$\mathcal{A}_e$ codimension 1 multigerms of immersions

by

Amit Kumar

MA4K9 Scholarly Report

Submitted to The University of Warwick

Mathematics Institute

April, 2017

Contents

"Being good in Mathematics is not about how much you know, it's about how you behave when you don't know." - Raoul Bott

1 Introduction

Let $f_{n,k} : (\mathbb{C}^n, S) \to (\mathbb{C}^{n+1}, 0)$ with $|S| = k$ be multigerms from the n-dimensional complex space to $n+1$ dimensional pointed complex space such that $f_{n,k}$ consists of k germs of immersions $f_i, i = 1, \ldots, k$. For every subset $I \subset \{1, \ldots, k\}$ with $|I| = k-1$, the germs f_i for $i \in I$ meet in general position in $\mathbb{C}^{n+1}$, so that the multigerms they define is stable, and $\bigcap_{i \in I} f_i(\mathbb{C}^n)$, is a smooth manifold of dimension $n-k+2$. The remaining germs $f_j, j \notin I$, is *not* in general position with respect to the $f_i, i \in I$ but instead makes tangential contact of minimal order with $\bigcap_{i \in I} f_i(\mathbb{C}^n)$. Of course, this description only makes sense (because of restriction of general position) if $k < n+2$. The relation between $f_{n,k}$ and the branches f_k is up to the choice of local co-ordinates i. e. $f_k = f_{n,k} \circ \phi$ where $\phi : \mathbb{C}^n \to \mathbb{C}^n$ is a bi-holomorphic (or diffeomorphic) map. My goal in this article is to study these multigerms and make a solid generalisation as much as possible.

I have begun with studying conditions for meeting in general position and finally I could be able to formulate one which was proved very helpful while making generalisation. One of my ambition while doing this project has been to find some rigorous way to both - calculation of $\mathcal{A}_e$ co-dimension and family of maps satisfying the above condition. Although, I agree with Klien that mathematical ideas should be guided by intuition and not by rigour, but in my opinion, the role of rigour in Mathematics is analogous to of that money in real life: it's not everything, of course, but living can be very difficult, in fact unmanageable, without it; a sufficient amount is always needed to make life run smoothly. Following this philosophy, I have formulated an equivalent condition for meeting in general position and have used ideas to express the dependence among the entries of matrix representing $\theta(f), tf$ or ωf. These are two, but I have pursued only one because the second one, although very promising in lower dimension, lead to an infinite array of higher order tensors as the dimension increases. At the end, I have made some observation and certain conjectures. These conjectures came to my brain in the last moment, so I couldn't have given much thought upon these and left as it came.

2 Some Basic Notions and Results

2.1 Transversality and General Position

We recall the definition of a family of manifolds meeting in *general position* :

A finite set $\{E_1, E_2, \ldots, E_r\}$ of vector subspaces of a finite dimensional vector space V is said *to be meeting in general position iff:*

$$\textit{codim} \bigcap_{i=1}^{r} E_i = \sum_{i=1}^{r} \textit{codim} \quad E_i$$

Proposition 2.1. *When $r = 2$ the condition of being in general position is same as E_1 and E_2 being transversal.*

Proof. We have the following identity:

$$\begin{aligned} \dim E_1 + E_2 &= \dim E_1 + \dim E_2 - \dim(E_1 \cap E_2) \\ \Leftrightarrow \dim V - \dim E_1 + E_2 &= \dim V - \dim E_1 + (\dim V - \dim E_2) - (\dim V - \dim(E_1 \cap E_2)) \\ \Leftrightarrow \text{codim} \quad E_1 + E_2 &= \text{codim} \quad E_1 + \text{codim} \quad E_2 - \text{codim} \quad E_1 \cap E_2 \end{aligned} \tag{2.1}$$

Now, if E_1 and E_2 meet in general position then

$$\text{codim} \quad E_1 \cap E_2 = \text{codim} \quad E_1 + \text{codim} \quad E_2$$

So, using 2.1 we get

$$\text{codim} \quad E_1 + E_2 = 0$$

which is the same thing as:

$$E_1 + E_2 = V$$

We can retrace the set of argument backward to prove the converse. □

In general:

A finite set $\{E_1, E_2, \ldots, E_r\}$ of vector subspaces of a finite dimensional vector space V is said to have an almost regular intersection of order k (with respect to V) iff:

$$\textit{codim} \bigcap_{i=1}^{r} E_i = \sum_{i=1}^{r} \textit{codim} \quad E_i - k$$

When $k = 0$ we recover our definition of meeting in general position [1] Where codim E of a vector subspace E of a finite dimensional vector space V equals $\dim V - \dim E$.

An equivalent formulation of meeting in general position is:

A finite set $\{E_1, E_2, \ldots, E_r\}$ of vector subspaces of a finite dimensional vector space V is said

[1] some author also use the term *regular intersection*

to be meeting in general position iff the natural map

$$V \to \frac{V}{E_1} \oplus \cdots \oplus \frac{V}{E_r}$$

is surjective.

Proposition 2.2. *The two definitions for the subspaces $\{E_1, E_2, \ldots, E_r\}$ of a finite dimensional vector space V are equivalent.*

Proof. Let πbe the natural map, that is:

$$\pi : V \to \frac{V}{E_1} \oplus \cdots \oplus \frac{V}{E_r}$$

$$x \mapsto (x + E_1, \ldots, x + E_r).$$

Claim: π is a linear map: obviously if x and y are two vectors in V then

$$\pi(x + y) = ((x + y) + E1, \ldots, (x + y) + E_r)$$

$$\pi(ax) = (ax + E_1, \ldots, ax + E_r)$$

but

$$(x + y) + E_i = (x + E_i) + (y + E_i) \ \&\ ax + E_i = a(x + Ei).$$

Now, using *rank-nullity theorem* we have dimension of V equals rank of π plus nullity of π - the dimension of kernel of π . Now, if $x \in \ker \pi$ then,

$$\pi(x) = 0 \Longleftrightarrow x \in \bigcap_{i=1}^{r} E_i$$

which, in turn, implies:

$$\ker \pi = \bigcap_{i=1}^{r} E_i$$

and, thus the nullity of π equals $\dim \bigcap_{i=1}^{r} E_i$, that is, rank of π equals codim $\bigcap_{i=1}^{r} E_i$. Now, if we assume the definition of general position using codimension then, considering that

$$\dim(\frac{V}{E_1} \oplus \cdots \oplus \frac{V}{E_r}) = \sum_{i=1}^{r} \dim(\frac{V}{E_i}) = \sum_{i=1}^{r} \text{codim} \ \ E_i$$

Which, by above, equals the rank of π and hence π is surjective, and conversely.

□

There's still another equivalent formulation of meeting in general position:

Proposition 2.3. *A finite set $\{E_1, E_2, \ldots, E_r\}$ of vector subspaces of a finite dimensional vector space V is said to be meeting in general position iff $E_1 \times \cdots \times E_r$ meets the diagonal*

$D = \{(v_1, \dots, v_r) \in V^r \mid v_1 = \cdots = v_r\}$ *in* V^r *transversally.*

Proof. Let $v = (v_1, \dots, v_r) \in V^r$ and $\pi_i : V \to V/E_i : x \mapsto x + E_i = \bar{x}$, then $\bar{v_i}$ is the image of v_i in V/E_i under πi . Therefore,

$$(\bar{v_1}, \dots, \bar{v_r}) \in \frac{V}{E_i} \oplus \cdots \oplus \frac{V}{E_r}$$

. By proposition 2.2

$$\pi : V \to \frac{V}{E_1} \oplus, \dots, \oplus \frac{V}{E_r}$$

is surjective. Hence, there exist $x \in V$ such that $\pi x = (\bar{v_1}, \dots, \bar{v_r})$, but this implies

$$(x + E_1, \dots, x + E_r) = (\bar{v_1}, \dots, \bar{v_r})$$

Which means that $x + E_i = v_i + E_i$ that is, $v_i - x \in E_i$. Thus, $(v_1 - x, \dots, v_r - x) \in E_1 \times \cdots \times E_r$ and hence

$$v = (v_1, \dots, v_r) = (v_1 - x, \dots, v_r - x) + (x, \dots, x)$$

and the latter is in D. □

Now, if N is a manifold of dimension n, then at any point p in N the tangent space T_pN is a vector space isomorphic to $\mathbb{C}^n$ (or $\mathbb{R}^n$ in $\mathcal{C}^\infty$ category) so, the definition of meeting in general position naturally translates to manifolds:

Definition 2.4. *Let* $\{f^i : M_i \to N\}_{i=1,\dots,r}$ *be a finite set of analytic (or smooth) functions then for* $p \in N, \{f^i : M_i \to N\}_{i=1,\dots,r}$ *is said to be in general position at* p *if* $p \in \bigcap_{i=1}^r f^i$ *and* T_pf^i *meet in general position in* T_pN *for every* $i = 1, \dots, r$. $\{f^i : M_i \to N\}_{i=1,\dots,r}$ *is said to be in general position if they meet in general position for every* $p \in \bigcap_{i=1}^r f^i$.

Proposition 2.5. *Let* $f^1 : M_1 \to N$ *and* $f^2 : M_2 \to N$ *be two maps, then* f^1 *and* f^2 *meet in general position iff* $\forall x_i \in M_i$ *such that* $f^1(x_1) = f^2(x_2)$ *, we have:*

$$d_{x_1}f^1(T_{x_1}M_1) + d_{x_2}f^2(T_{x_2}M_2) = T_pN$$

where $p = f^1(x_1) = f^2(x_2)$.

2.2 Tangent Space to The Zero Set

Suppose $F : \mathbb{C}^n \to \mathbb{C}$ with no critical point in its zero set $N = F^{-1}(0)$. Then N is a regular submanifold of $\mathbb{C}^n$ and thus a manifold in itself [1, Theorem-9.9]. For $p \in N$ let $X_p \in T_pN$, we want to find the condition that X_p satisfies under the isomorphism $T_p\mathbb{C}^n \cong \mathbb{C}^n$ which contains T_pN as a vector subspace. So, let's assume that $\{X^1, \dots, X^n\}$ is the standard coordinate on $\mathbb{C}^n$ and $\gamma = (\gamma^1, \dots, \gamma^n)$ is a curve in N such that $\gamma(0) = p$ and $\gamma'(0) = X_p$ where $p = (p^1, \dots, p^n)$

and $X_p = (v^1, \dots, v^n)$. Since, γ is a curve in N, therefore

$$0 = \frac{d}{dt}(F(\gamma(t))) = \sum_{i=1}^{n} \frac{\partial F}{\partial X^i}(\gamma(t))((\gamma^i)'(t))$$

at $t = 0$:

$$0 = \sum_{i=1}^{n} \frac{\partial F}{\partial X^i}(p) v^i$$

which is the condition satisfied by any vector in T_pN. Now, considering that X_p is a vector starting at p , we can write $(v^1, \dots, v^n) = (X^1 - p^1, \dots, X^n - p^i)$, which means

$$\sum_{i=1}^{n} \frac{\partial F}{\partial X^i}(p)(X^i - p^i) = 0 \tag{2.2}$$

The expression 2.2 is the equation of the tangent space to $N = F^{-1}(0)$ at p. Hence, we have the following result:

Proposition 2.6. *For $F : \mathbb{C}^n \to \mathbb{C}$ of which 0 is a regular value, the points $(X^1, \dots, X^n) \in \mathbb{C}^n$ is in the tangent space of $N = F^{-1}(0)$ iff $(X^1, \dots, X^n)$ satisfies 2.2. So, we obtain the following description of T_pN :*

$$T_pN = \left\{ (X^1, \dots, X^n) \in \mathbb{C}^n \mid \sum_{i=1}^{n} \frac{\partial F}{\partial X^i}(p)(X^i - p^i) = 0 \right\}$$

Theorem 2.7. *Let $F_1, F_2, \dots, F_k$ be functions from $\mathbb{C}^n$ to $\mathbb{C}$ $(k < n)$, such that 0 is the regular value of each of F_i then regular submanifolds $F_1^{-1}(0), \dots, F_k^{-1}(0)$ to meet in general position at $p \in \cap_{i=1}^{k} F_i^{-1}(0)$ if and only if $\nabla_p F_1, \dots, \nabla_p F_k$ are linearly independent. Where $\nabla_p F_i = \left(\frac{\partial F_i}{\partial x^1}(p), \dots, \frac{\partial F_i}{\partial x^n}(p) \right)$ stands for the gradient of the scaler field $F_i : \mathbb{C}^n \to \mathbb{C}$ evaluated at p.*

Proof. $\nabla_p F_1, \dots, \nabla_p F_k$ are linearly independent iff $c_1 \nabla_p F_1 + \dots + c_k \nabla_p F_k = 0 \Leftrightarrow c_1 = \dots = c_k = 0$ which is

$$c_1 \begin{pmatrix} \frac{\partial F_1}{\partial x^1}(p) \\ \vdots \\ \frac{\partial F_1}{\partial x^n}(p) \end{pmatrix} + \dots + c_k \begin{pmatrix} \frac{\partial F_k}{\partial x^1}(p) \\ \vdots \\ \frac{\partial F_k}{\partial x^n}(p) \end{pmatrix} = 0 \Leftrightarrow c_1 = \dots = c_k = 0$$

which is equivalent to the fact that the $n \times k$ matrix

$$A = \begin{pmatrix} \frac{\partial F_1}{\partial x^1}(p) & \cdots & \frac{\partial F_k}{\partial x^1}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_1}{\partial x^n}(p) & \cdots & \frac{\partial F_k}{\partial x^n}(p) \end{pmatrix}$$

has rank k.

