COLORING TRIVALENT GRAPHS: A DEFECT TFT APPROACH

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(*pre-preprint version*)

1. Introduction

1.1. **Overview.** The mathematics done in this manuscript accomplishes the reformulation of graph coloring problem as an obstruction problem with the thesis:

(1.1) *Certain problems in mathematics disguise itself as a local to global problem*

This is a thesis and not a theorem. Nonetheless, powerful enough to make one see the graph coloring problem as a problem of obstruction, and to view the *word problem* in group theory as a cobordism problem in certain bordism category. To see this, let us consider the problem of graph coloring first, where this thesis is easier to demonstrate. Fig. 1 shows the *Tait-coloring* process of a theta graph. Where, a Tait-coloring stands for a 3-edge admissible coloring of a graph: if the edges share a vertex then they all should get a different color. ([Bal18], [Tai80], [KR21], as well as [P^+71] are places where this definition can be found.)

When attempting to color such a graph Γ, embedded in some surface Σ, we begin locally: by choosing a stratified neighborhood (U, Γ_U) (see U and V in (i) in figure-1) and stack over it the same neighborhood but with an admissible coloring to the graph in it. The stratified neighborhood (U, Γ_U) is well covered by the collection of all admissible coloring of Γ_U. A coloring of the portion of the graph in this neighborhood, namely Γ_U, can be thought as a choice of a section over (U, Γ_U). We extend such sections over the union of two neighborhoods (U, Γ_U) and (V, Γ_V) by keeping only those sections (from all possible combinations) that agrees on the intersection. With this idea, we see immediately that the entire graph Γ admits an admissible coloring if this can be done globally, or if a global section exists. These are the kind of problems that the subjects of *obstruction theory* deals with. From this perspective, the famous formulation of the 4-*color theorem* by Tait in terms of 3-edge coloring of a trivalent graph can be restated as:

For a planar trivalent graph Γ, the only obstruction to its 3-edge coloring is given by a bridge, that is, if Γ is a trivalent, planar, and bridgless graph than a global section always exists.

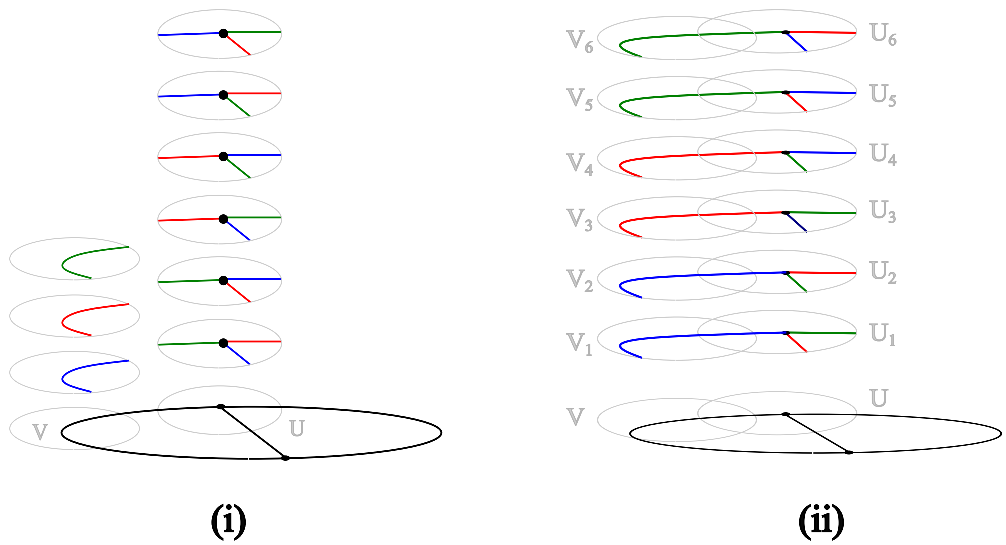


FIGURE 1. Caption

Note that I chose to state it as a meta-conjecture rather than a conjecture. The reason behind this is that we do not quite have a formulation in terms of the above 'sheaf' like gadget at this moment (see **??** for detail), but what we really have is more like a jigsaw puzzle perspective coming from a special kind of *cell-decomposition*. Consider again the graph Γ, embedded in some surface Σ, we use the piecewise linear cell-decomposition (or PLCW for short, introduced in [KJ12] inspired by the work in [Lur08]) to decompose this *stratified* space into cells. Next, color the graph inside these cells admissibly and put it back to its original place, as if we are solving some jigsaw puzzle. If we can do this to all the pieces (2-cells) and complete the surface with an admissibly colored copy of the same graph, then we say that the graph admits an admissible coloring. Before giving an account of how this is achieved, I want to point out, at this point only heuristically, that solving a word problem has a very similar local to global flavour: we can pick a spot in the string of generators and use the relation to untangle it, but whether this word is same as a given another word can only be decided after looking at the entire string. By the end of this subsection we will know that the theory developed to tackle the problem of graph coloring as a local to global problem gives this formulation of the word problem as a byproduct.

The main tool that is used is the concept of *field theory with defects*. Defects are everywhere in mathematical physics: from condensed matter physics to quantum field theories (see for instance [FMT22] and [BFM+22]). Justifying its name, they are defective in the sense that the theory that governs them is different than the surrounding theory. I refer to the excellent survey [CDZR23] for a decent knowledge and for now, only mention that a pair (Σ, Γ) where Σ is a surface and Γ is an embedded graph is an example of what I call a *surface with defects*. (It can be generalised for sufficiently nice higher dimensional stratified spaces, but I limit myself to dimension two.) There are two key steps:

- Given a group G and a presentation $\mathcal{P}_G$, construct a category $Bord_2^{def,cw}(\mathcal{P}_G)$. Objects in this category are given by a marked circles (marked by generators of $\mathcal{P}_G$). A morphism between two objects is given by a isotopy (relative to boundary) class of a surface with defects locally modelled by the relations in $\mathcal{P}_G$. Finally, the composition is given by gluing along the common boundary. Both objects and morphisms are equipped with a PLCW decomposition. As an example, when G is *Klein-four group* $\mathbb{K}_4$ and $\mathcal{P}_G = \langle a, b, c \mid a^2 = b^2 = c^2 = abc = 1\rangle$, the set of morphisms consists surfaces with trivalent graphs all of which are 3-edge colorable.
- Next, based on the work of [DKR11] construct a lattice TFT

$$\chi^{cw} : Bord_2^{def,cw}(\mathcal{D}^{\mathbf{3}}) \to \mathrm{Vect}_F(\mathbb{C})$$

  Where $\mathrm{Vect}_F(\mathbb{C})$ is the symmetric monoidal category of finite-dimensional $\mathbb{C}$-vector spaces, and $Bord_2^{def,cw}(\mathcal{D}^{\mathbf{3}})$ is the category whose objects are trivially marked circle and whose morphism are given by isotopy classes of surfaces with single-colored trivalent graphs. (A trivially marked circle can be thought of as unmarked circle, and similarly a single-colored graph can be thought as uncolored.)

