbitrary, it follows that every generator of $M \otimes \mathbf{Q}$ is 0.

(3) Let R be a commutative ring, $I \subseteq R$ an ideal, and M any R-module. Then

$$(R/I) \otimes_R M \cong M/IM. \tag{2.32}$$

To see this consider the exact sequence of R-modules

$$0 \longrightarrow I \xrightarrow{\iota} R \longrightarrow R/I \longrightarrow 0.$$

Tensor this sequence of R-modules with M to obtain an exact sequence

$$I \otimes_R M \xrightarrow{\iota\otimes 1} R \otimes_R M \longrightarrow (R/I) \otimes_R M \longrightarrow 0.$$

But according to Proposition 2.15, $R \otimes_R M \cong M$ (via the isomorphism $\Phi(r \otimes m) = rm$), and under this identification it is easy to see that $\operatorname{Im}(\iota \otimes 1) = IM$. Thus, $(R/I) \otimes_R M \cong M/IM$, as we wished to verify.

Example 2.32 (1) shows that even if a sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

is short exact, the tensored sequence (2.25) need not be part of a short exact sequence, i.e., the initial map need not be injective. For a simple situation where this occurs, take $m = n$ in Example 2.32 (1). Then exact sequence (2.30) becomes

$$\mathbf{Z}_n \xrightarrow{\phi\otimes 1} \mathbf{Z}_n \longrightarrow \mathbf{Z}_n \longrightarrow 0.$$

The map $\phi \otimes 1$ is the zero map, so it is certainly not an injection.

This example, plus our experience with Hom, suggests that we consider criteria to ensure that tensoring a short exact sequence with a fixed module produces a short exact sequence. We start with the following result, which is exactly analogous to Theorem 2.8 for Hom.

(2.25) Theorem. *Let N be a fixed (R, T)-bimodule. If*

$$0 \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \longrightarrow 0 \tag{2.33}$$

is a split short exact sequence of (S, R)-bimodules, then

$$0 \longrightarrow M_1 \otimes_R N \xrightarrow{\phi\otimes 1_N} M \otimes_R N \xrightarrow{\psi\otimes 1_N} M_2 \otimes_R N \longrightarrow 0 \tag{2.34}$$

is a split short exact sequence of (S, T)-bimodules, while if (2.33) *is a split short exact sequence of (T, S)-bimodules, then*

$$0 \longrightarrow N \otimes_T M_1 \xrightarrow{1_N\otimes\phi} N \otimes_T M \xrightarrow{1_N\otimes\psi} N \otimes_T M_2 \longrightarrow 0 \tag{2.35}$$

is a split short exact sequence of (R, S)-bimodules.

Proof. We will do sequence (2.34); (2.35) is similar and is left as an exercise. Let $\alpha : M \to M_1$ split ϕ, and consider the map

$$\alpha \otimes 1 : M \otimes_R N \to M_1 \otimes_R N.$$

Then

$$((\alpha \otimes 1)(\phi \otimes 1))\,(m \otimes n) = (\alpha\phi \otimes 1)(m \otimes n) = (1 \otimes 1)(m \otimes n) = m \otimes n$$

so that $\phi\otimes 1$ is an injection, which is split by $\alpha\otimes 1$. The rest of the exactness is covered by Theorem 2.23. $\square$

(2.26) *Remark.* Theorems 2.7 and 2.23 show that given a short exact sequence, applying Hom or tensor product will give a sequence that is exact on one end or the other, but in general not on both. Thus Hom and tensor product are both called **half exact**, and more precisely, Hom is called **left exact** and tensor product is called **right exact**. We will now investigate some conditions under which the tensor product of a module with a short exact sequence always produces a short exact sequence. It was precisely this type of consideration for Hom that led us to the concept of projective module. In fact, Theorem 3.5.1 (4) shows that if P is a projective R-module and

$$0 \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \longrightarrow 0$$

is a short exact sequence of R-modules, then the sequence

$$0 \longrightarrow \operatorname{Hom}_R(P,\, M_1) \xrightarrow{\phi_*} \operatorname{Hom}_R(P,\, M) \xrightarrow{\psi_*} \operatorname{Hom}_R(P,\, M_2) \longrightarrow 0$$

is short exact. According to Theorem 3.3.10, the crucial ingredient needed is the surjectivity of ψ_* and this is what projectivity of P provides. For the case of tensor products, the crucial fact needed to obtain a short exact sequence will be the injectivity of the initial map of the sequence.