Now, let

$$E_i = \{(x^1, \ldots, x^n) \mid \frac{\partial F_i}{\partial x^1}(p)(x^1 - p^1) + \cdots + \frac{\partial F_i}{\partial x^n}(p)(x^n - p^n) = 0\}$$

where the constraint $\frac{\partial F_i}{\partial x^1}(p)(x^1 - p^1) + \cdots + \frac{\partial F_i}{\partial x^n}(p)(x^n - p^n) = 0$ can be recognised, by the previous theorem, as the equation of the tangent space to the regular level set $F_i^(- 1)(0)$ and $p = (p^1, \ldots, p^n)$. Please note that all the E_i has codimension 1: since it's the tangent space to the level set which has dimension $n-1$. Then a point $(x^1, \ldots, x^n) \in \cap_{i=1}^k$ if and only if the equations

$$\frac{\partial F_i}{\partial x^1}(p)(x^1 - p^1) + \cdots + \frac{\partial F_i}{\partial x^n}(p)(x^n - p^n) = 0$$

is satisfied for every $i = 1, \ldots, k$ that is if and only if the matrix

$$B = \begin{pmatrix} \frac{\partial F_1}{\partial x^1}(p) & \cdots & \frac{\partial F_1}{\partial x^n}(p) \\ \vdots & \cdots & \vdots \\ \frac{\partial F_k}{\partial x^1}(p) & \cdots & \frac{\partial F_k}{\partial x^n}(p) \end{pmatrix}$$

has rank k , but the matrix B is nothing but A^T and hence they have the same rank. Now, since $\cap_{i=1}^k E_i$ is the solution set of the above system of n variables in k unknown, it's dimension is $n-k$. So, the codimension of $\cap_{i=1}^k E_i$ in $\mathbb{C}^n$ is k and therefore we have:

$$\text{codim} \ \ \cap_{i=1}^k E_i) = \sum_{i=1}^k \text{codim} \ \ E_i$$

□

2.3 Quick Revision of Singularity Theory

Let $\mathcal{O}_n^p$ be the vector space of *monogerms* with n variables and p components, i. e.

$$\mathcal{O}_n^p = \{f = (f^1(x^1, \ldots, x^n), \ldots, f^p(x^1, \ldots, x^n)) \mid \forall i, f^i = \sum a_{j_1, \ldots, j_n}(x^1)^{j_1} \ldots (x^n)^{j_n}\} \quad (2.3)$$

where the sum in 2.3 is an infinite sum, i. e. each f^i is a power series (or smooth functions if the underlying domain is $\mathbb{R}$) in the variables $x^1, \ldots, x^n$ and $f : (\mathbb{C}^n, x_0) \to (\mathbb{C}^p, X_0)$ is a germ of mapping. When $p = 1$, $\mathcal{O}_n^1 = \mathcal{O}_n$ is the ring of germs of functions in n - variables which is a local ring; let $\mathcal{M}$ is the maximal ideal. So, it means that the set $\mathcal{O}_n^p$ is a free module of rank p over $\mathcal{O}_n$.

A multigerm is a germ of analytic (over $\mathbb{C}$) or smooth (over $\mathbb{R}$) map $f = \{f_1, \ldots, f_k\}$: $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ where $S = \{x_1, \ldots, x_k\} \subset \mathbb{C}^n$. Each f_i is called a branch of f.

Definition 2.8. *Two germs f, g : $(\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ are $\mathcal{A}$ equivalent ($f \sim_{\mathcal{A}} g$) if there exists germs of bianalytic (diffeomorphism in case of $\mathbb{R}$) maps $\phi : (\mathbb{C}^n, S) \to (\mathbb{C}^n, S)$ and $\psi : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ such that $g = \psi \circ f \circ \phi^{-1}$.*

Remark 2.9. *When ψ is identity map we would have right equivalence ($f \sim_{\mathcal{R}} g$) and left* *equivalence* ($f \sim_{\mathcal{L}} g$) *in the case when* ϕ *is the identity map. In fact, if we denote the group* $Diff(\mathbb{C}^n, S) \times Diff(\mathbb{C}^p, 0)$ *by* $\mathcal{A}$ *then* $\mathcal{A} = \mathcal{R} \times \mathcal{L}$.

Definition 2.10. *An* r *- parameter unfolding of* $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *is a germ* $F : (\mathbb{C}^n \times \mathbb{C}^r, S \times \{0\}) \to (\mathbb{C}^p \times \mathbb{C}^r, 0)$ *such that* $F(x,t) = (f_t(x), t)$ *subject to the condition* $f_0(x) = f(x)\ \forall x \in (\mathbb{C}^n, S)$.

Definition 2.11. *Two* r *- parameter unfoldings* F, G *of* f *are equivalent if there exists germs of bi-analytic maps (or diffeomorphisms)*

$$\Phi : (\mathbb{C}^n \times \mathbb{C}^r, S \times \{0\}) \to (\mathbb{C}^n \times \mathbb{C}^r, S \times \{0\})$$

and

$$\Psi : (\mathbb{C}^p \times \mathbb{C}^r, 0) \to (\mathbb{C}^p \times \mathbb{C}^r, 0)$$

which are themselves unfoldings of the identity in $\mathbb{C}^n$ *and* $\mathbb{C}^p$ *respectively, such that*

$$G = \Psi \circ F \circ \Phi^{-1}$$

The unfolding F *is trivial if it is equivalent to* $f \times Id$ *where* $f \times Id : (x,t) \mapsto (f(x), t)$. *The map germ* $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *is stable iff every unfolding of* f *is trivial.*

Definition 2.12. *Let* F *be an* r *- parameter unfolding of* f *and* $h : (\mathbb{C}^d, 0) \to (\mathbb{C}^r, 0) : h(u) = t$. *Then*

$$G = h^*F : (\mathbb{C}^n \times \mathbb{C}^d, 0) \to (\mathbb{C}^p \times \mathbb{C}^d, 0)$$

$$(x, u) \mapsto (f(x, h(u)), u)$$

is a d-parameter unfolding of f *called the pullback of* F *by* h.

Definition 2.13. *An unfolding* F *of* f *is* $\mathcal{A}_d$***-versal*** *if every other d parameter unfolding of* f *is equivalent to* h^*F *for some base change map* $h : (\mathbb{C}^d, 0) \to (\mathbb{C}^r, 0)$. *A* ***miniversal*** *unfolding is a versal unfolding with a minimum number of parameters.*

Let $\theta_{\mathbb{C}^n, S}$ be the $\mathcal{O}_n$ module of germs (at S) of vector fields on $\mathbb{C}^n$ and $\theta_{\mathbb{C}^p, 0}$ is defined analogously; for brevity, we will denote these by θ_n and θ_p. Let $\theta(f)$ be the $\mathcal{O}_n$ module of germs of maps: $\bar{f} : (\mathbb{C}^n, S) \to T\mathbb{C}^p$ such that $\pi_p \circ \bar{f} = f$ where $\{T\mathbb{C}^p, \mathbb{C}^p, \pi_p\}$ is the tangent bundle of $\mathbb{C}^p$.

$$\begin{tikzcd}
T\mathbb{C}^n \arrow[r, "df"] \arrow[d, "\pi_n"'] & T\mathbb{C}^p \arrow[d, "\pi_p"] \\
\mathbb{C}^n \arrow[r, "f"'] \arrow[ur, "\bar{f}"] & \mathbb{C}^p
\end{tikzcd}$$

Therefore, $\theta(f) \cong \mathcal{O}_n^p \oplus \cdots \oplus \mathcal{O}_n^p$ (k times - as many branches of germs are there - $k = |S|$.) Now, $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ induces a map $f^* : \mathcal{O}_p \to \mathcal{O}_n$ such that $f^* : h \mapsto h \circ f$, that means, $\theta(f)$ is an $\mathcal{O}_p$ module via f^*. Define $tf : \theta_n \to \theta(f)$ as $tf : \xi \mapsto df \circ \xi$, where $df : T\mathbb{C}^n \to T\mathbb{C}^p$

is the component of the bundle map

$$f' = (f, df)(p, v) = (f(p), d_p f(v))$$

and $\omega f : \theta_p \to \theta(f)$ such that $\omega f : \eta \to \eta \circ f$.

$$\begin{tikzcd} T\mathbb{C}^n \arrow[r, "df"] & T\mathbb{C}^p \\ \mathbb{C}^n \arrow[u, "\xi"] \arrow[r, "f"] & \mathbb{C}^p \arrow[u, "\eta"'] \end{tikzcd}$$

The $\mathcal{A}_e$ tangent space $T\mathcal{A}_e f$ is defined as

$$T\mathcal{A}_e f = tf(\theta_n) + \omega f(\theta_p)$$

$tf(\theta_n)$ is an $\mathcal{O}_n$ submodule of $\theta(f)$, but, as we have mentioned above, it also has an $\mathcal{O}_p$ module structure, and similarly $\omega f(\theta_p)$ is also a $\mathcal{O}_p$ submodule via f^*. So, $T\mathcal{A}_e f$ is an $\mathcal{O}_p$ module via f^*.

Definition 2.14. *The $\mathcal{A}_e$ codimension of a germ f, denoted as $\mathcal{A}_e -$ codim (f) , is the $\mathbb{C}$ vector space dimension of*

$$T^1 f = \frac{\theta(f)}{T\mathcal{A}_e f}$$

.

We will state (without proof) a very important theorem due to John Mather, which expresses the stability of a germ in terms of it's $\mathcal{A}_e$ codimension.

Theorem 2.15. *(Mather's infinitesimal criterion for stability) [2] A germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ is stable iff $\mathcal{A}_e -$ codim $(f) = 0$.*

Let's do the following example where we wish to illustrate a general principle of calculating $\mathcal{A}_e -$ codim f which will be used in calculating the codimension of multigerms mentioned in the introduction. Here, I must mention that the monomials x^k do not generate the power series in x but If a multi-germ f has isolated instability, then it has finite codimension, which means that $T\mathscr{A}_e f \supset \mathfrak{m}_n^k \theta(f)$ for some finite k. It follows that we need not worry about infinite power series: modulo $T\mathscr{A}_e f$, every element of $\theta(f)$ can be represented by a polynomial map-germ, and thus the monomials $x^\alpha \partial/\partial y_j$ generate the quotient $\theta(f)/T\mathscr{A}_e f$.

Example 2.16. *The germs $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $x \mapsto x^n (n \in \mathbb{N})$ is stable if and only* *if* $n < 3$.

$$\begin{tikzcd} T\mathbb{C} \arrow[r, "df"] & T\mathbb{C} \\ \mathbb{C} \arrow[u, "\xi"] \arrow[r, "f"] & \mathbb{C} \arrow[u, "\eta"'] \end{tikzcd}$$

tf(ξ) $= (df \circ \xi)$ *So, for* $\xi = \xi(x)\frac{\partial}{\partial x}$ *we have* $df(\xi) = df(\xi(x)\frac{\partial}{\partial x})$. *But* ξ *is a vector field (a section of the tangent bundle) i. e.* $\xi : \mathbb{C} \to T\mathbb{C}$ *such that* $\xi : p \mapsto \xi_p \in T_p\mathbb{C}$ *for every* $p \in \mathbb{C}$ *i.e.* $\xi(p) = \xi(p)\frac{\partial}{\partial x}$ *Thus at every point* $p \in \mathbb{C}$, *since* $df\big|_p = d_p f : T_p\mathbb{C} \to T_{f(p)}\mathbb{C}$ *is linear, for* $\xi_p \in T_p\mathbb{C}$*:*

$$(df \circ \xi)(p) = \xi(p) d_p f(\frac{\partial}{\partial x})$$

[2] *and if*

$$d_p f(\frac{\partial}{\partial x}) = A\frac{\partial}{\partial X}$$

then applying $X(= f(x))$ *both sides we get*

$$A = \frac{\partial f}{\partial x}\bigg|_p$$

As p varies, this becomes a function and thus:

$$df(\xi) = \xi(x) df(\frac{\partial}{\partial x})$$

which by above becomes

$$df(\xi(x)\frac{\partial}{\partial x}) = \xi(x)\frac{\partial f}{\partial x}$$

In general, for $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$ *with* $f(x^1, \ldots, x^n) = (f^1, \ldots, f^p)$ *and under the isomoprhism* $T_p\mathbb{C}^n \cong \mathbb{C}^n$ *, for* $\xi = (\xi_1, \ldots, \xi_n)$ *we can prove in the same fashion*

$$df : \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \longmapsto \begin{bmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f^p}{\partial x^1} & \cdots & \frac{\partial f^p}{\partial x^n} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \tag{2.4}$$

which we will abbreviate as

$$df(\xi) = [J_f]_{p\times n}[\xi]_{n\times 1}$$

or even more concisely as

$$df : [\xi] \longmapsto [J_f][\xi] \tag{2.5}$$

where $[j_f]$ stands for the *jacobian matrix* of f. Coming back to our example, we see that