Then the main result can be stated as:

Theorem 1.1. Let Γ be a trivalent graph embedded in $\mathbb{S}^2$. Consider the surface with defect $(\mathbb{S}^2, \Gamma)$ in $\mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}^{\mathbf{3}}_+))(\emptyset, \emptyset)$. The action of the functor χ^{cw} on $(\mathbb{S}^2, \Gamma)$ is the assignment

$$\begin{aligned} \chi^{cw}(\mathbb{S}^2, \Gamma) : \mathbb{C} &\longrightarrow \mathbb{C} \\ \lambda &\mapsto \#\mathrm{Tait}(\Gamma)\lambda \end{aligned} \tag{1.2}$$

In other words the number $\chi^{cw}(\mathbb{S}^2, \Gamma)(1)$ is the number of Tait-coloring of the planar trivalent graph Γ.

The proof of this theorem uses the forgetful functor

$$\Pi^{cw} : Bord_2^{def,\ cw}(\mathcal{P}_{\mathbb{K}_4}) \to Bord_2^{def,\ cw}(\mathcal{D}^3)$$

to construct a stratified covering projection π^{cw} called *bleach* that forgets the color. A coloring process of a given trivalent graph is then a choice of local-section of π^{cw}. (Please note the notation 'cw'.)

1.2. **Acknowledgement.** First, the author would like to thank his advisor Scott Baldridge for patiently and passionately listening to the raw ideas and for his constant support. Next, the author is indebted to Ingo Runkel for his powerful suggestions, and Nils Carqueville for many fruitful discussions. Finally, the author wants to thank David Truemann, Eric Zaslow, Roger Casals, Mikhail Khovanov and Louis-Hadrien Robert whose works have been an inspiration for this project. In particular, the author thanks Mikhail Khovanov for many related discussions, especially in the early stages. The author would also like to thank Kevin Schreve for many informal discussions in connection with the word problem.

2. Category of 2-dimensional bordism with defects

The goal of this section is to define the category of smooth bordism with defects. We essentially follow [DKR11], but we give some new definitions and modify some old ones in order to set the ground for our work in the subsequent sections. By $Bord_2^{def}$ we actually mean the category $Bord_2^{def,\ top}$, but we omit the word 'top' as we are dealing with *topological defects* throughout this manuscript. The category $Bord_2^{def}(\mathcal{D})$ constitutes objects and morphisms that are *stratified spaces* with each strata labelled with elements of sets called *defect conditions*. We explore each of these concepts in the following subsections.

2.1. Defect Conditions.

Definition 2.1. Given an n-dimensional oriented manifold M, and a collection $\mathfrak{S} = \{M_0, \ldots, M_n\}$ of submanifolds of M, we say that $\mathfrak{S}$ is an *admissible decomposition* of M if the following conditions hold.

- (**covering**) $M = \bigcup_{i=0}^{n} M_i$,
- (**decomposition**) $\dim(M_i) = i$, $M_i \cap M_j = \emptyset$ for $i \neq j$, and the orientation of M_n is induced by the orientation of M, and
- (**admissibility**) each partial union $\bigcup_{i=0}^{k} M_i$ is a closed subset of M for every $k \leq n$.

Remark 2.1. (1) A consequence of admissibility condition is that $\bar{M}_k \setminus M_k$ is contained in the union $\bigcup_{i=0}^{k-1} M_i$ of lower dimensional pieces. Where $\bar{M}_k$ is the closure of M_k in M.

(2) Let $\mathfrak{M}^k$ denotes the partial union $\bigcup_{i=0}^{k} M_i$, then we have the filtration by closed subspaces

$$M = \mathfrak{M}^n \supset \mathfrak{M}^{n-1} \supset \cdots \supset \mathfrak{M}^0 \supset \mathfrak{M}^{-1} = \emptyset \tag{2.1}$$

and $\mathfrak{M}^k \setminus \mathfrak{M}^{k-1} = M_k$. Therefore, an admissible decomposition canonically leads to *stratification* of M. We will refer to the components of M_k as the

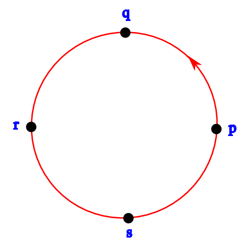


FIGURE 2. An admissible decomposition $\mathfrak{U} = \{U_0, U_1\}$ of $\mathbb{S}^1$, where $U_0 = \{p, q, r, s\}$ and $U_1 = \mathbb{S}^1 \setminus U_0$.

k-dimensional strata of M. We refer to the reader to [Fri17], (2.2) for the definition of a filtered space and stratification.

Example 2.1. (1) Let M be n-dimensional, the collection $\mathfrak{S}_0 = \{M\}$, which means $M_k = M$ if $k = n$, and $M_k = \emptyset$ otherwise, is an admissible decomposition of M.

(2) Let $U = \mathbb{S}^1$ denote the unit circle in the complex plane. Form the collection $\mathfrak{U}$ with $U_0 = \{p, q, r, s\}$, where $p = 1, q = \iota, r = -1, s = -\iota$, and $U_1 = \mathbb{S}^1 \setminus M_0 = \{(p, q), (q, r), (r, s), (s, p)\}$. Then, $\mathfrak{U}$ is an admissible decomposition of $\mathbb{S}^1$. See Fig. 2 below.

(3) (**non-example**) Let $\psi : \mathbb{S}^2 \to \mathbb{R}^2$ be the stereographic projection. The collection $\mathfrak{S}_1 = \{M_0, M_1, M_2\}$, where $M_0 = \{\psi^{-1}(0, 0)\}$, $M_1 = \{\psi^{-1}(\{(x, 0) \mid x \in \mathbb{R}, x \neq 0\})\}$, and $M_2 = \mathbb{S}^2 \setminus (M_0 \cup M_1)$ is not an admissible decomposition. M_0 is the south-pole, M_1 is the great circle containing the south-pole but missing the north-pole - ∞. We see that $M_0 \cup M_1$ is not closed in $\mathbb{S}^2$ as the north-pole, which is in the closure of $M_0 \cup M_1$, is missing.