(2.27) Proposition. *Let N be an (R,T)-bimodule that is projective as a left R-module. Then for any injection $\iota : M_1 \to M$ of (S,R)-bimodules,*

$$\iota \otimes 1 : M_1 \otimes_R N \longrightarrow M \otimes_R N$$

is an injection of (S,T)-bimodules. If N is projective as a right T-module and $\iota : M_1 \to M$ is an injection of (T,S)-bimodules, then

$$1 \otimes \iota : N \otimes_T M_1 \longrightarrow N \otimes_T M$$

is an injection of (R,S)-bimodules.

Proof. First suppose that as a left R-module N is free with a basis $\{n_j\}_{j\in J}$. Then $N \cong \oplus_{j\in J} R_j$ where each summand $R_j = Rn_j$ is isomorphic to R as a left R-module. Then by Proposition 2.18

$$M_1 \otimes_R N \cong \bigoplus_{j\in J} (M_1 \otimes_R R_j) = \bigoplus_{j\in J} M_{1j}$$

where each M_{1j} is isomorphic to M_1 as a left S-module, and similarly $M \otimes_R N \cong \oplus_{j\in J} M_j$, where each M_j is isomoprhic to M as a left S-module. Furthermore, the map $\iota \otimes 1 : M_1 \otimes_R N \to M \otimes_R N$ is given as a direct sum

$$\bigoplus_{j\in J} (\iota_j : M_{1j} \to M_j)$$

where each ι_j agrees with ι under the above identifications. But then, since ι is an injection, so is each ι_j, and hence so is $\iota \otimes 1$.

Now suppose that N is projective as a left R-module. Then there is a left R-module N' such that $N \oplus N' = F$ where F is a free left R-module. We have already shown that

$$\iota \otimes 1 : M_1 \otimes_R F \longrightarrow M \otimes_R F$$

is an injection. But using Proposition 2.18 again,

$$M_1 \otimes_R F \cong (M_1 \otimes_R N) \oplus (M_1 \otimes_R N')$$

so we may write $\iota \otimes 1 = \iota_1 \oplus \iota_2$ where (in particular) $\iota_1 = \iota \otimes 1 : M_1 \otimes_R N \to M \otimes_R F$. Since $\iota \otimes 1$ is an injection, so is ι_1, as claimed. Thus the proof is complete in the case that N is projective as a left R-module. The proof in case N is projective as a right T-module is identical. $\square$

Note that we have not used the right T-module structures in the above proof. This is legitimate, since if a homomorphism is injective as a map of left S-modules, and it is an (S, T)-bimodule map, then it is injective as an (S, T)-bimodule map.

(2.28) Corollary. *Let N be a fixed (R, T)-bimodule that is projective as a left R-module. If*

$$0 \longrightarrow M_1 \xrightarrow{\phi} M \xrightarrow{\psi} M_2 \longrightarrow 0 \tag{2.36}$$

is a short exact sequence of (S, R)-bimodules, then

$$0 \longrightarrow M_1 \otimes_R N \xrightarrow{\phi \otimes 1_N} M \otimes_R N \xrightarrow{\psi \otimes 1_N} M_2 \otimes_R N \longrightarrow 0 \tag{2.37}$$

is a short exact sequence of (S, T)-bimodules; while if (2.36) *is an exact sequence of (T, S)-bimodules and N is projective as a right T-module, then*

$$0 \longrightarrow N \otimes_T M_1 \xrightarrow{1_N \otimes \phi} N \otimes_T M \xrightarrow{1_N \otimes \psi} N \otimes_T M_2 \longrightarrow 0 \tag{2.38}$$

is a short exact sequence of (R, S)-bimodules.