[2] a linear map is determined by it's action on the basis elements

$tf([\xi]) = [nx^{n-1}\xi(x)]$ and for a vector field $\eta = [\eta(x)]$ on the target $\omega f([\eta]) = [(\eta \circ f)(x)]$ So,

$$\omega f : [\eta(x)] \longmapsto [(\eta \circ f)(x)] = [\eta(x^n)]$$

Now $\theta(f)$ is a power series in one variable x which means it's generated (over $\mathbb{C}$) by the sequance $\{x^k\}$ i. e. for $\{k = 0, 1, 2, \dots\}$ i. e. every element of $\theta(f)$ is of the form $\sum_k a_k x^k$ where all a_k might be non-zero [3] The operation tf and $\omega(f)$ have different generating sets: for tf it's $\{x^{n+k-1}\}$ and for $\omega(f)$ it's $\{x^{nk}\}$ which is very easy to see. Thus in order to calculate the $\mathcal{A}_e - codimension$ of f we would like to see that how much is the spread of $T\mathcal{A}_e f$,which is the span of these two generating sets together, in $\theta(f)$. One nice way to do this is to see how many elements of the generating set $\{x^k\}$ (call this set Θ) of $\theta(f)$ can be recollected back from the span of $\{x^{n+k-1}\}$ (call this set T) and $\{x^{kn}\}$ (call this set $\Omega(f)$). From the span of T we get all the terms of the set Θ whose power is greater than or equal to $n-1$. So, what we don't have yet is the finite set $\Theta_1 = \{1, x, x^2, \dots, x^{n-2}\}$ whose span is the quotient vector space $V = \frac{Sp_{\mathbb{C}}\Theta}{Sp_{\mathbb{C}}T}$ and is of finite dimension $n-1$

Theorem 2.17. *Suppose T is a vector space and S_1 and S_2 are two subsets of V, and let $\langle S \rangle$ stands for the span of S over the same field, then*

$$\frac{T}{\langle (S_1 \cup S_2) \rangle} = \frac{\frac{T}{\langle S_1 \rangle}}{\langle \bar{S_2} \rangle}$$

where $\bar{S}_2$ stands for set S_2 modulo $\langle S_1 \rangle$.

So, we have $\bar{\Omega} = \{1\}$ and

$$T^1 f = \frac{V}{\bar{\Omega}} = \{x, x^1, \dots, x^{n-2}\}$$

which has dimension $n-2$ thus if $n > 3$ the $\mathcal{A}_e -$codim f is non-zero and hence it follows from Theorem 2.15 that the germ is unstable. [4]

Alternatively, assume that $\mathbb{C}^{\mathbb{N}}$ denote the set of all sequences in $\mathbb{C}$, then there's a natural identification of the elements of $\mathcal{O}_1$, the ring of power series in one variable over $\mathbb{C}$, to the elements of $\mathbb{C}^{\mathbb{N}}$:

$$\psi : \mathcal{O}_1 \to \mathbb{C}^{\mathbb{N}}; \sum_n a_n x^n \longmapsto \{a_n\}$$

Under this identification, we can write a power series as as sequence $(a_0, a_1, \dots, a_k, \dots)$, and then we have:

$$\Theta = \{(a_0, a_1, \dots) \mid a_i \in \mathbb{C} \forall i\}$$

$$T = \{(x_0, x_1, \dots) \mid x_i = 0 \ \forall \ i < n-1\}$$

$$\Omega = \{(x_0, x_1, \dots) \mid x_i = 0 \ \forall \ i \neq 0 \mod n\}$$

We will abbreviate a power series, under this identification by $\{x\} Thus,$T$and \Omega$ are subspaces

[3] this is why we have chosen to call $\{x^k\}$ a generating set rather than *a basis*

[4] for $n = 1$ this formula doesn't hold, rather in that case we have $V = 0$ where V is as in the example.

of the vector space Θ. Choose an arbitrary element of Θ , say $f = \{a_k\}$ and in the direction of Theorem 2.17, reduce it module T:

$$(a_0, a_1, \dots) = (0, \dots, 0, a_{n-1}, a_n, \dots) + (a_0, \dots, a_{n-2})$$

So, modulo T we get $f = (a_0, \dots, a_{n-2})$ that is

$$V = \frac{\Theta}{T} = \{(a_0, \dots, a_{n-2}) \mid a_i \in \mathbb{C} \ \forall \ i = 0, \dots, n-2\}$$

which is a finite dimensional vector space over $\mathbb{C}$. Similarly, if we reduce $(a_0, 0, \dots, 0, a_n, 0, \dots, 0, a_{2n}, \dots)$ modulo T, we get

$$\bar{\Omega} = \{(a_0, 0, \dots, 0)\}$$

which is a 1-dimensional subspace of V, and therefore, by Theorem 2.17 gives:

$$T^1 f = \{(0, a_1, \dots, a_{n-2}) \mid a^i \in \mathbb{C} \ \forall \ i = 1, \dots, n-2\}$$

and a basis of $T^1 f$ is the set $\{e_i\}_{i=1,\dots,n-2}\}$ where e_i stands for the row vector with 1 at the i position and zero otherwise, where $i = 0, 1, 2, \dots$ Please note that under the identification ψ the above basis of $T^1 f$ will correspond to $\{x^i\}$. Let $\mathbb{C}^{\mathbb{N}}$ is as above, we define a map "shift operator", denoted σ from $\mathbb{C}^{\mathbb{N}}$ to itself as follows:

$$\sigma : (a_0, a_1, a_2, \dots) \longmapsto (0, a_0, a_1, \dots)$$

σ thus defined is linear and it's composition by itself causes as many shifts as many times it's composed. So, now, we can write our set T (above) as: $\sigma^{n-2}(\{x_k\})$. It will simply our notation.

Another, very useful result is (again due to Mather) [3]:

Theorem 2.18. *Let $F : (\mathbb{C}^n \times \mathbb{C}^r, S \times 0) \to (\mathbb{C}^p \times \mathbb{C}^r, 0)$ be an r - parameter unfolding of the germ f, such that $F(x,t) = (f_t(x), t)$ with $f_0(x) = f(x)$. Then f is a versal unfolding of f iff*

$$T\mathcal{A}_e f + Sp_{\mathbb{C}} \left\{ \frac{\partial f_t}{\partial t_1}, \dots, \frac{\partial f_t}{\partial t_r} \right\} = \theta(f)$$

An unfolding is miniversal if $r = \mathcal{A}_e - codim \ \ f$

Definition 2.19. *Let $F : (\mathbb{C}^n \times \mathbb{C}^r, S \times 0) \to (\mathbb{C}^p \times \mathbb{C}^r, 0)$ be an r - parameter unfolding of the germ $f : (\mathbb{C}^n, S) \to (\mathbb{C}^p, 0)$, such that $F(x,t) = (f_t(x), t)$ with $f_0(x) = f(x)$. Then, $f_t(x)$ is called an r* ***- parameter deformation*** *of f. That is if $\pi_p : (\mathbb{C}^p \times \mathbb{C}^r) \to \mathbb{C}^p$ is the projection map $((x^1, \dots, x^p), (t^1, \dots, t^r)) \mapsto (x^1, \dots, x^p)$ then the r - parameter deformation of f is given by the composite $\pi_p \circ F = f_t(x)$.*

In that case we would say that the r - parameter deformation $f_t(x)$ is ***induced*** *from the r - parameter unfolding F. For brevity, where there's no scope of confusion, we would simply use the phrase "f_t is the deformation of f induced from F" thus would drop the terms like parameter and unfolding.*

Definition 2.20. *A deformation f_t of f is called a* ***versal deformation*** *if it is induced from a versal unfolding.*

Referring to example 2.16, we have:

$$T\mathcal{A}_e f + Sp_{\mathbb{C}}\left\{x, x^2, \ldots, x^{n-2}\right\} = \theta(f)$$

Thus a miniversal deformation of f is an $n-2$ parameter deformation

$$F(x, t_1, \ldots, t_{n-2}) = f_{\mathbf{t}}(x) = (x^n + t_{n-2}x^{n-2} + \cdots + t_1 x)$$

where $\mathbf{t}$ stands for the vector $(t_1, \ldots, t_{n-2})$ [5]

3 the case when $n = 1$

Here, we have only one choice, namely, $k = 2$, since the maximum possible codimension of $\cap T_p f_i$ is two, and since our germs are immerssions, the codimension of $T_p f_i$ is one. Therefore, if $i > 2$ germs will never meet in general position. So, let's consider the situation when this is the case. $f_{1,2} : (\mathbb{C}, \{0, 0'\}) \to (\mathbb{C}^2, 0)$ can be parametrised as:

$$\begin{cases} f_1 : x & \mapsto & (x, 0) \\ f_2 : x & \mapsto & (x, x^2) \end{cases}$$

f^1 is clearly an immersion; in fact it's an embedding of $\mathbb{C}$ in $\mathbb{C}^2$. To see that f^2 is an immersion, consider a point $p \in \mathbb{C}$, then

$$d_p f_2 : T_p\mathbb{C} \to T_{f_2(p)}\mathbb{C}^2$$
$$p \mapsto (1, 2p)$$

where, denoting the co-ordinates in the target as X^1 and X^2 [6], the above means that for the basic vector fields $\{d/dx\}$ on the source - $\mathbb{C}$ and $\{\partial/\partial X^1, \partial/\partial X^2\}$ on the target - $\mathbb{C}^2$

$$d_p f^2(\frac{d}{dx}) = \frac{\partial}{\partial X^1} + 2p\frac{\partial}{\partial X^2}$$

which means that $d_p f^2$ has rank 1 for every $p \in \mathbb{C}$. Hence, f^2 is also an immersion.

If there is a point $q \in \mathbb{C}^2$ which is the intersection point of images of f^1 and f^2 then there exist $x_1, x_2 \in \mathbb{C}$ such that $f^1(x_1) = f^2(x_2)$ which means $(x_1, 0) = (x_2, x_2^2)$, which gives $x_2 = 0, x_1 = 0$. Thus $(0, 0)$ is the only point in $f^1 \cap f^2$.

If two manifolds, (say) γ_1 and γ_2 ,need not be of same dimension, in the n - dimensional ambient space admits global parametrisation $\gamma_1 = \gamma_1(x^1, \ldots, x^{n_1})$ and $\gamma_2 = \gamma_2(y^1, \ldots, y^{n_2})$ (say) then finding their point of intersection is same as solving the system of equation $(\gamma_1^1, \ldots, \gamma_1^n) = (\gamma_2^1, \ldots, \gamma_2^n)$ [7] However, most of the time, one ends up solving the same equation after eliminating sufficient number of variables as the manifolds would have given in terms of co-ordinates in the target space. So, it's better to eliminate the parameters and find the equation, if possible. For instance, in our case, the image of f^1 is given by the equation $X^2 = 0$ [8] and the image of f^2 by the equation $X^2 = X^1$; solving we get $(X^1, X^2) = (0, 0)$.

[5] we will always follow the convention to denote the co-ordinates by superscripts and branches of multigerms by subscript, except in the case of parameter space co-ordinates will be denoted by subscript.

[6] $\{1_{\mathbb{C}^2}, X^1, X^2\}$ is the standard chart on $\mathbb{C}^2$; we will follow the convention that lowercase letters are co-ordinates on the source and the uppercase letter are co-ordinates on the target.

[7] γ_1 and γ_2 are maps in to n spaces and therefore have n component functions.

[8] this is an abbreviation for $\{(X^1, X^2) \in \mathbb{C}^2 \mid X^2 = 0\}$

Now, using proposition 2.6, we have, for $p = (p^1, p^2) \in f_1 \cap f_2$ and $F = X^2 - (X^1)^2$ (for the image set of f_2),

$$T_pF = \left\{(X^1, X^2) \in \mathbb{C}^2 \mid \frac{\partial F}{\partial X}(p)(X^1 - p^1) + \frac{\partial F}{\partial Y}(p)(X^2 - p^2)\right\}$$

Since, $O = (0,0)$ is the only point in the intersection, we get:

$$T_O f_2 = T_O F = \{(X^1, X^2) \mid X^2 = 0\}$$

For the image of f_1, since it is itself a linear subspace, we have

$$T_O f_1 = \{(X^1, X^2) \mid X^2 = 0\}$$

which is same as $T_O f_2$. Thus we see that at $(0,0)$ the two curves fails to meet in general position, as required by the condition.