(4) (turning (3) into an example) However, the collection $\mathfrak{S}_2 = \{M_0', M_1', M_2'\}$, where $M_0' = \{\psi^{-1}(0, 0), \infty\}, M_1' = M_1$ and $M_2' = \mathbb{S}^2 \setminus (M_0' \cup M_1')$, is admissible.

(5) Given a link $L \subset \mathbb{S}^3$, the collection $M_0 = M_2 = \phi$, $M_1 = L$, and $M_3 = \mathbb{S}^3 \setminus (M_0 \cup M_1 \cup M_2)$ is an admissible decomposition of $\mathbb{S}^3$

(6) Let Σ be a surface and Γ an embedded graph in Σ with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ such that $V(\Gamma) \cap \partial\Sigma = \emptyset$. The collection $\Sigma_0 = V(\Gamma), \Sigma_1 = E(\Gamma)$ and $\Sigma_2 = \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$ is an admissible decomposition of Σ.

Given a set D_i, let $\bar{D}_i$ be the set of formal inverses of elements in D_i, and $X_i = D_i \cup \bar{D}_i$. For example if $D_1 = \{x, y, z\}$ then $\bar{D}_1 = \{x^{-1}, y^{-1}, z^{-1}\}$, and $X_1 = \{x, y, z, x^{-1}, y^{-1}, z^{-1}\}$. With this convention, we define:

Definition 2.2. A *defect condition* is a class $\{D_2, D_1, D_0\}$ together with two maps

$$\psi_{1,2} : X_1 \to D_2 \times D_2 \quad \text{and} \quad \psi_{0,1} : X_0 \to \sqcup_{m=0}^{\infty}((X_1)^m / C_m)$$

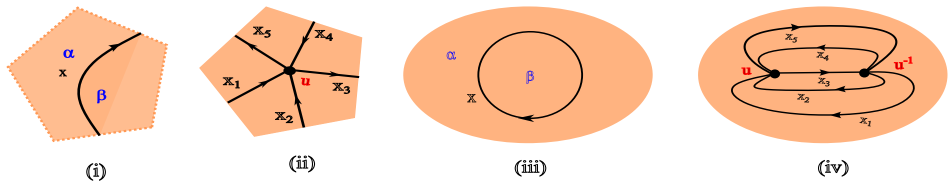


FIGURE 3. The map $\psi_{1,2}$ can be visualised as (i) where (in this case) it is: $x \mapsto (\alpha, \beta)$. (ii) represnts $\psi_{0,1} : u \mapsto [(x_1, x_2, x_3^{-1}, x_4, x_5^{-1})]$. (iii) (respectively (iv)) represents the orientation consistency condition for $\psi_{1,2}$ (respectively $\psi_{0,1}$).

Where, by $(X_1)^m$ we mean m-fold cartesian product of the set X_1. These maps are subject to the following two *orientation consistency* conditions:

- If $\psi_{1,2}(x^\epsilon) = (\alpha, \beta)$, then $\psi_{1,2}(x^{-\epsilon}) = (\beta, \alpha)$. And,
- If $\psi_{0,1}(u^\epsilon) = [(x^{\epsilon_1}, \ldots, x^{\epsilon_m})]$, then $\psi_{0,1}(u^{-\epsilon}) = [(x^{-\epsilon_m}, \ldots, x^{-\epsilon_1})]$

Usually, in the literature, the map $\psi_{1,2}$ is given in terms to two maps $s, t : X_1 \to D_2$ such that for $x^\epsilon \in X_1, \psi_{1,2}(x^\epsilon) = (t(x^\epsilon), s(x^\epsilon))$.

See Fig. 3 for a pictorial representation of the maps $\psi_{1,2}, \psi_{0,1}$, and the orientation consistency conditions. We will see in later sections that these are not just representations and carry deeper meaning.

At this point, if we introduce the groups

$$F[D_0] \coloneqq \langle X_0 \mid \emptyset \rangle \ , \ F[D_1] \coloneqq \langle X_1 \mid \emptyset \rangle \ , \ \Sigma[D_2] \coloneqq \langle X_2 \mid \alpha = \alpha^{-1} \rangle$$

and identify an ordered tuple $(x_1, \ldots, x_m)$ with the (unreduced) word $x_1 \ldots x_m$, then the two orientable consistency conditions in 2.2 can be written as:

$$\begin{aligned} \psi_{1,2}(x^{-\epsilon}) &= (\psi_{1,2}(x^\epsilon))^{-1} \\ \psi_{0,1}(x^{-\epsilon}) &= (\psi_{0,1}(x^\epsilon))^{-1} \end{aligned} \tag{2.2}$$

The reason that the corresponding group on D_2 is not free, in contrast with its counterparts on D_0 and D_1, has to with the orientation condition in [2.1, (**decomposition**)] as we will see in the subsequent sections.

After fixing a defect condition $\mathcal{D} \coloneqq \{D_2, D_1, D_0, \psi_{0,1}, \psi_{1,2}\}$, we define objects and morphisms in the category $\text{Bord}_2^{\text{def}}(\mathcal{D})$ in the next few subsections.