Proof. This follows immediately from Theorem 2.23 and Proposition 2.27. $\square$

(2.29) *Remark.* A module satisfying the conclusion of Proposition 2.27 is said to be **flat**. That is, a left R-module N is flat if tensoring with all short exact sequences of right R-modules produces a short exact sequence, with a similar definition for right R-modules. Given Theorem 2.23, in order to prove that a left R-module N is flat, it is sufficient to prove that for all right R-modules M and submodules K, the inclusion map $\iota : K \to M$ induces an injective map

$$\iota \otimes 1 : K \otimes_R N \longrightarrow M \otimes_R N.$$

Thus, what we have proven is that projective modules are flat.

In Section 6.1 we discussed duality for free modules over commutative rings. Using the theory developed in the current section, we will extend portions of our discussion of duality to the context of projective (bi-)modules.

(2.30) Definition. *Let M be an (R, S)-bimodule. The* **dual module** *of M is the (S, R)-bimodule M^* defined by*

$$\mathrm{Hom}_R(M,\, R).$$

In particular, if M is a left R-module, i.e., take $S = \mathbf{Z}$, then the dual module M^* is a right R-module. The **double dual** *of M is defined to be*

$$M^{**} = \mathrm{Hom}_{-R}(M^*,\, R).$$

As in Section 6.1, there is a homomorphism $\eta : M \to M^{**}$ of (R, S)-bimodules defined by

$$(\eta(v))(\omega) = \omega(v) \quad \textit{for all } v \in M,\ \omega \in M^*$$

and if η is an isomorphism, then we will say that M is **reflexive**.

If M is an (R, S)-bimodule, which is finitely generated and free as a left R-module, then given any basis $\mathcal{B}$ of M, one may construct a basis of M^* (as a right R-module) exactly as in Definition 6.1.3 and the proof of Theorem 6.1.7 goes through verbatim to show that finitely generated free R-modules are reflexive, even when R need not be commutative. Furthermore, the proofs of Theorems 3.5.8 and 6.1.13 go through without difficulty if one keeps track of the types of modules under consideration. We will simply state the following result and leave the details of tracing through the module types as an exercise.

(2.31) Proposition. *Let M be an (R, S)-bimodule, which is finitely generated and projective as a left R-module. Then the dual module M^* is finitely generated and projective as a right R-module. Furthermore, M is reflexive as an (R, S)-bimodule.*

Proof. Exercise. See the comments above. □

If M is an (R, S)-bimodule and P is an (R, T)-bimodule, then define

$$\zeta : M^* \times P \to \mathrm{Hom}_R(M,\, P)$$

by

$$(\zeta(\omega,\, p))\,(m) = \omega(m)p \qquad \text{for} \quad \omega \in M^*,\ p \in P, \text{ and } m \in M.$$

Then ζ is S-middle linear and hence it induces an (S, T)-bimodule homomorphism

$$\widetilde{\zeta} : M^* \otimes_R P \longrightarrow \mathrm{Hom}_R(M,\, P)$$

given by

$$(\widetilde{\zeta}(\omega \otimes p))(m) = \omega(m)p \tag{2.39}$$

for all $\omega \in M^*$, $p \in P$, and $m \in M$.

(2.32) Proposition. *Let M be an (R, S)-bimodule, which is finitely generated and projective as a left R-module, and let P be an arbitrary (R, T)-bimodule. Then the map*

$$\widetilde{\zeta} : M^* \otimes_R P \longrightarrow \operatorname{Hom}_R(M,\, P)$$

defined by Equation (2.39) *is an* (S,T)*-bimodule isomorphism.*

Proof. Since $\widetilde{\zeta}$ is an (S,T)-bimodule homomorphism, it is only necessary to prove that it is bijective. To achieve this first suppose that M is free of finite rank k as a left R-module. Let $\mathcal{B} = \{v_1, \ldots, v_k\}$ be a basis of M and let $\{v_1^*, \ldots, v_k^*\}$ be the basis of M^* dual to $\mathcal{B}$. Note that every element of $M^* \otimes_R P$ can be written as $x = \sum_{i=1}^k v_i^* \otimes p_i$ for $p_1, \ldots, p_k \in P$. Suppose that $\widetilde{\zeta}(x) = 0$, i.e., $(\widetilde{\zeta}(x))(m) = 0$ for every $m \in M$. But $\widetilde{\zeta}(x)(v_i) = p_i$ so that $p_i = 0$ for $1 \le i \le k$. That is, $x = 0$ and we conclude that $\widetilde{\zeta}$ is injective.