Using the notations of section 2.3, let $\xi = (\xi_1, \xi_2) \in \theta_1$ then $\xi = (a(x)\frac{\partial}{\partial x}, b(x)\frac{\partial}{\partial x})$ and, and

$$tf(\xi) = (df_1 \circ \xi_1, df_2 \circ \xi_2)$$

$$tf(a(x), b(x)) = \left(\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}a(x)\end{bmatrix}\begin{bmatrix}1\\2x\end{bmatrix}\begin{bmatrix}b(x)\end{bmatrix}\right) = \begin{pmatrix}a(x) & b(x)\\ 0 & 2xb(x)\end{pmatrix}$$

and for $\eta = \begin{bmatrix}\eta_1(X^1, X^2)\\ \eta_2(X^1, X^2)\end{bmatrix}$ we have

$$\omega f(\eta) = \begin{pmatrix}\eta_1(x,0) & \eta_1(x, x^2)\\ \eta_2(x,0) & \eta_2(x,x^2)\end{pmatrix}$$

Now, as in section-2.3, the set θf can be represented as a 2×2 matrix, each of whose entry is a power series in x. A generating set, using the Mather-Gaffney criterion for the finite $\mathcal{A}_e$ co-dimension, for θf is:

$$\{x^k e_{11}, x^k e_{12}, x^k e_{21}, x^k e_{22}\}$$

where e_{ij} is the usual matrix units which form a basis for the space $M_{m\times n}$ [9]. Using this generating set

$$tf = \langle\{x^k e_{11}, x^k e_{12} + x^{k+1} e_{22}\}\rangle$$

where $\langle S\rangle$ means span of S for any set S.[10] In the generating set of two variables η has a generating set of the form $\{e^{k,l}_{1,1} + e^{k,l}_{1,2}, e^{k,l}_{2,1} + e^{k,l}_{2,2}\}$. Let's choose η as the monomial $\eta_1(X^1, X^2) = (X^1)^k(X^2)^l$ then after composing with f we get for $l \neq 0$ $\begin{pmatrix}0 & x^{1+2l}\\ 0 & 0\end{pmatrix}$ if we choose $k = 1$ and $\begin{pmatrix}0 & x^{2l}\\ 0 & 0\end{pmatrix}$ if we choose $k = 0$. Therfore, we have have got every odd and non-zero even powers

[9] we will also write $x^k y^l e_{i,j}$ as $e^{k,l}_{i,j}$ *sometimes*

[10] We can write tf as:

$$\begin{bmatrix}\sum_k a_k x^k & \sum_k b_k x^k\\ 0 & \sum_k a_k x^{k+1}\end{bmatrix} = \sum_k a_k \begin{bmatrix}x^k & 0\\ 0 & 0\end{bmatrix} + \sum_k b_k \begin{bmatrix}0 & x^k\\ 0 & x^{k+1}\end{bmatrix}$$

and similarly for $\omega(f)$

of x at (1,2). We get $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ when $l = 0, k = 0$, but using tf we get $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ Therefore, we get $x^k e_{1,2}$, we already have $x^k e_{1,1}$ from tf. Similarly, using the same monomial for η_2 we get for $l \neq 0$ entries of the type $\begin{pmatrix} 0 & 0 \\ 0 & x^{1+2l} \end{pmatrix}$ if $k = 1$ and of the type $\begin{pmatrix} 0 & 0 \\ 0 & x^{2l} \end{pmatrix}$ when $k = 0$, and $\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$ when $k = 0, l = 0$. Therefore, we have got $x^{k+1} e_{2,2}$ i.e. entries with non-zero powers of x at (2,2) together with $e_{2,1} + e_{2,2}$. Now using the vector field of the type $\eta_1 = 0$ and $\eta_2(X^1, X^2) = (X^1)^k$ we get entries of the type $\begin{pmatrix} 0 & 0 \\ x^k & x^k \end{pmatrix}$, but for $k > 0$ we have $x^k e_{2,2}$ from the above discussion, therefore we get $x^{k+1} e_{2,1}$. Therefore, among the generating sets of θ we have got everything except $x^0 e_{2,1} = e_{2,1}$ and $x^0 e_{2,2} = e_{2,2}$ but these are not linearly independent in $T^1 f$ since $e_{2,1} + e_{2,2}$ is in $T\mathcal{A}_e f$. Hence, we have got:

$$T\mathcal{A}_e f_{1,2} + \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} = T^1 f_{1,2}$$

Now, using the identification of $\mathcal{O}$ with $\mathbb{C}^{\mathbb{N}}$, let's denote

$$\Theta = \left\{ \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \right\}$$

where each X_{ij} is a sequence in $\mathbb{C}$ i. e.

$$X_{ij} = ((x_{11})_0, (x_{11})_1, (x_{11})_2, \dots)$$

under which we have

$$T = \left\{ \begin{bmatrix} X_{11} & X_{12} \\ 0 & \sigma(X_{12}) \end{bmatrix} \right\}$$

and

$$\Omega = \left\{ \begin{bmatrix} X_{11} & X_{11} \\ X_{21} & X_{21} \end{bmatrix} \right\}$$

To see this, let $\eta_1(X^1, X^2) = \sum_{k,l} a_{kl} (X^1)^k (X^2)^l$ then, at (1,1) we have $(a_{00}, a_{10}, \dots, a_{k0}, \dots)$ and at (2,2) we would choose $l = 0$ to get the same sequence. Let's start with an arbitrary element

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

of Θ and reduce step by step, first modulo $\langle \Omega \rangle$. We can write P as:

$$\begin{bmatrix} A_{11} & A_{11} \\ A_{21} & A_{22} \end{bmatrix} + \begin{bmatrix} 0 & A_{12} - A_{11} \\ 0 & A_{22} - A_{21} \end{bmatrix}$$

So,

$$\bar{P} = \begin{bmatrix} 0 & X_{12} - X_{11} \\ 0 & X_{22} - X_{21} \end{bmatrix}$$

and reducing a member of T we get

$$\bar{T} = \left\{ \begin{bmatrix} 0 & X_{12} - X_{11} \\ 0 & \sigma(X_{12}) \end{bmatrix} \right\}$$

Now $\bar{P}$ is clearly same as the the following set:

$$\bar{Q} = \left\{ \begin{bmatrix} 0 & X \\ 0 & Y \end{bmatrix} \right\}$$

and the members of $\bar{T}$ are matrices of the form:

$$\left\{ \begin{bmatrix} 0 & X \\ 0 & \sigma(X+B) \end{bmatrix} \right\}$$

for some matrix B. So, we can write $\bar{A}$ as:

$$\begin{bmatrix} 0 & X \\ 0 & \sigma(X+B) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Y - \sigma(X+B) \end{bmatrix}$$

Which is a set of the form:

$$\left\{ \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \right\}$$

which is a 1-dimensional subspace of $M_{2\times 2}$ has a basis as $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore:

$$T\mathcal{A}_e f + Sp_{\mathbb{C}} \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = T^1 f$$

Now, using Theorem 2.18, a miniversal deformation of $f_{1,2}$ can be given as:

$$F_{1,2}(x,t) = \left\{ \begin{array}{lll} f_1 : x & \mapsto & (x, 0) \\ f_2 : x & \mapsto & (x, x^2 + t) \end{array} \right.$$

The equation of the first image is $X^2 = 0$ as before, and the equation of the second image is a family of curves parametrised by t : $X^2 = (X^1)^2 + t$ which is just the translates of the previous curve. A real picture of which will be represented by a translates of the parabola $Y = X^2$ in the XY-plane. The parabola intersects the X-axis, transversally, at two points if $t < 0$ and not otherwise. However, the complex picture will be different: There are two intersection points for every value of t but 0. At these points $p = (\pm i\sqrt{t}, 0)$ the equation of tangent is:

$$(X^2 - 0) - 2i\sqrt{t}(X^1 - \sqrt{t}) \ \& \ (X^2 - 0) + 2i\sqrt{t}(X^1 + \sqrt{t})$$

which is transverse to the curve given by $X^1 = 0$.

4 The case $n = 2$

4.1 $n = 2, k = 2$

A parametrisation for $f_{2,2}$ can be given by the following:

$$\left\{ \begin{array}{lll} f_1 : (x^1, x^2) & \mapsto & (x^1, x^2, 0) \\ f_2 : (x^1, x^2) & \mapsto & (x^1, x^2, (x^1)^2 + (x^2)^2) \end{array} \right.$$

is clearly an immersion [11], to see that f_2 is an embedding, let $p = (a, b) \in \mathbb{C}^2$, then,

$$d_p f_2 : T_p\mathbb{C}^2 \to T_{f_2(p)}\mathbb{C}^3$$

[11] just like the case $n = 1$ this is an embedding of $\mathbb{C}^2$ into $\mathbb{C}^3$

is given by

$$X_p \mapsto (J_p f_2) X_p$$

where $J_p f_2$ is the *jacobian matrix* of f_2 at p, given by:

$$J_p f_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2a & 2b \end{pmatrix}$$

It is apparent from the expression of $J_p f_2$ that this matrix has rank 2 for every value of a and b and hence f_2 is a immersion.

In order to find the intersection point of the images of f_1 and f_2 let's write the equation of the images of f_1 and f_2 :

$$f_1(\mathbb{C}^2) = \{(X^1, X^2, X^3,) \mid X^3 = 0\}$$
$$f_2(\mathbb{C}^2) = \{(X^1, X^2, X^3) \mid X^3 = (X^1)^2 + (X^2)^2\}$$

Solving the two system of equations

$$X^3 = 0, X^3 = (X^1)^2 + (X^2)^2$$

over $\mathbb{C}$ we get

$$f_1 \cap f_2 = \{(X^1, X^2, 0) \mid X^2 = \pm i X^1\}$$

which we would abbreviate as: $\{(a, ia, 0)\}$where it is understood that $a \in \mathbb{C}$ and $i = \sqrt{(-1)}$.

The tangent at any point of the image of f_1 is the same as the image and hence given by the set of points in $\mathbb{C}^3$ specified by the equation $X^3 = 0$. To find the tangent space at any point of the image of f_2 let's write the equation of this as $F(X^1, X^2, X^3) = 0$ where $F \equiv (X^1)^2 + (X^2)^2 - X^3$, so that image of f_2 is now given by the zero set of the function F. For the moment, let's denote $F^{-1}(0)$ (or the image of f_2) by N , then, for $q \in \mathbb{C}^3$ the tangent vector $X_q \in T_q N$ will be given by:

$$X_q = v^1 \frac{\partial}{\partial X^1} + v^2 \frac{\partial}{\partial X^2} + v^3 \frac{\partial}{\partial X^3}$$

which we will write as $\langle v^1, v^2, v^3 \rangle$ under the isomorphism $T_q N \cong \mathbb{C}^3$. Now, assume that $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is a curve in $N = F^{-1}(0)$ such that $\gamma(0) = q$ and $\gamma'(0) = X_q$ then:

$$\frac{d}{dt}(F(\gamma(t))) = \frac{\partial F}{\partial X^1}(\gamma(t))(\gamma_1'(t)) + \frac{\partial F}{\partial X^2}(\gamma(t))(\gamma_2'(t)) + \frac{\partial F}{\partial X^3}(\gamma(t))(\gamma_3'(t)) = 0$$

Since, $X_q \in T_q N$ we can write the above expression at $t = 0$ as

$$\frac{\partial F}{\partial X^1}(q)(X^1 - q^1) + \frac{\partial F}{\partial X^2}(q)(X^2 - q^2) + \frac{\partial F}{\partial X^3}(q)(X^3 - q^3) = 0$$

where $q = (q^1, q^2, q^3)$ For $q \in f_1 \cap f_2$ we have $q^2 = iq^1, q^3 = 0$, the equation of tangent space at q is (setting $q^1 = a$)

$$2a(X^1 - a) + 2ia(X^2 - ia) - 1(X^3 - 0) = 0$$

For $a = 0$ this gives $X^3 = 0$ as the equation of the tangent plane, which is same as the equation of the tangent plane at $(0, 0, 0)$ to the image of f_1, and, thus f_1 and f_2 doesn't meet in general position at $(0, 0, 0)$. For any other point of intersection $a \neq 0$ and hence the two loci meet in general position.

Now, $\theta f_{2,2} = \theta f_1 \oplus \theta f_2$ which is generated by the elements of the type $(x^1)^m(x^2)^n e_{i,j}$; for brevity, we will write $x^m y^n e_{i,j}$ as $e^{m,n}_{i,j}$. Therefore, the set of generators for $\theta f_{2,2}$ will be

$$\{e^{m,n}_{1,1}, e^{m,n}_{1,2}, e^{m,n}_{2,1}, e^{m,n}_{2,2}, e^{m,n}_{3,1}, e^{m,n}_{3,2}\}$$

Now, for $\xi \in \theta_{\mathbb{C}^2,S}$ such that $\xi = \xi_1 \oplus \xi_2$ where $\xi_i = (a_i(x^1,x^2), b_i(x^1,x^2))$

$$tf(\xi) = \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a_1(x^1,x^2) \\ b_1(x^1,x^2) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x^1 & 2x^2 \end{pmatrix} \begin{pmatrix} a_2(x^1,x^2) \\ b_2(x^1,x^2) \end{pmatrix}\right]$$

which gives:

$$tf(\xi) = \begin{bmatrix} a_1(x^1,x^2) & a_2(x^1,x^2) \\ b_1(x^1,x^2) & b_2(x^1,x^2) \\ 0 & 2xa_2(x^1,x^2) + 2yb_2(x^1,x^2) \end{bmatrix}$$

which has a generating set of the form

$$\{e^{m,n}_{1,1}, e^{m,n}_{2,1}, e^{1,2m,n} + e^{m+1,n}_{3,2}, e^{m,n}_{2,2} + e^{m,n+1}_{3,2}\}$$

and for $\eta = (\eta_1(X^1,X^2,X^3), \eta_2(X^1,X^2,X^3), \eta_3(X^1,X^2,X^3)) \in \theta_{\mathbb{C}^3,0}$ we have

$$\omega(f)(\eta) = \begin{bmatrix} \eta_1(x^1,x^2,0) & \eta_1(x^1,x^2,X^3) \\ \eta_2(x^1,x^2,0) & \eta_2(x^1,x^2,X^3) \\ \eta_3(x^1,x^2,0) & \eta_3(x^1,x^2,X^3) \end{bmatrix}$$

where $X^3 = (x^1)^2 + (x^2)^2$, which has the generating set borrowed from the matrix before it's composed with f:

$$\{e^{m,n}_{1,1} + e^{m,n}_{1,2}, e^{m,n}_{2,1} + e^{m,n}_{2,2}, e^{m,n}_{3,1} + e^{m,n}_{3,2}\}$$