2.2. **Objects.** Naively, an object in the category $\text{Bord}_2^{\text{def}}(\mathcal{D})$ is a circle with marked points and arcs. Points are marked with the elements of D_1, and arcs are marked with the elements of D_2. If we denote the circle by $\mathbb{S}^1$ then these points give rise to admissible decomposition of $\mathbb{S}^1$ as in Example- 2.1(2). More generally, given sets of

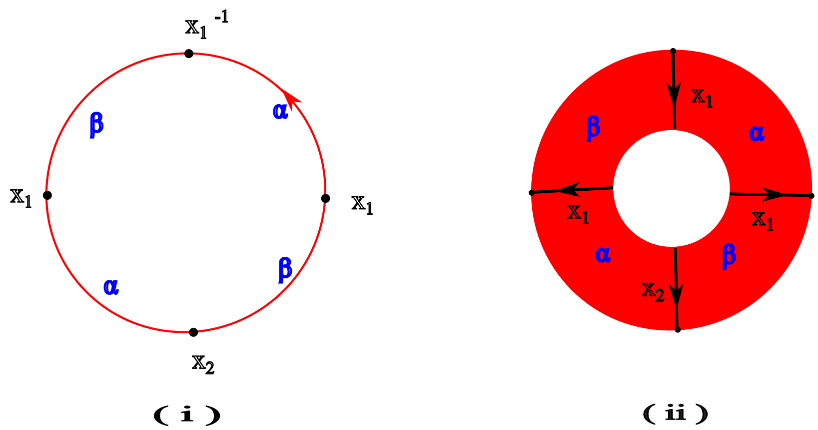


FIGURE 4. (i) shows a circle with defects. The map $d : \mathbb{S}^1 \to X$ takes $p \mapsto x_1, q \mapsto x_1^{-1}, r \mapsto x_1, s \mapsto x_2, (p,q) \mapsto \alpha,\ (q,r) \mapsto \beta,\ (r,s) \mapsto \alpha,\ (s,p) \mapsto \beta$, and $\psi_{0,1}$ maps x_1 to (α, β) and x_2 to (β, α). (ii) shows a cylinder on this circle with defects. If we agree to call $p(\text{respectively} q, r) \times [0,1]$ as $p[1]$ (respectively $q[1], r[1]$), then there is no change in the map d as p (respectively q, r) lies in the unique component $p[1]$ (respectively $q[1], r[1]$). Same holds for the two dimensional strata.

defect conditions $\mathcal{E} \coloneqq \{E_0, E_1, \phi_{0,1}\}$, where $E_0 = D_1, E_1 = D_2$ and $\phi_{0,1} = \psi_{1,2}$ we define:

Definition 2.3. a 1-manifold with defects with defect conditions $\mathcal{E}$ is a tuple $(\mathbb{L}, \mathfrak{L}, d)$ where

(1) $\mathbb{L}$ is an oriented 1-manifold,
(2) $\mathfrak{L}$ is an admissible decomposition of $\mathbb{L}$. It consists of a set of points L_0 in the interior of $\mathbb{L}$ and its complement $L_1 = \mathbb{L} \setminus L_0$.
(3) For $Y_i = E_i \sqcup \bar{E}_i$ and $Y = Y_0 \sqcup Y_1$, $d : \mathbb{L} \to Y$ such that
  - $d(L_0) \subset Y_0$,
  - $d(L_1) \subset E_1$, and
  - $d|_{\pi_0(L_i)}$ is constant
(4) The map $d : \mathbb{L} \to Y$ respects $\phi_{0,1}$. More precisely, if $p_0 \in L_0$ is such that it is out-boundary of l_1 and in-boundary of r_1 for $l_1, r_1 \in L_1$, then $(t \circ d)(p_0) = d(l_1)$ and $(s \circ d)(p_0) = d(r_1)$.

Example 2.2. We upgrade Example 2.1, (2)] to a 1-manifold with defects in Figure-4, where $E_0 = \{x_1, x_2\}, E_1 = \{\alpha, \beta\}$.

Remark 2.2. We call a closed 1-manifold with defect, a *circle with defects*. In this case, we further assume that the underlying circle comes with a *distinguished point*, which we denote by -1. No 0-dimensional stratum is allowed to pass through this point under isotopy preserving the defect structure. In other words, all the marked points lies in $\mathbb{S}^1 \setminus \{-1\}$. See [[Car16], 2.1] and [DKR11], 2.3] for details behind this convention.

2.3. **Morphism.** A morphism in the category $\mathrm{Bord}_2^{\mathrm{def}}(\mathcal{D})$ is given by an equivalence class of bordism between two circles with defects. We proceed to define this carefully.

Definition 2.4. Given a set of defect conditions $\mathcal{D} \coloneqq \{D_2, D_1, D_0, \psi_{0,1}, \psi_{1,2}\}$ and associated sets $\{X_i\}$, a *surface with defects* with defect conditions $\mathcal{D}$ consists of the following data.

(1) An orientable surface Σ, possibly with boundary.
(2) An admissible decomposition $\mathfrak{S} \coloneqq \{\Sigma_2, \Sigma_1, \Sigma_0\}$ of Σ such that Σ_0 lies in the interior of Σ.
(3) For $X = X_0 \sqcup X_1 \sqcup X_2$ a map $d : \Sigma \to X$ such that
   - $d(\Sigma_i) \subset X_i$ for all $i \neq 2$,
   - $d(\Sigma_2) \subset D_2$, and
   - $d|_{\pi_0(\Sigma_i)}$ is constant.
(4) The map $d : \Sigma \to X$ respects the maps $\psi_{0,1}$ and $\psi_{0,2}$. More precisely:
   - For $l \in \Sigma_1$ and $A, B \in \Sigma_2$ such that l is the out-boundary of A and in-boundary of B, $(t \circ d)(l) = d(A)$ and $(s \circ d)(l) = d(B)$.
   - For a sequence $l_1, \dots l_m \in \Sigma_1$ such that $\partial l_i = \pm p$ where $p \in \Sigma_0$, the tuple $(d(l_1), \dots, d(l_m))$ is in the same equivalence class as $\psi_{0,1}(d(p))$.

We will denote a surface with defect by a tuple $(\Sigma, \mathfrak{S}, \mathcal{D}, d)$ or just by $(\Sigma, \mathfrak{S}, d)$ when the defect condition $\mathcal{D}$ is clear from the context. We will refer to Σ as the *underlying surface.*

Remark 2.3. A consequence of [2.4, (4)] is that the ordered basis $(s(x) - t(x), x)$ gives the orientation of the underlying surface.

Convention 2.1. Let u^{ϵ} be the value of d at a given zero dimensional stratum. If $\epsilon = +1$, the sign of the defect at a 1-dimensional stratum is $+1$ if the zero dimensional stratum is the out-boundary of it; negative otherwise. For example, in Figure- 6 the map $\psi_{0,1}(u) = [(x_1, x_2, x_5^{-1}, x_3^{-1})]$. The sign ϵ in the expression x^{ϵ} is positive for x_1 because the arrow is going into u (which means u is an out-boundary) while it is negative for x_5. The reason for this kind of convention can be inferred from [[Car16] , (2.16)]

Example 2.3. Figure- 5 shows both an example and a non-example of a surface with defects. here, the underlying surface is a disk which is a surface with boundary.