Given any $f \in \operatorname{Hom}_R(M,\, P)$, let

$$x_f = \sum_{i=1}^k v_i^* \otimes f(v_i).$$

Then $(\widetilde{\zeta}(x_f))(v_i) = f(v_i)$ for $1 \le i \le k$, i.e., $\widetilde{\zeta}(x_f)$ and f agree on a basis of M; hence, $\widetilde{\zeta}(x_f) = f$ and $\widetilde{\zeta}$ is a surjection, and the proof is complete in case M is free of rank k.

Now suppose that M is finitely generated and projective, and let N be a left R-module such that $F = M \oplus N$ is finitely generated and free. Then $\widetilde{\zeta} : F^* \otimes_R P \to \operatorname{Hom}_R(F,\, P)$ is a $\mathbf{Z}$-module isomorphism, and

$$F^* \otimes_R P = (M \oplus N)^* \otimes_R P \cong (M^* \oplus N^*) \otimes_R P \cong (M^* \otimes_R P) \oplus (N^* \otimes_R P)$$

while

$$\operatorname{Hom}_R(F,\, P) = \operatorname{Hom}_R(M \oplus N,\, P) \cong \operatorname{Hom}_R(M,\, P) \oplus \operatorname{Hom}_R(N,\, P)$$

where all isomorphisms are $\mathbf{Z}$-module isomorphisms. Under these isomorphisms,

$$\begin{aligned} \widetilde{\zeta}_F &= \widetilde{\zeta}_M \oplus \widetilde{\zeta}_N \\ \widetilde{\zeta}_M &: M^* \otimes_R P \longrightarrow \operatorname{Hom}_R(M,\, P) \\ \widetilde{\zeta}_N &: N^* \otimes_R P \longrightarrow \operatorname{Hom}_R(N,\, P). \end{aligned}$$

Since $\widetilde{\zeta}_F$ is an isomorphism, it follows that $\widetilde{\zeta}_M$ and $\widetilde{\zeta}_N$ are isomorphisms as well. In particular, $\widetilde{\zeta}_M$ is bijective and the proof is complete. □

(2.33) Corollary. *Let M be an (R,S)-bimodule, which is finitely generated and projective as a left R-module, and let P be an arbitrary (T,R)-bimodule. Then*

$$M^* \otimes_R P^* \cong (P \otimes_R M)^*$$

as (S,T)*-bimodules.*

Proof. From Proposition 2.32, there is an isomorphism

$$\begin{aligned} M^* \otimes_R P^* &\cong \mathrm{Hom}_R(M,\, P^*) \\ &= \mathrm{Hom}_R(M,\, \mathrm{Hom}_R(P,\, R)) \\ &\cong \mathrm{Hom}_R(P \otimes_R M,\, R) \qquad \text{(by adjoint associativity)} \\ &= (P \otimes_R M)^* . \end{aligned}$$

□

(2.34) *Remark.* The isomorphism of Corollary 2.33 is given explicitly by

$$\phi(f \otimes g)(p \otimes m) = f(m)g(p) \in R$$

where $f \in M^*$, $g \in P^*$, $p \in P$, and $m \in M$.