Now, in the span of tf and ωf which is $T\mathcal{A}_e f$, we already have $e^{m,n}_{1,1}$ and $e^{m,n}_{2,1}$ from tf, using ωf we get $e^{m,n}_{1,2}$ and $e^{m,n}_{2,2}$ using which with tf we get $e^{m+1,n}_{3,2}$ that is at (3,2) we have everything but monomial of the form $x^0 y^k$. Since, $e^{m,n}_{3,1} + e^{m,n}_{3,2}$ is in the generating set of ωf we get $e^{m+1,n}_{3,1}$ and similarly we get $e^{m,n+1}_{3,1}$ which means that at (3,1) we get everything but constant terms, and the same is true for (3,2). Therefore it seems that the elements:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix} \;\&\; \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

forms a basis for the space $\theta(f)f/T\mathcal{A}_e f$ but they are not linearly independent since their sum is the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

which is in $T\mathcal{A}_e f$ if we choose $\eta_3(X^1,X^2,X^3) = 1$ and therefore, we have, instead:

$$T\mathcal{A}_e f + \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} = T^1 f$$

Hence, a miniversal deformation for $f_{2,2}$ can be given, using Theorem 2. 18, as follows:

$$F_{1,2}(x^1,x^2,t) = \begin{cases} f_1 : (x^1,x^2) & \mapsto \quad (x^1,x^2,0) \\ f_2 : (x^1,x^2) & \mapsto \quad (x^1,x^2,(x^1)^2 + (x^2)^2 + t) \end{cases}$$

which is again translates of the surface given by the equation $X^3 = (X^1)^2 + (X^2)^2$.

4.2 $n = 2, k = 3$

- A parametrisation of $f_{2,3}$ can be given as:

$$\begin{cases} f_1 : (x^1, x^2) & \longmapsto & (x^1, x^2, 0) \\ f_2 : (x^1, x^2) & \longmapsto & (x^1, 0, x^2) \\ f_3 : (x^1, x^2) & \longmapsto & (x^1, x^2 - (x^1)^2, -x^2 - (x^1)^2) \end{cases}$$

Another parametrisation of the same is:

$$\begin{cases} g_1 : (x^1, x^2) & \longmapsto & (x^1, x^2, -x^2) \\ g_2 : (x^1, x^2) & \longmapsto & (x^1, x^2, x^2) \\ g_3 : (x^1, x^2) & \longmapsto & (x^1, x^2, (x^1)^2) \end{cases}$$

Denote the former by f and the latter by g, then these two parametrisation are indeed right-left equivalent: set a multigerm of diffeomorphism $\phi^{-1} : (\mathbb{C}^2, S) \to (\mathbb{C}^2, S)$

$$\begin{cases} (x^1, x^2) & \mapsto & (x^1, \frac{1}{2}x^2) \\ (x^1, x^2) & \mapsto & (x^1, -\frac{1}{2}x^2) \\ (x^1, x^2) & \mapsto & (x^1, x^2) \end{cases}$$

and define the germ of diffeomorphism $\psi : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ as:

$$\psi : (X^1, X^2, X^3) \longmapsto (X^1, X^2 - X^3, -X^2 - X^3)$$

Action of ψ on the image of g is:

$$\begin{array}{rcl} (X^1, X^2, -X^2) & \mapsto & (X^1, 2X^2, 0) \\ (X^1, X^2, X^2) & \mapsto & (X^1, 0, -2X^2) \\ (X^1, X^2, (X^1)^2) & \mapsto & (X^1, X^2 - (X^1)^2, -X^2 - (X^1)^2) \end{array}$$

thus we eventually have;

$$\psi \circ g \circ \phi = f$$

Given two map germs $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ and $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is there any way, other than the direct computation (since, it can be ardently tiresome) to decide that these two are left-right equivalent?

Or alternatively we could ask for a sufficient condition which would determine that the two germs are not right-left (or any other type) equivalent. For instance, if the rank of f is not same as the rank of g then the two germs can never be $\mathcal{A}$ (or $\mathcal{R}$ or $\mathcal{L}$) equivalent. Since, at any point $p \in \mathbb{C}^n$ and with the germ of diffeomorphisms $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ and $\psi : (\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ such that

$$g(p) = \psi \circ f \circ \phi^{-1}(p)$$

we also have, by the chain rule,

$$d_p g = d_{g(\phi^{-1}(p))}\psi \circ d_{\phi^{-1}(p)} f \circ d_p \phi^{-1}$$

and since ϕ and ψ are diffeomorphisms their differentials are isomorphisms which would have no effect on the rank of f i. e. f and g have the same rank. **Is the converse true?** Consider $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0), x \mapsto x^2$ and $g : (\mathbb{C}, 0) \to (\mathbb{C}, 0), x \mapsto x^3$ then they both have the rank zero at $x = 0$ and one at every other point, i.e. they have the same rank at every point $x \in \mathbb{C}$ **Are**

f **and** g **right-left equivalent?** We will answer this question, together with related theories, in a later section, for now, let me demonstrate a special case of this problem by showing that the two parametrisation for $f_{2,3}$ above is not right equivalent via a linear transformation.

Suppose, if possible, there exist a linear map

$$\psi : \mathbb{C}^3 \to \mathbb{C}^3 \text{ taking } X \longmapsto AX$$

where A is 3×3 matrix, such that $g = \psi \circ f$. Then we must have:

$$g_1 = \psi \circ f_1, g_2 = \psi \circ f_2, g_3 = \psi \circ f_3$$

which means

$$dg_1 = d\psi df_1, dg_2 = d\psi df_2, dg_3 = d\psi df_3$$

but ψ is linear, therefore, $d\psi = A$, where A is as above. Hence,

$$dg_1 = Adf_1, dg_2 = Adf_2, dg_3 = Adf_3$$

$dg_1 = Adf_1$ implies:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

which gives

$$\begin{array}{lll} a_{1,1} = 1 & a_{2,1} = 0 & a_{3,1} = 0 \\ a_{1,2} = 0 & a_{2,2} = 1 & a_{3,2} = -1 \end{array}$$

Using these values in $A, dg_2 = Adf_2$ we get

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_{1,3} \\ 0 & 1 & a_{2,3} \\ 0 & -1 & a_{3,3} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

which gives

$$a_{1,3} = 0 \quad a_{2,3} = 1 \quad a_{3,3} = 1$$

So, by now, i. e. after solving these set of simple equations, we get

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

But

$$Adf_3 : \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2x^1 & 1 \\ -2x^1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -4x^1 & 0 \\ 0 & -2 \end{pmatrix} \neq dg_3$$

for any point in the domain, since

$$dg_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2x & 0 \end{pmatrix}$$

Therefore we saw that this is not possible to achieve the second parametrisation for $f_{2,3}$ from the parametrisation only using the diffeomorphism given by the linear map in the target. This's true that in general, derivatives will always be given by the form $dg = AdfB$ where A is the Jacobian of the diffeomorphism in the target and B in the source, but the system of equation forces us to solve a large number of equations, which has to be checked at everypoint in order to find a point where this system comes out as an inconsistent system; another possibility is to find out the condition under which the system is inconsistent, but again this would require an enormous amount of labour.

Using the first parametrisation, we see that the image of the first germ f_1 is the set of points of $\mathbb{C}^3$ given by the equation $X^3 = 0$ and that of f_2 is $X^2 = 0$ which are embeddings of $\mathbb{C}^2$ in $\mathbb{C}^3$. Moreover, these are linear subspaces of $\mathbb{C}^3$ and hence the tangent space at any point will coincide with the space itself. The image set of f_3 is given by the equation $X^3 + X^2 + (X^1)^2$, at any point $p = (p^1, p^2) \in \mathbb{C}^2$ we have:

$$d_p f_3([v^1, v^2]^T) = [J_p f_3]\,\big|_p [v^1, v^2]^T$$

where the superscript t stands for transpose. But

$$[J_p f_3]\,\Bigg|_p = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2p^1 & 0 \end{bmatrix}$$

which clearly has rank 2 no matter what the point p is. Hence, f_3 is also an immersion. Image of f_1 and f_2 intersect along the set $\{(X^1, X^2, X^3) \mid X^3 = 0 = X^2\}$ which is the set of points $\{(X^1, 0, 0)\}$ along which the two curves meet in general position. Let's denote the image of f_3 by the equation

$$F = X^3 + X^2 + (X^1)^2$$

the by proposition 2.6 the equation of tangent at $p = (p^1, p^2, p^3)$ is

$$2p^1(X^1 - p^1) + (X^2 - p^2) + (X^3 - p^3) = 0$$

Now

$$f_1 \cap f_3 = \{(\pm i\sqrt{X^2}, X^2, 0)\} \ \& \ f_2 \cap f_3 = \{(\pm i\sqrt{X^3}, 0, X^3)\}$$

The equation of tangents at these points are, respectively

$$\pm 2i\sqrt{p^2}(X^1 \mp i\sqrt{p^2}) + (X^2 - p^2) + X^3 = 0$$

and

$$\pm 2i\sqrt{p^3}(X^1 \mp i\sqrt{p^3}) + X^2 + (X^3 - p^3) = 0$$

which, respectively, meet the curves $X^3 = 0$ and $X^2 = 0$ transversally. $O = (0, 0, 0)$ is the only point in $f_1 \cap f_2 \cap f_3$. At O the equation of tangent to f_3 is $X^2 + X^3 = 0$. Let

$$\begin{aligned} E_1 &= \{(X^1, X^2, 0)\} \\ E_2 &= \{(X^1, 0, X^3)\} \\ E_3 &= \{(X^1, X^2, -X^2)\} \end{aligned}$$

Please note that E_1, E_2 and E_3 are the set specified by the tangent spaces (at O) of the image set of germs f_1, f_2 , and f_3 respectively. E_1 and E_2 are spanned by $\{e_1, e_2\}$ and $\{e_1, e_3\}$ respectively, and therefore have codimensions 1. $(X^1, X^2, -X^2) \in E_3$ can be written as

$$(X^1, X^2, -X^2) = X^1(1, 0, 0) + X^2(0, 1, -1)$$

i. e. a basis for E_3 is $\{e_1, e_2 - e_3\}$, and, thus, E_3 too has codimension 1. Now

$$E_1 \cap E_2 \cap E_3 = \{(X^1, 0, 0)\}$$

which has codimension 2, i. e.

$$\text{codim } E_1 \cap E_2 \cap E_3 \neq \text{codim } E_1 + \text{codim } E_2 + \text{codim } E_3$$

Hence, at O the three germs don't meet in general position. Let's use the first parametrisation to compute $\mathcal{A}_e$ codimension, since we have already proved that they are right-left equivalent the outcome would be same. So, we have:

$$\begin{cases} f_1 : (x^1, x^2) & \longmapsto & (x^1, x^2, 0) \\ f_2 : (x^1, x^2) & \longmapsto & (x^1, 0, x^2) \\ f_3 : (x^1, x^2) & \longmapsto & (x^1, x^2 - (x^1)^2, -x^2 - (x^1)^2) \end{cases}$$

$\theta(f) = \theta(f_1) \oplus \theta(f_2) \oplus \theta(f_3)$ is generated (since the $\mathcal{A}_e$ co-dimension is finite) by the set $\{e^{k,l}_{i,j} \mid k, l \in \mathbb{N} \& 1 \leq i, j \leq 3\}$ If we get $e^{k,l}_{i,j}$ $\forall\, k, l \in \mathbb{N}$ we will sometimes denote it by $\mathcal{O}_{i,j}$.

For $\xi = \xi_1 \oplus \xi_2 \oplus \xi_3 \in T\mathbb{C}^2$ where $\xi_i = (a_i(x), b_i(x))^T$ we have $tf(\xi)$, where $x = (x^1, x^2)$

$$df_{2,3} \circ \xi = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a_1(x) \\ b_1(x) \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_2(x) \\ b_2(x) \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ -2x^1 & 1 \\ -2x & -1 \end{bmatrix} \begin{bmatrix} a_3(x) \\ b_3(x) \end{bmatrix} \right)$$

after simplifying we get

$$df_{2,3} \circ \xi = \begin{pmatrix} a_1(x) & a_2(x) & a_3(x) \\ b_1(x) & 0 & -2x^1 a_3(x) + b_3(x) \\ 0 & b_2(x) & -2x^1 a_3(x) - b_3(x) \end{pmatrix}$$

So, an arbitrary element of $tf(\xi)$ can be written as:

$$\begin{pmatrix} a_1(x) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_2(x) & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ b_1(x) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_2(x) & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & a_3(x) \\ 0 & 0 & -2x^1 a_3(x) \\ 0 & 0 & -2x^1 a_3(x) \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & b_3(x) \\ 0 & 0 & -b_3(x) \end{pmatrix}$$

Therefore it has the generating set of the form

$$T = \{e^{k,l}_{1,1}, e^{k,l}_{1,2}, e^{k,l}_{2,1}, e^{k,l}_{3,2}, e^{k,l}_{1,3} - 2e^{k+1,l}_{2,3} - 2e^{k+1,l}_{3,3}, e^{k,l}_{2,3} - e^{k,l}_{3,3}\}$$

Please note that we already have got $\mathcal{O}_{1,1}, \mathcal{O}_{1,2}, \mathcal{O}_{2,1}$ and $\mathcal{O}_{3,2}$ from tf.