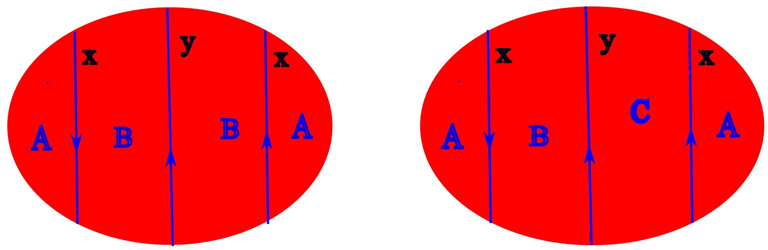


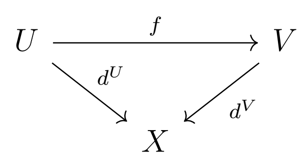
FIGURE 5. Shows an example of a surface (discs) with defect (left) and a non-example (right).

Next, we define the concept of isomorphism between two surfaces with defects with the same defect conditions $\mathcal{D}$.

Definition 2.5. Given two surfaces with defects $(U, \mathfrak{U}, d^U)$ and $(V, \mathfrak{V}, d^V)$, an isomorphism between them is a map f such that

(1) $f : U \to V$ is an orientation preserving diffeomorphism, together with the property that for all i, $U_i = f^{-1}(V_i)$. In other words, f preserves orientation and admissible decomposition.

(2) $d^U = f^* d^V$, or the following diagram commute:



I aim to define it as the isotopy (isomorphism) of underlying stratified space that preserves orientation and labels on the strata.

— Amit

Given a circle with defect $(\mathbb{S}^1, \mathfrak{C}, d)$ with defect conditions $\mathcal{E}$ as in 2.3, there is a canonical way to put a surface with defect structure on the surface $\mathbb{S}^1 \times I$ for any interval I. We denote this surface with defects by $(\mathbb{S}^1 \times I, \mathfrak{C}[1], \mathcal{D}, d)$. The motivation behind such a notation comes from the observation that the admissible decomposition, defect conditions, and the incident map is given by just shifting the index by $+1$, while there is no change in the map d. Note that the resulting surface with defects does not have a zero dimensional stratum.

Definition 2.6. Given a 1-manifold with defect $\hat{\mathbb{S}} := (\mathbb{S}^1, \mathfrak{C}, d)$ with defect conditions $\mathcal{E}$ as in Definition- 2.3,

(1) A *cylinder* on $\hat{\mathbb{S}}$ is a surface with defect isomorphic to

$$(\mathbb{S}^1 \times [0, 1], \mathfrak{C}[1], \mathcal{D}, d)$$

(2) a *collar* of $\hat{\mathbb{S}}$ is a surface with defects isomorphic to

$$(\mathbb{S}^1 \times [0,1), \mathfrak{C}[1], \mathcal{D}, d)$$

.

In figure- 4 , (ii) shows a cylinder on (i).

Finally, we define what does it mean for two circles with defects to be cobordant via a surface with defects.

Definition 2.7. Let $(U, \mathfrak{U}, d_U)$ and $(V, \mathfrak{V}, d_V)$ be two circles with defects with defect conditions $\mathcal{E}$ coming from the defect conditions $\mathcal{D}$ as in the Definition- 2.3. A *bordism with defect* is a surface with defects $(\Sigma, \mathfrak{S}, d)$ with defect conditions $\mathcal{D}$ such that

(1) (Oriented bordism) $\partial\Sigma = (-U) \sqcup V$ where a minus (-) denotes the reverse orientation.
(2) (compatible with admissible decomposition)$U_j \sqcup V_j \subset \partial\Sigma_{j+1}$ for all $j = 0, 1, 2$.
(3) Let $\mathfrak{i}$ (respectively $\mathfrak{j}$) be the inclusion map for the inclusion of U (respectively V) into $\partial\Sigma$. The maps d_U and d_V is related to d via $d_U = \mathfrak{i}^* d$ and $d_V = \mathfrak{j}^* d$. These maps fit together in the following commutative diagram:

$$\begin{array}{ccccc} & & \Sigma & & \\ & \overset{\mathfrak{i}}{\nearrow} & & \overset{\mathfrak{j}}{\nwarrow} & \\ U & & \Big\downarrow d & & V \\ & \underset{d_U}{\searrow} & & \underset{d_V}{\swarrow} & \\ & & X & & \end{array}$$

Convention 2.2. We denote an oriented bordism Σ with in-boundary U and out-boundary V by $\begin{array}{ccccc} & & \Sigma & & \\ & \overset{i}{\nearrow} & & \overset{o}{\searrow} & \\ U & & & & V \end{array}$.

Example 2.4. Figure- 6 shows an example of a bordism from a disjoint union of two cirlce with defects to a single circle with defects. Here, we have chosen $D_2 = \{\alpha, \beta, \gamma, \delta\}, D_1 = \{x_1, x_2, x_3, x_4, x_5, x_6\}, D_0 = \{u\}; \psi_{1,2} : x_1 \mapsto (\alpha, \beta), x_2 \mapsto (\beta, \alpha), x_3 \mapsto (\alpha, \gamma), x_4 \mapsto (\alpha, \gamma), x_5 \mapsto (\gamma, \alpha), x_6 \mapsto (\delta, \alpha); \psi_{1,2} : u \mapsto [(x_2, x_5^{-1}, x_3^{-1}, x_1)]$.

Finally, we collect all the constituent data of the category $\mathrm{Bord}_2^{\mathrm{def}}(\mathcal{D})$ at one place:

Definition 2.8. The category $Bord_2^{def}(\mathcal{D})$ consists of the following data:

- The **defect conditions** $\mathcal{D}$, which constitutes sets D_2, D_1, D_0, and maps $\psi_{1,2}$ and $\psi_{0,1}$ defined in - 2.2.