We will conclude this section by studying the matrix representation of the tensor product of R-module homomorphisms. Thus, let R be a commutative ring, let M_1, M_2, N_1, and N_2 be finite rank free R-modules, and let $f_i : M_i \to N_i$ be R-module homomorphisms for $i = 1, 2$. Let m_i be the rank of M_i and n_i the rank of N_i for $i = 1, 2$. If $M = M_1 \otimes M_2$ and $N = N_1 \otimes N_2$, then it follows from Proposition 2.18 that M and N are free R-modules of rank $m_1 n_1$ and $m_2 n_2$, respectively. Let $f = f_1 \otimes f_2 \in \mathrm{Hom}_R(M,\, N)$. We will compute a matrix representation for f from that for f_1 and f_2. To do this, suppose that

$$\begin{aligned} \mathcal{A} &= \{a_1, \, \dots \, , a_{m_1}\} \\ \mathcal{B} &= \{b_1, \, \dots \, , b_{n_1}\} \\ \mathcal{C} &= \{c_1, \, \dots \, , c_{m_2}\} \\ \mathcal{D} &= \{d_1, \, \dots \, , c_{n_2}\} \end{aligned}$$

are bases of M_1, N_1, M_2, and N_2, respectively. Let

$$\begin{aligned} \mathcal{E} = \{ & a_1 \otimes c_1,\ a_1 \otimes c_2,\ \dots\ , a_1 \otimes c_{m_2}, \\ & a_2 \otimes c_1,\ a_2 \otimes c_2,\ \dots\ , a_2 \otimes c_{m_2}, \\ & \quad \vdots \\ & a_{m_1} \otimes c_1,\ a_{m_1} \otimes c_2,\ \dots\ , a_{m_1} \otimes c_{m_2} \} \end{aligned}$$

and

$$\begin{aligned} \mathcal{F} = \{ & b_1 \otimes d_1,\ b_1 \otimes d_2,\ \dots\ , b_1 \otimes d_{n_2}, \\ & b_2 \otimes d_1,\ b_2 \otimes d_2,\ \dots\ , b_2 \otimes d_{n_2}, \\ & \quad \vdots \\ & b_{n_1} \otimes d_1,\ b_{n_1} \otimes d_2,\ \dots\ , b_{n_1} \otimes d_{n_2} \}. \end{aligned}$$

Then $\mathcal{E}$ is a basis for M and $\mathcal{F}$ is a basis for N. With respect to these bases, there is the following result:

(2.35) Proposition. *With the notation introduced above,*

$$[f_1 \otimes f_2]_{\mathcal{F}}^{\mathcal{E}} = [f_1]_{\mathcal{B}}^{\mathcal{A}} \otimes [f_2]_{\mathcal{D}}^{\mathcal{C}}.$$

Proof. Exercise. □

Recall that the notion of tensor product of matrices was introduced in Definition 4.1.16 and has been used subsequently in Section 5.5. If $[f_1]_{\mathcal{B}}^{\mathcal{A}} = A = [\alpha_{ij}]$ and $[f_2]_{\mathcal{D}}^{\mathcal{C}} = B = [\beta_{ij}]$, then Proposition 2.35 states that (in block matrix notation)

$$[f_1 \otimes f_2]_{\mathcal{F}}^{\mathcal{E}} = \begin{bmatrix} \alpha_{11}B & \alpha_{12}B & \cdots & \alpha_{1m_1}B \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n_11}B & \alpha_{n_12}B & \cdots & \alpha_{n_1m_1}B \end{bmatrix}.$$

There is another possible ordering for the bases $\mathcal{E}$ and $\mathcal{F}$. If we set

$$\mathcal{E}' = \{a_i \otimes c_j : 1 \le i \le m_1,\ 1 \le j \le m_2\}$$

and

$$\mathcal{F}' = \{b_i \otimes d_j : 1 \le i \le n_1,\ 1 \le j \le n_2\}$$

where the elements are ordered by first fixing j and letting i increase (lexicographic ordering with j the dominant letter), then we leave it to the reader to verify that the matrix of $f_1 \otimes f_2$ is given by

$$[f_1 \otimes f_2]_{\mathcal{F}'}^{\mathcal{E}'} = \begin{bmatrix} \beta_{11}A & \beta_{12}A & \cdots & \beta_{1m_2}A \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{n_21}A & \beta_{n_22}A & \cdots & \beta_{n_2m_2}A \end{bmatrix}.$$