For $\eta = (\eta_1(X), \eta_2(X), \eta_3(X))^T \in \theta_{\mathbb{C}^3,0}$ we get

$$\omega f(\eta) = \eta \circ f_{2,3} = \begin{pmatrix} \eta_1(x^1, x^2, 0) & \eta_1(x^1, 0, x^2) & \eta_1(x^1, x^2 - (x^1)^2, -x^2 - (x^1)^2) \\ \eta_2(x^1, x^2, 0) & \eta_2(x^1, 0, x^2) & \eta_2(x^1, x^2 - (x^1)^2, -x^2 - (x^1)^2) \\ \eta_3(x^1, x^2, 0) & \eta_3(x^1, 0, x^2) & \eta_3(x^1, x^2 - (x^1)^2, -x^2 - (x^1)^2) \end{pmatrix}$$

If we choose $\eta(X)$ such that $\eta_1(X) = (X^1)^k$ i. e. independent of X^2 and X^3 and $\eta_2(X) = \eta_3(X) = 0$ then we get

$$\omega f(\eta) \begin{pmatrix} (x^1)^k & (x^1)^k & (x^1)^k \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

but because we have
$$\begin{pmatrix}(x^1)^k & (x^1)^k & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$
from tf we get every power of x^1 at (1,3) i. e. $(x^1)^k e_{1,3}$. Now, let's choose $\eta_1(X) = (X^1)^k X^2$, and $\eta_2(X) = \eta_3(X) = 0$ which would give

$$\omega f(\eta) = \begin{pmatrix}(x^1)^k x^2 & 0 & (x^1)^k x^2 - (x^1)^{k+2}\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

but since we have got $\mathcal{O}_{1,1}$ from tf and $(x^1)^k e_{1,3}$ from the argument made in the last sentence, we get $(x^1)^k x^2 e_{1,3}$. Now assume that we have got $(x^1)^k (x^2)^l e_{1,3}$ for every $l_1 < l$, then after expanding using the binomial theorem and using the deduction from the last sentence, we get $e_{i,3}^{k,l}$ for every l by induction. Therefore, we have got $\mathcal{O}_{1,3}$. Now, using this with $e_{1,3}^{k,l} - 2e_{2,3}^{k+1,l} - 2e_{3,3}^{k+1,l}$ we get $-2e_{2,3}^{k+1,l} - 2e_{3,3}^{k+1,l}$ or (after absorbing the minus sign and 2) $e_{2,3}^{k+1,l} + e_{3,3}^{k+1,l}$ which together with $e_{2,3}^{k,l} - e_{3,3}^{k,l}$ would give $e_{2,3}^{k+1,l}$ and $e_{3,3}^{k+1,l}$ That means, we have got $\mathcal{O}_{2,3} - \{1, (x^2), (x^2)^2, \dots\}$ and $\mathcal{O}_{3,3} - \{1, (x^2), (x^2)^2, \dots\}$ Now using the vector field of the type $\eta_1(X) = \eta_3(X) = 0$ and $\eta_2(X) = (X^2)^l$ and using the fact that we have $\mathcal{O}_{2,1}$ and $\mathcal{O}_{2,3} - \{1, (x^2), (x^2)^2, \dots\}$ and binomial expansion, we get $\mathcal{O}_{2,3} - 1$, using which with $\mathcal{O}_{2,1}$ we get $\mathcal{O}_{2,2} - 1$ as well. Similarly, using the vector field of the type $\eta_1(X) = \eta_2(X) = 0$ and $\eta_3(X) = (X^3)^l$ and with similar argument we get $\mathcal{O}_{3,3} - 1$ and $\mathcal{O}_{3,1} - 1$. So, by now, we have got

$$\begin{pmatrix}\mathcal{O}_{1,1} & \mathcal{O}_{1,2} & \mathcal{O}_{1,3}\\ \mathcal{O}_{2,1} & \mathcal{O}_{2,2} - 1 & \mathcal{O}_{2,3} - 1\\ \mathcal{O}_{3,1} - 1 & \mathcal{O}_{3,2} & \mathcal{O}_{3,3} - 1\end{pmatrix}$$

Therefore,

$$T\mathcal{A}_e f_{2,3} + \left\{ \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} \right\} = T^1 f_{2,3}$$

But
$$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}$$
are not linearly independent, since, if we choose a vector field of the type $\eta_2(X) = 1, \eta_1(X) = \eta_3(X) = 0$ and using the fact that we have $\mathcal{O}_{2,1}$ from tf we have

$$\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix} + \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} = \begin{pmatrix}0&0&0\\0&1&1\\0&0&0\end{pmatrix} \in T\mathcal{A}_e f_{2,3}$$

Similarly using the vector field of the type $\eta_1(X) = \eta_2(X) = 0, \eta_3(X) = 1$ ans using the fact that we have $\mathcal{O}_{3,2}$ from tf we get

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix} + \begin{pmatrix}0&0&0\\0&0&0\\1&0&0\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&0\\1&0&1\end{pmatrix} \in T\mathcal{A}_e f_{2,3}$$

But, then, since we have $e^{k,l}_{2,3} - e^{k,l}_{3,3}$ among the generators of tf we see that

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \in T\mathcal{A}_e f_{2,3}$$

Hence, we get

$$T\mathcal{A}_e f_{2,3} + \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} = T^1 f_{2,3}$$

proving that $f_{2,3}$ has $\mathcal{A}_e$ co-dimension 1.

In the light of theorem 2.19, a miniversal deformation of $f_{2,3}$ can be given by

$$F_{2,3}(x^1, x^2, t) = \left\{ \begin{array}{lcl} f_1 : (x^1, x^2) & \longmapsto & (x^1, x^2, 0) \\ f_2 : (x^1, x^2) & \longmapsto & (x^1, 0, x^2) \\ f_3 : (x^1, x^2) & \longmapsto & (x^1, x^2 - (x^1)^2, -x^2 - (x^1)^2 + t) \end{array} \right.$$

which is again a family of translates of the original curve, since $X^3 + X^2 = -(X^1)^2 + t$

5 A Step Towards Generalisation

5.1 n = 3, k =2

$f_{3,2}$ can be parametrised as:

$$\left\{ \begin{array}{lcl} f_1 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3, 0) \\ f_2 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3, (x^1)^2 + (x^2)^2 + (x^3)^2) \end{array} \right.$$

Being embedding of $\mathbb{C}^3$ into $\mathbb{C}^4$ f_1 is clearly an immersion. And to see that f_2 is an immersion, we write down it's Jacobian:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x^1 & 2x^2 & 2x^3 \end{pmatrix}$$

which has rank 3 and therefore it's an immerssion.

The multigerm represent the following reducible curve in $\mathbb{C}^4$

$$\{(X^1, X^2, X^3, X^4) \in \mathbb{C}^4 \mid X^4(X^4 - (X^1)^2 - (X^2)^2 - (X^3)^2) = 0\}$$

So, the image of f_2 is given by the equation

$$F := X^4 - (X^1)^2 - (X^2)^2 - (X^3)^2 = 0$$

For a point $p = (p^1, p^2, p^3, p^4) \in F^{-1}(0)$ the equation of tangent at p is given by:

$$-2p^1(X^1 - p^1) - 2p^2(X^2 - p^2) - 2p^3(X^3 - p^3) + (X^4 - p^4) = 0$$

which is transverse to the set given by $X^4 = 0$ for every point $p \neq (0,0,0,0)$. At $(0,0,0,0)$ it fails to meet the the curve $X^4 = 0$ in general position, as required.

$$\begin{tikzcd} T\mathbb{C}^3 \arrow[r, "df_{3,2}"] & T\mathbb{C}^4 \\ \mathbb{C}^3 \arrow[u, "\xi"] \arrow[r, "f"', "3{,}2"] & \mathbb{C}^4 \arrow[u, "\eta"'] \end{tikzcd}$$

where $\xi = (\xi_1, \xi_2) \in \theta_{\mathbb{C}^3,0} \oplus \theta_{\mathbb{C}^3,0}$. We assume the usual generating set for $\theta(f) : \{e_{i,j}^{m,n,p}\}$. For

$$\xi = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}$$

where each a_i, b_i, c_i are power series in x^1, x^2, x^3, we get:

$$tf(\xi) = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \\ 0 & 2x^1a_2 + 2x^2b_2 + 2x^3c_2 \end{pmatrix}$$

and for $\eta(X^1, X^2, X^3, X^4) = (\eta_1, \eta_2, \eta_3, \eta_4)^T$ we get

$$\omega f(\eta) = \begin{pmatrix} \eta_1(x^1,x^2,x^3,0) & \eta_1(x^1,x^2,x^3,X^4) \\ \eta_2(x^1,x^2,x^3,0) & \eta_2(x^1,x^2,x^3,X^4) \\ \eta_3(x^1,x^2,x^3,0) & \eta_3(x^1,x^2,x^3,X^4) \\ \eta_4(x^1,x^2,x^3,0) & \eta_4(x^1,x^2,x^3,X^4) \end{pmatrix}$$

For such a subspace a general generating set i. e. one considering it as a function of co-ordinates in the target is

$$\{e_{1,1}^{m,n,p,q} + e_{1,2}^{m,n,p,q}, e_{2,1}^{m,n,p,q} + e_{2,2}^{m,n,p,q}, e_{3,1}^{m,n,p,q} + e_{3,2}^{m,n,p,q}, e_{4,1}^{m,n,p,q} + e_{4,2}^{m,n,p,q}\}$$

.

From tf we already have $e_{1,1}^{m,n,p}, e_{2,1}^{m,n,p}, e^{3,1m,n,p}$. Choosing a vector field of the type $\eta_1 = \eta_1(X^1, X^2, X^3), \eta_2 = \eta_3 = \eta_4 = 0$ and using the result stated in the last sentence, we also get $e_{1,2}^{m,n,p}, e_{2,2}^{m,n,p}, e_{3,2}^{m,n,p}$ which, using the fact that $e_{1,2}^{m,n,p} + e_{4,2}^{m+1,n,p}$ is among the generators of tf we have $e_{4,2}^{m+1,n,p}$ and similarly we get $e_{4,2}^{m,n+1,p}$ and $e_{4,2}^{m,n,p+1}$ as well. Therefore, we get everything but constants at (4,2), and the same for (4,1) after using the contribution from ωf. So, the missing elements are : $e_{4,1}$ and $e_{4,2}$, but we have $e_{4,1} + e_{4,2}$ after setting $\eta_1 = \eta_2 = \eta_3 = 0, \eta_4 = 1$. Therefore:

$$T\mathcal{A}_e f_{3,2} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} = T^1 f_{3,2}$$

Therefore, a miniversal deformation of $f_{3,2}$ can be given by

$$F_{3,2}(x^1, x^2, x^3, t) = \left\{ \begin{array}{lll} f_{1,t} : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3, 0) \\ f_{2,t} = (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3, (x^1)^2 + (x^2)^2 + (x^3)^2 + t) \end{array} \right.$$

5.2 $n = 3, k = 3$

A parametrisation for $f_{3,3}$ can be given as:

$$\begin{cases} f_1 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3, -x^3) \\ f_2 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3, x^3) \\ f_3 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3, (x^1)^2 + (x^2)^2) \end{cases}$$

Let $p = (p^1, p^2, p^3) \in \mathbb{C}^3$ be an arbitrary point. $d_p f_1$ is given by the matrix (jacobian)

$$J_p f_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

which has rank 3 irrespective of the value of p, and therefore is an immersion. Similarly, we have

$$J_p f_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

of rank 3 irrespective of what p is, and hence f_2 is an immersion. For f_3 we get

$$J_p f_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2x^1 & 2x^2 & 0 \end{pmatrix}$$

the upper 3×3 matrix is a constant matrix of rank 3, and hence f_3 is also an immersion.