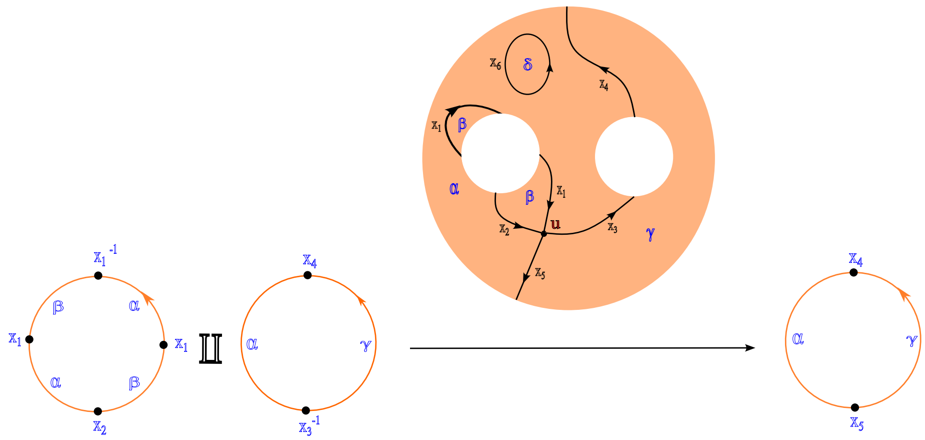


FIGURE 6. The pair of pants with defects is a bordism between two circles with defects in the left and one in the right. The sign convention on 1 strata is taken positive if the arrow flows in the direction of time. Thus an in-boundary gets positive (respectively negative) sign if the arrow is out of (respectively into) it. The opposite is true for the out-boundary.

- an **Objects** is a disjoint union of circles with defects, defined in 2.3, together with a germ of collars and distinguished points. As mentioned in [[Fre19], 1.2] we would like to think these marked circles as coming with a germ of an embedding into a surface with defects. Further, the germ of collars makes sure that the gluing is well behaved.
- Given two objects U and V, a **morphism** between them is either a permutation of markings on a given circle with defects or a surface with defect Σ such that $U \xrightarrow{i} \Sigma \xrightarrow{o} V$ and respects the condition on distinguished points. See Definition 2.10 and Remark 2.4 below. We consider two such bordisms to be equivalent if there is a boundary preserving isomorphism of surface with defects. More precisely, a morphism between two objects is given by a defect bordism class between them.

Example 2.5. Figure- 6 shows an example of a morphism from marked circles in the left to the marked circle in the right. Please note the convention about the sign of 1-defect conditions.

Given a surface with defects $(\Sigma, \mathfrak{S}, \mathcal{D}, d)$, one would expect its cross-sections to be object in the category $Bord_2^{def}(\mathcal{D})$, but this requires some care as an arbitrary cross-section may not have a germ of collar around it. Our strategy is to we only want to consider those cross sections which admits a collar around it. This is accomplished in two steps. First, we note the existence of a forgetful functor into the category $Bord_2$. Next, we use Morse theory to define a *generic cross-section*.

Definition 2.9. For any set of defect conditions $\mathcal{D}$, there exists a forgetful functor $\mathfrak{D} : Bord_2^{def}(\mathcal{D}) \to Bord_2$ defined by its actions on objects and morphisms as follows:

1. (On objects) $\mathfrak{D}((O, \mathfrak{O}, d_O)) = O$,
2. (On morphism) $\mathfrak{D}((\Sigma, \mathfrak{S}, \mathcal{D}, d)) = \Sigma$

That is, the functor $\mathfrak{D}$ maps a circle (surface) with defects to the underlying circle (surface) by forgetting all stratification and defects.

Next, for a surface Σ (or an underlying surface of a surface with defects) choose a height function $f : \Sigma \to \mathbb{R}$. Since Σ is compact, we can assume that $f(\Sigma) \subset [0, 1]$.

Definition 2.10. Let $(\Sigma, \mathfrak{S}, \mathcal{D}, d)$ be a surface with defects, and $f : \Sigma \to [0, 1]$ be a height function. For $t \in [0, 1]$, we say that $f^{-1}(\{t\})$ is a *generic cross-section* of the surface with defects $(\Sigma, \mathfrak{S}, \mathcal{D}, d)$ if there exists $\epsilon > 0$ such that $f^{-1}([t - \epsilon, t + \epsilon])$ is isomorphic to the cylinder over $f^{-1}(\{t\})$.

We refer to [ADE14] , (1.2) for the existence of Morse function on a given surface Σ. The existence of a height function follows from the existence of Morse functions.

Remark 2.4. To respect the condition on distinguished point, we demand that a generic cross-section is a circle with defects with a distinguished point.

We end this section by mentioning that we have kept ourselves limited to two-dimensional defect data as it is best suited for our objective, but it is possible to talk about higher dimensional defects and with structures. One such generalised picture is presented in '[Lur08], Section-4.3. The other approach, as suggested by our definition of a surface with defects, comes from the introduction of *constructible sheaf*. This latter approach has been developed well in [FMT22], Section-2.4 and 2.5 in relation with topological symmetries of QFT.

3. Topological field theories with defect

Let $\mathbb{K}$ be a field and $\mathrm{Vect}_{\mathbb{K}}$ denotes the symmetric monoidal category of $\mathbb{K}$ vector spaces. A *topological field theory* or TFT with defect is a symmetric monoidal functor

$$T : \mathrm{Bord}_2^{\mathrm{def}}(\mathcal{D}) \to \mathrm{Vect}_{\mathbb{K}}$$

We are interested in the category of $\mathbb{K}$ vector spaces with a trace pairing, which restricts the target category of T to $\mathrm{Vect}_F(\mathbb{K})$ - the category of finite dimensional $\mathbb{K}$ vector spaces. One example of such a funtor comes from the lattice TFT construction. We do not give this construction in detail, but highlight only the essential ingredients and steps. A detail of this construction can be found in [[DKR11], (3)], which is closest to the spirit of this manuscript. Other references are: [[FFRS07] , [FRS02]], together with an earlier work [[Tur99]].

3.1. **Category of bordism with PLCW decomposition.** The most essential ingredient for lattice TFTs is the category $\mathrm{Bord}_2^{\mathrm{def,\,cw}}(\mathcal{D})$. It has same objects and morphisms as the category $\mathrm{Bord}_2^{\mathrm{def}}(\mathcal{D})$ but they come equipped with an extra structure, namely a PLCW decomposition. We only collect the key feature of PLCW decomposition, and refer to [KJ12] for details. The main feature of PLCW decomposition is that although, it is more general than triangulation, it is less general than a CW decomposition. More precisely, for a cell-decomposition of a compact n-dimensional manifold M into generalised k-cells for $k = 0, \ldots, n$, if $\phi : B^k \to M$ is the characteristic map then ϕ is a homeomorphism when restricted to the interior of each $(k-1)$-cell of ∂B^k.