7.3 Exercises

1. Let M be a simple R-module, and let N be any R-module.
   (a) Show that every nonzero homomorphism $f : M \to N$ is injective.
   (b) Show that every nonzero homomorphism $f : N \to M$ is surjective.
2. Let F be a field and let $R = \left\{\left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right] : a,\, b,\, c \in F\right\}$ be the ring of upper triangular matrices over F. Let $M = F^2$ and make M into a (left) R-module by matrix multiplication. Show that $\mathrm{End}_R(M) \cong F$. Conclude that the converse of Schur's lemma is false, i.e., $\mathrm{End}_R(M)$ can be a division ring without M being a simple R-module.
3. Suppose that R is a D-algebra, where D is a division ring, and let M be an R-module which is of finite rank as a D-module. Show that as an R-module, $\ell(M) \le \mathrm{rank}_D(M)$.
4. An R-module M is said to satisfy the decending chain condition (DCC) on submodules if any strictly decreasing chain of submodules of M is of finite length.

(a) Show that if M satisfies the DCC, then any nonempty set of submodules of M contains a minimal element.
(b) Show that $\ell(M) < \infty$ if and only if M satisfies both the ACC (ascending chain condition) and DCC.

5. Let $R = \{\left[\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right] : a, b \in \mathbf{R}; c \in \mathbf{Q}\}$. R is a ring under matrix addition and multiplication. Show that R satisfies the ACC and DCC on left ideals, but neither chain condition is valid for right ideals. Thus R is of finite length as a left R-module, but $\ell(R) = \infty$ as a right R-module.

6. Let R be a ring without zero divisors. If R is not a division ring, prove that R does not have a composition series.

7. Let $f : M_1 \to M_2$ be an R-module homomorphism.
(a) If f is injective, prove that $\ell(M_1) \le \ell(M_2)$.
(b) If f is surjective, prove that $\ell(M_2) \le \ell(M_1)$.

8. Let M be an R-module of finite length and let K and N be submodules of M. Prove the following length formula:

$$\ell(K + N) + \ell(K \cap N) = \ell(K) + \ell(N).$$

9. (a) Compute $\ell(\mathbf{Z}_{p^n})$.
(b) Compute $\ell(\mathbf{Z}_{p^n} \oplus \mathbf{Z}_{q^m})$.
(c) Compute $\ell(G)$ where G is any finite abelian group.
(d) More generally, compute $\ell(M)$ for any finitely generated torsion module over a PID R.

10. Compute the length of $M = F[X]/\langle f(X)\rangle$ as an $F[X]$-module if $f(X)$ is a polynomial of degree n with two distinct irreducible factors. What is the length of M as an F-module?

11. Let F be a field, let V be a finite-dimensional vector space over F, and let $T \in \mathrm{End}_F(V)$. We shall say that T is semisimple if the $F[X]$-module V_T is semisimple. If $A \in M_n(F)$, we shall say that A is semisimple if the linear transformation $T_A : F^n \to F^n$ (multiplication by A) is semisimple. Let $\mathbf{F}_2$ be the field with 2 elements and let $F = \mathbf{F}_2(Y)$ be the rational function field in the indeterminate Y, and let $K = F[X]/\langle X^2 + Y\rangle$. Since $X^2 + Y \in F[X]$ is irreducible, K is a field containing F as a subfield. Now let

$$A = C(X^2 + Y) = \begin{bmatrix} 0 & Y \\ 1 & 0 \end{bmatrix} \in M_2(F).$$

Show that A is semisimple when considered in $M_2(F)$ but A is not semisimple when considered in $M_2(K)$. Thus, semisimplicity of a matrix is not necessarily preserved when one passes to a larger field. However, prove that if L is a subfield of the complex numbers $\mathbf{C}$, then $A \in M_n(L)$ is semisimple if and only if it is also semisimple as a complex matrix.

12. Let V be a vector space over $\mathbf{R}$ and let $T \in \mathrm{End}_{\mathbf{R}}(V)$ be a linear transformation. Show that $T = S + N$ where S is a semisimple linear transformation, N is nilpotent, and $SN = NS$.