The multigerm $f_{3,3}$ is the parametric representation of $\{(X^1, X^2, X^3, X^4) \mid (X^4 + X^3)(X^4 - X^3)(X^4 - (X^1)^2 - (X^2)^2) = 0\} \subset \mathbb{C}^4$ which means the image of the first branch f_1 is given by:

$$\{(X^1, X^2, X^3, X^4) \mid (X^4 + X^3) = 0\}$$

which is a linear subspace, and therefore, the equation of the tangent space at any point $p = (p^1, p^2, p^3, p^4)$ will be $X^4 + X^3 - p^4 - p^3 = 0$. Same is true for the set

$$\{(X^1, X^2, X^3, X^4) \mid (X^4 - X^3) = 0\}$$

which is the image of the second branch f_2 , having the equation of the tangent space at p as $X^4 - X^3 - p^4 + p^3 = 0$ The set $\{(X^1, X^2, X^3, X^4) \mid (X^4 - (X^1)^2 - (X^2)^2) = 0\}$ can be written as $F^(- 1)(0)$ where

$$F = X^4 - (X^1)^2 - (X^2)^2$$

and therefore by proposition-2.6 the equation of tangent space at any point $p = (p^1, p^2, p^3, p^4)$ is given by

$$-2p^1(X^1 - p^1) - 2p^2(X^2 - p^2) + (X^4 - p^4) = 0$$

Now,

$$\cap_{i=1}^3 f_i = \{(X^1, X^2, X^3, X^4) \mid X^3 = X^4 = 0 \& X^2 = \pm X^1\}$$

which we will abbreviate as $\{X^1, \pm X^1, 0, 0)\}$ then for $p = (a, \pm ia, 0, 0) \in \cap_{i=1}^{3} f_i$ we have the equation of tangent space to f_3 at p is

$$-2a(X^1 - a) \mp 2a(X^2 \mp ia) + X^4 = 0$$

that to f_1 is:

$$X^4 + X^3 = 0$$

and to f_2 is

$$X^4 - X^3 = 0$$

Let

$$E_1 = \{(X^1, X^2, X^3, X^4) \mid X^4 + X^3 = 0\}$$
$$E_2 = \{(X^1, X^2, X^3, X^4) \mid X^4 - X^3 = 0\}$$
$$E_3 = \{(X^1, X^2, X^3, X^4) \mid -2a(X^1 - a) \mp 2a(X^2 \mp ia) + X^4 = 0\}$$

then E_1, E_2, E_3 all have codimension 1 since they are restricted by one linear equation (so that the value of one can be substituted in terms of others.) For $a \neq 0$ (which is same thing as $p \neq (0, 0, 0, 0)$) we get

$$E_1 \cap E_2 \cap E_3 = \{(X^1, X^2, X^3, X^4) \mid X^3 = X^4 = 0 \& X^1 + X^2 = a\}$$

which has codimension 3, and hence, the germs meet in general position for $p \in \cap f_i, p \neq (0, 0, 0, 0)$. For $p = (0, 0, 0, 0)$ we get

$$E_3 = \{(X^1, X^2, X^3, X^4) \mid X^4 = 0\}$$

which means

$$E_3 \subset E_1 \cap E_2 = \{(X^1, X^2, X^3, X^4) \mid X^3 = X^4 = 0\}$$

which has codimension 2 instead of 3, and hence the germs don't meet in general position.

In order to demonstrate the $\mathcal{A}_e$ - co-dimension of $f_{3,3}$ we will use the another right-left equivalent parametrisation:

$$\begin{cases} f_1 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3, 0) \\ f_2 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, 0, x^3) \\ f_3 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3 - (x^1)^2 - (x^2)^2, -x^3 - (x^1)^2 - (x^2)^2) \end{cases}$$

Using this parametrisation, it will follow from a result in the next following subsection that $f_{3,3}$ indeed have $\mathcal{A}_e$ co-dimension 1.

5.3 $n = 3, k = 4$

A parametrisation for $f_{3,4}$ can be given by:

$$\begin{cases} f_1 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3, 0) \\ f_2 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, 0, x^3) \\ f_3 : (x^1, x^2, x^3) & \longmapsto & (x^1, 0, x^2, x^3) \\ f_4 : (x^1, x^2, x^3) & \longmapsto & (x^1, x^2, x^3 - x^2, -x^3 - (x^1)^2) \end{cases}$$

Being embeddings of $\mathbb{C}^3$ into $\mathbb{C}^4$ f_1, f_2 & f_3 are clearly immersions; at any point $p = (p^1, p^2, p^3) \in \mathbb{C}^3$, we have:

$$J_p f_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \\ -2p^1 & 0 & -1 \end{pmatrix}$$

which have rank 3 irrespective of the value of p^1 (hence p) and, for this reason, f_4 is an immersion as well.

The image of this multigerms is the set

$$\{(X^1, X^2, X^3, X^4) \mid X^4 X^3 X^2 (X^4 + X^3 + X^2 + (X^1)^2) = 0\}$$

where $X^4 = 0$ is the equation of the image of f_1, $X^3 = 0$ is the equation of the image of f_2, $X^2 = 0$ is the equation of the image of f_3, and the image of f_4 is given by the equation

$$X^4 + X^3 + X^2 + (X^1)^2 = 0$$

Note that $(0, 0, 0, 0)$ is the only point in $\cap_{i=1}^4 f_i$

At any point $P = (P^1, P^2, P^3, P^4) \in \mathbb{C}^4$ the tangent space to the image of $f_1, f_2,$ & f_3 is given by (Proposition 2.6) $X^4 - P^4 = 0, X^3 - P^3 = 0$ and $X^2 - P^2 = 0$ respectively; the tangent space to the image of f_4 is given by

$$2P^1(X^1 - P^1) + (X^2 - P^2) + (X^3 - P^3) + (X^4 - P^4)$$

In the light of Theorem 2.7, the normals at $P = (P^1, P^2, P^3, P^4)$ are $(0, 0, 0, 1), (0, 0, 1, 0), (0, 1, 0, 0)$ and $(2P^1, 1, 1, 1)$. Consider the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -2P^1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

each of which column is formed by the normals mentioned above. This matrix has rank 3 when P^1 equals zero, i.e. at $(0, 0, 0, 0)$ the curves do not meet in general position since the normals are not linearly independent. However, if we take any three of them then they are clearly linearly independent since the matrices:

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -2P^1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & -2P^1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & -2P^1 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

has rank 3 in each case, no matter what the value of P is.

5.4 for arbitrary n and k

Let, $f_{n,k} : \mathbb{C}^n \to \mathbb{C}^{n+1}$ be multigerms of co-dimension 1 which satisfied all the properties mentioned in Introduction; we would like to develop an algorithm to achieve such branches $f_1, \ldots, f_k$ with the help of Theorem 2.7. Let's start with the case when $k = 2$: we will choose f_1 as any embedding Let's choose a very simple one as other examples in the text:

$$f_1 : (x^1, \ldots, x^n) \longmapsto (x^1, \ldots, x^n, 0)$$

In order to find f_2 we will express the image of f_1 as the zero set and use Theorem 2.7.

The equation of the image of f_1 is $F_1 \equiv X^{n+1} = 0$, therefore, at any point $p = (p^1, \ldots, p^{n+1}) \in \mathbb{C}^{n+1}, \nabla_p F_1 = (0, \ldots, 0, 1)$ Now we want to choose F_2such that$F_2^{-1}(0) \subset \mathbb{C}^{n+1}$ is a regular submanifold and the vector $\nabla_p F_2 = (\frac{\partial F_2}{\partial x^1}(p), \ldots, \frac{\partial F_2}{\partial x^{n+1}}(p))$ is linearly independent of the vector $\nabla_p F_1 = (0, \ldots, 0, 1)$, or in other words the $(n+1) \times 2$ matrix $(\nabla_p F_1, \nabla_p F_2)$ has the rank 2 at every point of $F_1^{-1}(0) \cap F_2^{-1}(0)$ except $O = (0, \ldots, 0)$.

$$(\nabla_p F_1, \nabla_p F_2) = \begin{pmatrix} 0 & \frac{\partial F_2}{\partial x^1}(p) \\ \vdots & \vdots \\ 1 & \frac{\partial F_2}{\partial x^{n+1}(p)} \end{pmatrix}$$

A simple choice would be $\frac{\partial F_2}{\partial x^{n+1}}(p) = -1$ and $\frac{\partial F_2}{\partial x^i}(p) = 2p^i$ Solving this system of equation and using the fact that $O \in F_2^{-1}(0)$ we get:

$$F_2 = -X^{n+1} + (X^n)^2 + \cdots + (x^1)^2$$

Now we need to find a parametrisation f_2 of $F_2^{-1}(0)$ which is an immersion, and the multigerm $f_{n,2}$ thus obtained has $\mathcal{A}_e$ co-dimension 1.

Let M, N be C^∞ or analytic manifolds, then the image $f(M)$ of every embedding $f : M \to N$ is a regular submanifold of N , and conversely every regular submanifold of N is the image of some embedding, namely the inclusion. Although, this is not true that every regular submanifold admits a global parametrisation from $\mathbb{C}^n$ or $\mathbb{R}^n$ (we assume that the submanifold is locally euclidean of dimension n) which is an immersion? Sphere is such an example, which can be given as $f^{-1}(0)$ where $F : (x^1)^2 + \cdots + (x^n)^2 - 1$, but we are dealing with a neighaborhood of $O = (0, \ldots, 0)$ and by the definition of manifold we always have such a parametrisation which is given by a diffeomorphism from a neighaborhood U of O and hence, an immersion.

So, in the above case, a parametrisation for $F_2^{-1}(0)$ can be given by

$$f_2 : (x^1, \ldots, x^n) \longmapsto (x^1, \ldots, x^n, (x^1)^2 + \cdots + (x^n)^2)$$

which has Jacobian at any point $q \in \mathbb{C}^n$ as the matrix having rows containing 1 in i^{th} position, and therefore it's an immersion.

To see that $f_{n,2}$ thus defined as:

$$\begin{cases} f_1 : (x^1, \ldots, x^n) & \longmapsto & (x^1, \ldots, x^n, 0) \\ f_2 : (x^1, \ldots, x^n) & \longmapsto & (x^1, \ldots, x^n, (x^1)^2 + \cdots + (x^n)^2) \end{cases}$$

has $\mathcal{A}_e$ co-dimension 1, let's recall that $\theta(f)$ will be generated by elements of the type $e_{i,j}^{k_1,\ldots,k_n} = (x^1)^{k_1} \ldots (x^n)^{k_n} e_{i,j}$ where $k_i \in \mathbb{N}$. Let $\xi = (\xi_1, \xi_2) \in \theta_{n,\{0,0\}}$ and $\xi_i = (a_{1,i}(x^1, \ldots, x^n), \ldots, a_{n,i}(x^1, \ldots, x^n))^T$

be the vector field in the source of i^{th} branch of the multigerm, then $tf(\xi)$ is

$$\begin{pmatrix} a_{1,1}(x^1,\dots,x^n) & a_{1,2}(x^1,\dots,x^n) \\ \vdots & \vdots \\ a_{n,1}(x^1,\dots,x^n) & a_{n,2}(x^1,\dots,x^n) \\ 0 & 2x^1a_{1,2}+\dots+2x^na_{n,2} \end{pmatrix}$$

thus tf is generated by the set

$$\{e_{1,1}^{k_1,\dots,k_n},\dots,e_{n,1}^{k_1,\dots,k_n}\}\cup\{e_{1,2}^{k_1,\dots,k_n}+e_{n+1,2}^{k_1+1,k_2,\dots,k_n},\dots e_{n,2}^{k_1,\dots,k_n}+e_{n+1,2}^{k_1,\dots,k_n+1}\}$$

and for $\eta=(\eta_1(X^1,\dots,X^{n+1}),\dots,\eta_{n+1}(X^1,\dots,X^{n+1}))^T$ we have $\omega f(\eta)$

$$\begin{pmatrix} \eta_1(x^1,\dots,x^n,0) & \eta_1(x^1,\dots,x^n,X^{n+1}) \\ \vdots & \vdots \\ \eta_{n+1}(x^1,\dots,x^n,0) & \eta_{n+1}(x^1,\dots,x^n,X^{n+1}) \end{pmatrix}$$

which, with the generating set for tf gives every element of the generating set except at $(n+1,1)$ and at $(n+1,2)$ where using the dependence relation from tf we have everything except constants, and therefore, just like previous cases with two branches, a basis for $\frac{\theta(f)}{T\mathcal{A}_ef_{n,2}}$ is $e_{n+1,2}$ i. e.

$$T\mathcal{A}_ef_{n,2}+\begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 1 \end{pmatrix}=T^1f_{n,2}$$

In this case when $n=2$ there's a result due to David Mond [4] which guarantees that the converse is also true. [3, Example 2.0.19]

Conjecture 5.1. *If $g_{n,2}$ is a bigerms from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ which satisfies all the conditions above, with the first branch $g_1:(x^1,\dots,x^n)\longmapsto(x^1,\dots,x^n,0)$ same as the first branch of our f_1 then $g_{n,2}\sim_{\mathcal{A}}f_{n,2}$ that is they are left-right equivalent.*

Now, suppose $k=3$. We will choose $f_1:(x^1,\dots,x^n)\longmapsto(x^1,\dots,x^n,0)$ and $f_2:(x^1,\dots,x^n)\longmapsto(x^1,\dots,0,x^n)$ with equations of the image set $F_1:X^{n+1}=0$ and $F_2:X^n=0$ respectively. We want to choose $F_3:\mathbb{C}^{n+1}\to\mathbb{C}$ such that at any point $p=(p^1,\dots,p^{n+1})$ other than $(0,\dots,0)$ the vectors $\nabla_pF_3,(0,\dots,0,1)\text{and}(0,\dots,1,0)$ are linearly independent, i. e. the matrix

$$\begin{pmatrix} 0 & 0 & \frac{\partial F_3}{\partial x^1}(p) \\ \vdots & \ddots & \vdots \\ 0 & 1 & \frac{\partial F_3}{\partial x^n}(p) \\ 1 & 0 & \frac{\partial F_3}{\partial x^{n+1}}(p) \end{pmatrix}$$

has rank 3 when $p\neq O=(0,\dots,0)$ and fails to have rank 3 at O. Such a function can be given if we choose

$$\nabla_pF_3=(2p^1,2p^2,\dots,1,1)^T$$

for in that case

$$M=(\nabla_pF_1,\nabla_pF_2,\nabla_pF_3)=\begin{pmatrix} 0 & 0 & 2p^1 \\ \vdots & \ddots & \vdots \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Then we can see that M has the desired property, and, then

$$F_3 : (X^1)^2 + \cdots + (X^{n-1})^2 + X^n + X^{n+1}$$

and a parametrisation for F_3 can be given as:

$$f_3 : (x^1, \ldots, x^n) \longmapsto (x^1, \ldots, x^{n-1}, x^n - (x^1)^2 - \cdots - (x^{n-1})^2, -x^n - (x^1)^2 - \cdots - (x^{n-1})^2)$$

i. e. $X^i = x^i \ \forall\ 1 \leq i \leq (n-1)$ and $X^n = x^{n-1}, x^n - (x^1)^2 - \cdots - (x^{n-1})^2$ and $X^{n+1} = -x^{n-1}, x^n - (x^1)^2 - \cdots - (x^{n-1})^2$

f_3 is clearly an immersion, because for any point $p = (p^1, \ldots, p^n) \in \mathbb{C}^n$ we have:

$$J_p f_3 = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -2x^1 & -2x^2 & \cdots & -2x^{n-1} & 1 \\ -2x^1 & -2x^2 & \cdots & -2x^{n-1} & -1 \end{pmatrix}$$

which has rank n, at any point p. Now, let $e_{i,j}^{k_1,\ldots,k_n}$ stands for the monomial $(x^1)^{k_1} \ldots (x^n)^{k_n}$ at (i, j) position of $n \times 3$ matrix, then since the multigerm is of finite co-dimension $\theta(f_{n,3}) = \theta(f_1) \oplus \theta(f_2) \oplus \theta(f_3)$ has $\{e_{i,j}^{k_1,\ldots,k_n}\}$ as it's generating set.