Possibly I should write it in slightly more detail.

— Amit

Example 3.1. cell-decomposition of $\mathbb{S}^2$ into a single 0-cell and a single 2-cell is a CW-decomposition but not a PLCW decomposition since for no cell-decomposition of $\mathbb{S}^1$ the attaching map is a homeomorphism.

Convention 3.1. We follow [DKR11] for notations and conventions. In particular, by a *cell-decomposition* of a manifold M, we mean a PLCW decomposition of M, and by a cell, we mean a generalised cell.

Convention 3.2. For a space M with a PLCW decomposition, we will denote the collection of cells by $C(M)$, and by $C_k(M)$ the collection of k-cells.

Definition 3.1. The category $\mathit{Bord}_2^{\mathrm{def,\,cw}}(\mathcal{D})$ consists of the following data:

(1) The set of **defect conditions** $\mathcal{D}$ as defined in Definition 2.2.
(2) **Objects** are disjoint union of circles with defects U together with a cell-decomposition $C(U)$, such that each point of the set U_0 lies in a 1-cell and each 1-cell contains at most one such point. (Fig. 7, (i).)
(3) A **morphism** is a surface with defects equipped with a PLCW decomposition $C(\Sigma)$ that is homeomorphic to one of the configurations in Fig. 7 (ii), (iii), or (iv)(after ignoring the labels.) More precisely,
    - the 1-dimensional submanifolds Σ_1 only intersects 1-cells and 2-cells, but not 0-cells. Moreover, each 1-cell intersects only one 1-stratum of Σ_1 .
    - Each 0-dimensional stratum lies inside a 2-cell, and each 2-cell contains at most one such strata. It may only contain a star-shaped configuration of 1-strata such that each edge of this cell is traversed by exactly one 1-stratum.

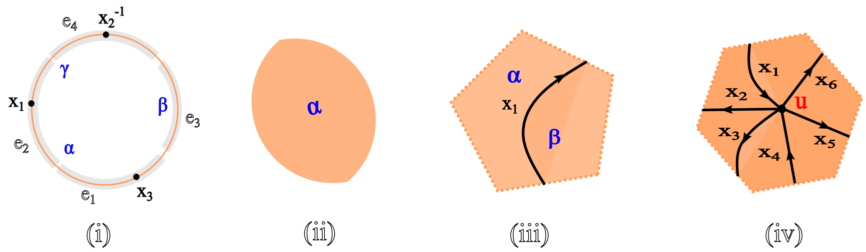


FIGURE 7. (i) shows an object in the category $Bord_2^{\text{def, cw}}(\mathcal{D})$. Note that the 1-cell e_3 does not contain any defect. (ii) is not a convex cell, but a generalised cell and does not contain any defect. We will refer to (ii), (iii) and (iv) as *basic-gons*. The map $\psi_{1,2}$ follow the orientation of the 1-stratum to call the left of it $t(x)$ and the right $s(x)$. For example, from (iii) we read $\psi_{1,2}(x) = (\alpha, \beta)$. To read the map $\psi_{0,1}$ at u^ϵ, if $\epsilon = +1$, then follow the orientation of the surface treating the boundary as the in-boundary. On the other hand, if $\epsilon = -1$, then follow the opposite orientation treating the boundary as the out-boundary. For instance, in (iv) $\psi_{0,1}(u) = [(x_1, x_2^{-1}, x_3^{-1}, x_4, x_5^{-1}, x_6^{-1})]$. Where we have oriented the surface in anti-clockwise manner.

- If a 2-cell contains no 0-stratum but only 1-strata then it must be homeomorphic to the configuration shown in Fig. 7, (iii).

Convention 3.3. We will refer to Fig. 7 (ii), (iii) and (iv) as *basic-gons*. We would like to think our surfaces with defects as assembled from them.

Remark 3.1. It is better to think the basic-gons as *cups* and *caps*. One can check that under orientation consistency conditions of Definition 2.2, caps transforms to cups and vice-versa.

We note that there is a forgetful functor $F : Bord_2^{\text{def, cw}}(\mathcal{D}) \to Bord_2^{\text{def}}(\mathcal{D})$, which is full and surjective. The lattice TFT construction uses the PLCW decomposition to construct a symmetric monoidal functor

$$T^{\text{CW}} : Bord_2^{\text{def, cw}}(\mathcal{D}) \to \text{Vect}_F(\mathbb{K})$$

and then F is used to show that T^{CW} is independent of the cell-decomposition by showing the existence of a unique symmetric monoidal functor T that makes the following diagram commute:

(3.1)
$$\begin{array}{ccc} Bord_2^{\text{def, cw}}(\mathcal{D}) & \xrightarrow{\quad T^{\text{CW}} \quad} & \mathrm{Vect}_F(\mathbb{K}) \\ & {\scriptstyle F}\searrow \quad \nearrow{\scriptstyle \exists!\, T} & \\ & Bord_2^{\text{def}}(\mathcal{D}) & \end{array}$$

We do not prove this here but refer to [[DKR11]], Section-3.6.

3.2. **Lattice TFT with defects.** In short, a lattice TFT assigns a Frobenius algebra A_a to 2-dimensional stratum labelled with defect a, a $(A_a - A_b)$-bimodule X_x to the 1-stratum labelled with x such that $t(x) = a$ and $s(x) = b$, and a bimodule intertwiner to 0-dimensioanl stratum. We refer to [DKR11] Section 3.3 for an overall algebraic preliminaries, with supplements [Koc] for Frobenius algebra, and [ML13] for bimodules. We employ the following convention for the rest of this manuscript as it will prove very handy for our purpose:

Convention 3.4. Let A and B be a unital, associative algebra over $\mathbb{K}$, and X be a $\mathbb{K}$ vector space. We write X for an $A-B$-bimodule X, and X^{-1} for the $B-A$-bimodule X^* - the dual of X. This way, we can denote a bimodule by X^ϵ where $\epsilon = \pm 1$.