13. Prove that the modules M_i and N_j in the proof of Lemma 1.33 are simple, as claimed.

14. Prove Lemma 1.37.

15. If D is a division ring and n is a positive integer, prove that $\mathrm{End}_D(D^n)$ is a simple ring.

16. Give an example of a semisimple commutative ring that is not a field.

17. (a) Prove that if R is a semisimple ring and I is an ideal, then R/I is semisimple.
(b) Show (by example) that a subring of a semisimple ring need not be semisimple.

18. Let R be a ring that is semisimple as a left R-module. Show that R is simple if and only if all simple left R-modules are isomorphic.
19. Let M be a finitely generated abelian group. Compute each of the following groups:
    (a) $\mathrm{Hom}_{\mathbf{Z}}(M, \mathbf{Q}/\mathbf{Z})$.
    (b) $\mathrm{Hom}_{\mathbf{Z}}(\mathbf{Q}/\mathbf{Z}, M)$.
    (c) $M \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z}$.
20. Let M be an (R, S)-bimodule and N an (S, T)-bimodule. Suppose that $\sum x_i \otimes y_i = 0$ in $M \otimes_S N$. Prove that there exists a finitely generated (R, S)-bisubmodule M_0 of M and a finitely generated (S, T)-bisubmodule N_0 of N such that $\sum x_i \otimes y_i = 0$ in $M_0 \otimes_S N_0$.
21. Let R be an integral domain and let M be an R-module. Let Q be the quotient field of R and define $\phi : M \to Q \otimes_R M$ by $\phi(x) = 1 \otimes x$. Show that $\mathrm{Ker}(\phi) = M_\tau =$ torsion submodule of M. (Hint: If $1 \otimes x = 0 \in Q \otimes_R M$ then $1 \otimes x = 0$ in $(Rc^{-1}) \otimes_R M \cong M$ for some $c \neq 0 \in R$. Then show that $cx = 0$.)
22. Let R be a PID and let M be a free R-module with N a submodule. Let Q be the quotient field and let $\phi : M \to Q \otimes_R M$ be the map $\phi(x) = 1 \otimes x$. Show that N is a pure submodule of M if and only if $Q \cdot (\phi(N)) \cap \mathrm{Im}(\phi) = \phi(N)$.
23. Let R be a PID and let M be a finitely generated R-module. If Q is the quotient field of R, show that $M \otimes_R Q$ is a vector space over Q of dimension equal to $\mathrm{rank}_R(M/M_\tau)$.
24. Let R be a commutative ring and S a multiplicatively closed subset of R containing no zero divisors. Let R_S be the localization of R at S. If M is an R-module, then the R_S-module M_S was defined in Exercise 6 of Chapter 3. Show that $M_S \cong R_S \otimes_R M$ where the isomorphism is an isomorphism of R_S-modules.
25. If S is an R-algebra, show that $M_n(S) \cong S \otimes_R M_n(R)$.
26. Let M and N be finitely generated R-modules over a PID R. Compute $M \otimes_R N$. As a special case, if M is a finite abelian group with invariant factors $s_1, \ldots, s_t$ (where as usual we assume that s_i divides s_{i+1}), show that $M \otimes_{\mathbf{Z}} M$ is a finite group of order $\prod_{j=1}^{t} s_j^{2t-2j+1}$.
27. Let F be a field and K a field containing F. Suppose that V is a finite-dimensional vector space over F and let $T \in \mathrm{End}_F(V)$. If $\mathcal{B} = \{v_i\}$ is a basis of V, then $\mathcal{C} = \{1\} \otimes \mathcal{B} = \{1 \otimes v_i\}$ is a basis of $K \otimes_F V$. Show that $[1 \otimes T]_{\mathcal{C}} = [T]_{\mathcal{B}}$. If $S \in \mathrm{End}_F(V)$, show that $1 \otimes T$ is similar to $1 \otimes S$ if and only if S is similar to T.
28. Let V be a complex inner product space and $T : V \to V$ a normal linear transformation. Prove that T is self-adjoint if and only if there is a real inner product space W, a self-adjoint linear transformation $S : W \to W$, and an isomorphism $\phi : \mathbf{C} \otimes_{\mathbf{R}} W \to V$ making the following diagram commute.