For $\xi = \xi_1 \oplus \xi_2 \oplus \xi_3$ where $\xi_i = (a_{1,i}, \ldots, a_{n,i})^T$ we get

$$tf(\xi) = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ \vdots & \ddots & \vdots \\ a_{n-1,1} & a_{n-1,2} & a_{n-1,3} \\ a_{n,1} & 0 & -2\sum_{i=1}^{n-1} x^i a_{i,3} + a_{n,3} \\ 0 & a_{n,2} & -2\sum_{i=1}^{n-1} x^i a_{i,3} - a_{n,3} \end{pmatrix}$$

and for $\eta(X) = (\eta_1(X), \ldots, \eta_{n+1}(X))$ we get

$$\omega f(\eta) = \begin{pmatrix} \eta_1(x^1, \ldots, x^n, 0) & \eta_1(x^1, \ldots, 0, x^n) & \eta_1(x^1, \ldots, X^n, X^{n+1}) \\ \vdots & \ddots & \vdots \\ \eta_{n+1}(x^1, \ldots, x^n, 0) & \eta_{n+1}(x^1, \ldots, 0, x^n) & \eta_{n+1}(x^1, \ldots, X^n, X^{n+1}) \end{pmatrix}$$

Just like the case of $f_{2,3}$ we can decompose an element of tf to see that it has a generating set of the form $\{e_{i,1}^{k_1,\ldots,k_n} 1\ \forall\ i \ \mid i \neq n+1\} \cup \{e_{i,2}^{k_1,\ldots,k_n}\ \forall\ i \ \mid i \neq n\} \cup \{e_{i,3}^{k_1,\ldots,k_n} - 2e_{n,3}^{k_1,\ldots,k_i+1,\ldots,k_n} - 2e_{n+1,3}^{k_1,\ldots,k_i+1,\ldots,k_n} \mid 1 \leq i \leq n-1\} \cup \{e_{n,3}^{k_1,\ldots,k_n} - e_{n+1,3}^{k_1,\ldots,k_n}\}$. Thus we see that we already have $\mathcal{O}_{1,1}, \mathcal{O}_{1,2}, \ldots \mathcal{O}_{n-1,1}, \mathcal{O}_{n-1,2}$ from tf. Choosing, $\eta(X)$ such that $\eta_1(X) = (X^1)^{k_1}$ and others as zero, we see, after using the fact that we have $\mathcal{O}_{1,1}$ and $\mathcal{O}_{1,2}$, that we have all the powers of x^1 at (1,3), and similarly using the vector field of the type $\eta_1(X) = (X^i)^{k_i}, \eta_2(x) = \cdots = \eta_{n+1}(x) = 0$ we see that we have all the positive integral exponents of $X^i, i = 1, \ldots, n$. Now, choosing $\eta_1(X) = (X^1)^{k_1} \ldots (X^{n-1})^{k_{n-1}} X^n, \eta_2 = \cdots = \eta_{n+1} = 0$ we get

$$\begin{pmatrix} (x^1)^{k_1} \ldots (x^{n-1})^{k_{n-1}} x^n & 0 & (x^1)^{k_1} \ldots (x^{n-1})^{k_{n-1}} x^n - \sum_{i=1}^{n-1} (x^1)^{k_1} \ldots (x^i)^{k_i+2} \ldots (x^{n-1})^{k_{n-1}} \\ 0 & 0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0 & 0 \end{pmatrix}$$

Now, using the fact that we have all the exponents of x^i for $i = 1, \ldots, n-1$ and $\mathcal{O}_{1,1}$ we get that we have $(x^1)^{k_1} \ldots (x^{n-1})^{k_{n-1}} x^n$ at (1,3), therefore we can apply induction just as in previous sections to get $\mathcal{O}_{1,3}$. Similarly, we also get $\mathcal{O}_{2,3}, \ldots, \mathcal{O}_{n-1,3}$, using what we get $-2e_{n,3}^{k_1,\ldots,k_i+1,\ldots,k_n} - 2e_{n+1,3}^{k_1,\ldots,k_i+1,\ldots,k_n}$, which after absorbing -2 gives $e_{n,3}^{k_1,\ldots,k_i+1,\ldots,k_n} + e_{n+1,3}^{k_1,\ldots,k_i+1,\ldots,k_n}$. Since, we also have $e_{n,3}^{k_1,\ldots,k_n} - e_{n+1,3}^{k_1,\ldots,k_n}$ from tf we get $e_{n,3}^{k_1,\ldots,k_i+1,\ldots,k_n}$ as well as $e_{n,3}^{k_1,\ldots,k_i+1,\ldots,k_n}$, and since this is true for every $i = 1, \ldots, n-1$ at (n, 3), we get everything except the exponents of x^n. Now using a vector field of the type $\eta_n(X) = (X^n)^{k_n}, k_n > 0, \eta_i(X) = 0$ if $i \neq n$ and the fact that we have $\mathcal{O}_{n,1}$ together with the availability of $e_{n,3}^{k_1,\ldots,k_i+1,\ldots,k_n}$ for every $i = 1, \ldots, n-1$ we also get $(x^n)^{k_n}, k_n \neq 0$ and similarly for (n+1, 3). Proceeding as in the subsection $f_{2,3}$ we get:

$$\begin{pmatrix} \mathcal{O}_{1,1} & \mathcal{O}_{1,2} & \mathcal{O}_{1,3} \\ \vdots & \vdots & \vdots \\ \mathcal{O}_{n,1} & \mathcal{O}_{n,2}-\{1\} & \mathcal{O}_{n,3}-\{1\} \\ \mathcal{O}_{n+1,1}-\{1\} & \mathcal{O}_{n+1,2} & \mathcal{O}_{n+1,3}-\{1\} \end{pmatrix}$$

Let $e_{i,j}$ denotes the standard matrix units i.e. 1 at (i,j) and 0 otherwise, then we see that we have got everything except $e_{n,2}, e_{n,3}, e_{n+1,1} \& e_{n+1,3}$, but since, $e_{n,2} + e_{n,3} \in T\mathcal{A}_e f_{n,3}$ because we can choose $\eta(X)$ with $\eta_n(X) = 1$ and others as zero, which with $\mathcal{O}_{n,1}$ from tf will give $e_{n,2} + e_{n,3}$. Similarly, $e_{n+1,1} + e_{n+1,3} \in T\mathcal{A}_e f_{n,3}$, and $e_{n,3} - e_{n+1,3} \in tf \subset T\mathcal{A}_e f_{n,3}$. Therefore, we get

$$T\mathcal{A}_e f_{n,3} + \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{pmatrix} = T^1 f_{n,3}$$

and hence, $f_{n,3}$ is of $\mathcal{A}_e$ co-dimension 1. We seek something analogus to Mond's result in this case as well:

Conjecture 5.2. *If $g_{n,3}$ is a bigerms from $\mathbb{C}^n$ to $\mathbb{C}^{n+1}$ which satisfies all the conditions above, with the first branch $g_1 : (x^1, \ldots, x^n) \longmapsto (x^1, \ldots, x^n, 0)$ and also the second branch $g_2 : (x^1, \ldots, x^n) \longmapsto (x^1, \ldots, 0, x^n)$ i. e. same as the first and second branch of our $f_{n,3}$ then $g_{n,3} \sim_{\mathcal{A}} f_{n,3}$ that is they are left-right equivalent.*

We have seen in both the cases $f_{n,2}$ and $f_{n,3}$ that if we use this algorithm to find out the required multigerm then how the analogous results in lower dimension have been proved a guide in calculating $\mathcal{A}_e$ co-dimension. In fact, we could observe that the last rows looked similar. Therefore:

Conjecture 5.3. *Assume $k < n$, suppose we have proved that the multigerms $f_{n-1,k}$ found using the algorithm above has $\mathcal{A}_e$ co-dimension 1, then the multigerm $f_{n,k}$ (found using the same algorithm) also have $\mathcal{A}_e$ co-dimension 1.*

6 Conclusion

Let me begin this section with some observation. First is in the direction of seeking some analogue of theorem 2.7 in arbitrary case: let $E_1, \ldots, E_k$ be vector subspaces of an n-dimensional vector space ^{n}, and assume that $E_i = Sp_{\mathbb{C}}\{v_{1,i}, \ldots, v_{m_i,i}\}$, then

Conjecture 6.1. *$E_1, \ldots, E_k$ meet in general position if and only if $\{v_{1,i} \wedge \cdots \wedge v_{m_i,i}\}$ are linearly independent in $\bigwedge^{N\,n}$ where N is the maximum of the dimensions of E_i.*

If this is not true, is this true when each E_i is of dimension n and the dimension of the ambient space is $n+1$?

Second is the following observation: consider the real picture of the miniversal deformation of the multigerm $f_{1,2}$

$$\begin{cases} f_1 : (x) & \longmapsto & (x, 0) \\ f_2 : (x) & \longmapsto & (x, x^2 + t) \end{cases}$$

which represents the curve $X(Y - X^2 - t) = 0$ As t varies the parabola shifts from downward to upward. For $t < 0$ it encloses a non-zero area with the $X - axis$ i. e. the image of the first branch, which vanishes when $t = 0$ and in the real case there's no intersection for $t > 0$ but if we consider this phenomenon over $\mathbb{C}$ then the point of singularity is charecterised by the vanishing of the measure. In general, if we have a differential 2-form defined on our manifolds we can talk about such phenomenon in general as well.

Coming back to the the problem of right-left equivalence, it's not difficult to see that the $\mathcal{A}_e$ co-dimension is invariant under the left-right equivalence, but sadly, this's not the complete invarient. In dimension 1, this's true because of the following theorem:

Theorem 6.2. *Let $f : \mathbb{C} \to \mathbb{C}$ is holomorphic $w = f(z)$, and for $z_0 \in \mathbb{C}$ the first $k-1$ derivatives of $f(z)$ vanish at z_0 but $f^k(z_0) \neq 0$ then there exists biholomorphic change of variables $\bar{z} : \mathbb{C} \to \mathbb{C}$ and $\bar{w} : \mathbb{C} \to \mathbb{C}$ such that in new variables we have $\bar{w} = (\bar{z})^k$.*

A proof can be found in [5]. So, even if we don't want to use the fact that $x \mapsto x^2$ and $x \mapsto x^3$ have different $\mathcal{A}_e$ co-dimension, we can see that they are not right-left equivalent. In fact, our result about $\mathcal{A}_e$ co-dimension of the map $x \mapsto x^n$ together with the above theorem states

If $f : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ is a monogerm with $\mathcal{A}_e$ co-dimension k, then f is right-left equivalent to x^{k+2}

Sadly, there's no such classification in higher dimension. Let me end this article by mentioning that all the conjectures made in the last section is an approach to find a normal form for $\mathcal{A}_e$ co-dimension 1 multigerm with the condition mentioned in the introduction.

"Make things as simple as possible, but not simpler." - Einstein

References

[1] [Tu] An introduction to manifolds. Universitext, Springer: 9781441973993

[2] [John N. Mather]Stability of $\mathbb{C}^\infty$ mappings II. Infinitesimal stability implies stability. Ann. of Math. (2) 891969254-291

[3] Classification of multigerms (from a modern viewpoint)

[4] On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$. Proc. London Math. Soc. (3), 50, 333-369, (1983).

[5] Riemann Surfaces and Algebraic Curves, LMS student texts 87