$$\begin{tikzcd} \mathbf{C} \otimes_{\mathbf{R}} W \arrow[r, "1\otimes S"] \arrow[d, "\phi"] & \mathbf{C} \otimes_{\mathbf{R}} W \arrow[d, "\phi"] \\ V \arrow[r, "T"] & V \end{tikzcd}$$

29. Let R be a commutative ring.
    (a) If I and J are ideals of R, prove that
    $$R/I \otimes_R R/J \cong R/(I + J).$$
    (b) If S and T are R-algebras, I is an ideal of S, and J is an ideal of T, prove that
    $$S/I \otimes_R T/J \cong (S \otimes_R T)/\langle I,\, J\rangle,$$

where $\langle I,\, J\rangle$ denotes the ideal of $S \otimes_R T$ generated by $I \otimes_R T$ and $S \otimes_R J$.

30. (a) Let F be a field and K a field containing F. If $f(X) \in F[X]$, show that there is an isomorphism of K-algebras:

$$K \otimes_F (F[X]/\langle f(X)\rangle) \cong K[X]/\langle f(X)\rangle.$$

(b) By choosing F, $f(X)$, and K appropriately, find an example of two fields K and L containing F such that the F-algebra $K \otimes_F L$ has nilpotent elements.

31. Let F be a field. Show that $F[X,\, Y] \cong F[X] \otimes_F F[Y]$ where the isomorphism is an isomorphism of F-algebras.

32. Let G_1 and G_2 be groups, and let $\mathbf{F}$ be a field. Show that

$$\mathbf{F}(G_1 \times G_2) \cong \mathbf{F}(G_1) \otimes_{\mathbf{F}} \mathbf{F}(G_2).$$

33. Let R and S be rings and let $f : R \to S$ be a ring homomorphism. If N is an S-module, then we may make N into an R-module by restriction of scalars, i.e., $a \cdot x = f(a) \cdot x$ for all $a \in R$ and $x \in N$. Now form the S-module $N_S = S \otimes_R N$ and define $g : N \to N_S$ by

$$g(y) = 1 \otimes y.$$

Show that g is injective and $g(N)$ is a direct summand of N_S.

34. Let F be a field, V and W finite-dimensional vector spaces over F, and let $T \in \operatorname{End}_F(V)$, $S \in \operatorname{End}_F(W)$.
(a) If α is an eigenvalue of S and β is an eigenvalue of T, show that the product $\alpha\beta$ is an eigenvalue of $S \otimes T$.
(b) If S and T are diagonalizable, show that $S \otimes T$ is diagonalizable.

35. Let R be a semisimple ring, M an (R, S)-bimodule that is simple as a left R-module, and let P be an (R, T)-bimodule that is simple as a left R-module. Prove that

$$M^* \otimes_R P = \begin{cases} \operatorname{End}_R(M) & \text{if } P \cong M \text{ as left } R\text{-modules} \\ 0 & \text{otherwise.} \end{cases}$$

36. Let R be a commutative ring and M an R-module. Let

$$M^{\otimes k} = M \otimes \cdots \otimes M,$$

where there are k copies of M, and let S be the submodule of $M^{\otimes k}$ generated by all elements of the form $m_1 \otimes \cdots \otimes m_k$ where $m_i = m_j$ for some $i \neq j$. Then $\Lambda^k(M) = M^{\otimes k}/S$ is called an *exterior algebra*.
(a) Show that if M is free of rank n, then $\Lambda^k(M)$ is free and

$$\operatorname{rank}(\Lambda^k(M)) = \begin{cases} \binom{n}{k} & \text{if } k \leq n \\ 0 & \text{if } k > n. \end{cases}$$

(b) As a special case of part (a),

$$\operatorname{rank}(\Lambda^n(M)) = 1.$$

Show that $\operatorname{Hom}_R(\Lambda^n(M),\, R)$ may be regarded as the space of determinant functions on M.