Recall, that for A - an associative, unital algebra over $\mathbb{K}$, a right A-module X, and a left A-module Y, the tensor product $X \otimes_A Y$ is defined as follows:

$$X \otimes A \otimes Y \xrightarrow{l-r} X \otimes Y \xrightarrow{p} X \otimes_A Y, \tag{3.2}$$

where l is the left-multiplication map given on pure tensors by $l(x \otimes a \otimes y) = x \otimes (ay)$ and extended by linearity. Similarly, r is the right-multiplication map given by $r(x \otimes a \otimes y) = (xa) \otimes y$. Where $x \in X, y \in Y$ and $a \in A$ and $\otimes$ denote the tensor product of $\mathbb{K}$-vector space - $\otimes_{\mathbb{K}}$. When X is an $A - A$-bimodule, the *cyclic tensor product* is defined:

$$A \otimes X \xrightarrow{l-r} X \xrightarrow{p} \circlearrowleft X, \tag{3.3}$$

where $l(a \otimes x) = ax$ and $r(a \otimes x = xa)$.

Definition 3.2. A topological field theory $T_0^{cw} : Bord_2^{def,\ cw}(\mathcal{D}) \to \mathrm{Vect}_F(\mathbb{K})$ is a *trivial surrounding theory* if $D_2 = \{*\}$ and T_0^{cw} assigns $\mathbb{K}$ to $*$.

Such a theory is characterised by the fact that the non-trivial part of the theory lies solely on the 1-dimensional strata. It assigns a $\mathbb{K}$ vector space X_x for each $x \in D_1$, which is naturally a $\mathbb{K} - \mathbb{K}$-bimodule. Note that it is enough to consider the assignment for D_1 as it can be extended on entire X_1 using the orientation consistency and Convention 3.4 via the rule: $T_0^{cw}(x^{-1}) = T_0^{cw}(x)^{-1}$. In other words if T_0^{cw} assigns x a $\mathbb{K}$-vector space X_x then it assigns x^{-1} its dual - X_x^*.

We state the following proposition, which is very important as all the calculations we are going to do is based on it:

Proposition 3.1. For a $\mathbb{K}$ vector space X, Y, the following identities hold:

(1) $X \otimes_{\mathbb{K}} Y \cong X \otimes Y$
(2) $\circlearrowleft_{\mathbb{K}} X \cong X$

Do I need to call this a proposition; it's quite trivial?
— Amit

Where the tensor product $\otimes_{\mathbb{K}}$ on the left of (1) is the tensor product in the sense of Eq. (3.2), and the tensor product $\otimes$ on the right is the tensor product of $\mathbb{K}$-vector space.

The proof of Proposition 3.1 is fairly straightforward. So, we omit it.

We summarise below the input data for a trivial surrounding theory before giving its lattice TFT construction. The reader should consult [[DKR11], 3.4, 3.5] for a more general theory.

Definition 3.3. A lattice TFT, which is a trivial surrounding theory assigns:

(1) The field $\mathbb{K}$ for $* \in D_2$,
(2) a $\mathbb{C}$-vector space X_x for each $x \in D_1$, extended to X_1 by the rule $T_0^{cw}(x^{-1}) = T_0^{cw}(x)^{-1}$.
(3) for $u \in D_0$ such that $\psi_{0,1}(u) = [(x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n})]$ a linear map $\mu_u \in \mathrm{Hom}_{\mathbb{K}}(X_{x_1}^{\epsilon_1} \otimes \cdots \otimes X_{x_n}^{\epsilon_n}, \mathbb{K})$ with the property that μ_u is invariant under the induced action on $X_{x_1}^{\epsilon_1} \otimes \cdots \otimes X_{x_n}^{\epsilon_n}$ of the action of the cyclic group C_n on the tuple $(x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n})$. We will denote the set of such maps by $\circlearrowleft_{\mathrm{Inv}} \mathrm{Hom}_{\mathbb{K}}(X_{x_1}^{\epsilon_1} \otimes \cdots \otimes X_{x_n}^{\epsilon_n}, \mathbb{K})$.

Remark 3.2. We just state that Definition 3.3-(3) is a consequence of the definition of general lattice TFT data and Proposition 3.1.

Example 3.2. Let $X \coloneqq \mathbb{C}\langle a, b, c\rangle$. The map $\mu : X \otimes X \otimes X \to \mathbb{C}$ defined on the bases by the rule

$$\mu(x \otimes y \otimes z) = \begin{cases} 1 & \text{if } x, y, z \text{ are all different} \\ 0 & \text{otherwise} \end{cases}$$

and extended by linearity satisfies the condition [Definition 3.3, (3)]. In fact, it is invariant under the transposition of factors.

Remark 3.3. We set $\mathbb{K} = \mathbb{C}$ for the rest of this manuscript, and emphasize that the assignment $\mathbb{C}$ to two-dimensional starta by a trivial surrounding theory should be viewed as a Frobenius algebra. Indeed the non-degenerate pairing $\beta : \mathbb{C} \otimes \mathbb{C} \to \mathbb{C}$ given by $\beta(a \otimes b) = ab$ makes $\mathbb{C}$ a Frobenius algebra with the *counit* $\epsilon_{\mathbb{C}}$ as the identity $\mathbf{1}_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$. The copairing $\gamma : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$ is also the identity map.

Construction 1. Fix the defect condition $\mathcal{D}$ with $D_2 = \{*\}$. We proceed to explain the trivial surronding theory as a symmetric monoidal functor

$$T_0^{cw} : Bord_2^{def,\ cw}(\mathcal{D}) \to \mathrm{Vect}_F(\mathbb{C})$$

defined on objects and morphisms using the PLCW decomposition as follows:

- **On objects.** Let U be an object in $Bord_2^{def,cw}(\mathcal{D})$ which is a single circle. By the definition of the category $Bord_2^{def,cw}(\mathcal{D})$, it comes equipped with a cell-decomposition as in Fig. 7-(i). Let $e \in C(U)$ be such a 1-cell. We assign to it the vector space:

$$R_e = \begin{cases} \mathbb{C} & \text{if } e \text{ contains no } 0-\text{defect} \\ X_x^{\epsilon} & \text{if } e \text{ contains a } 0-\text{defect with label } x^{\epsilon} \end{cases} \tag{3.4}$$

then the action of T_0^{cw} on this single object U is given by

$$T_0^{cw}(U) = \bigotimes_{e \in C_1(U)} R_e \tag{3.5}$$

Here we are using Convention 3.2. For a general object $O = U_1 \sqcup \cdots \sqcup U_n$, we extend the definition of T_0^{cw} by 'monoidal property' namely: $T_0^{cw}(O) = T_0^{cw}(U_1) \otimes \cdots \otimes T_0^{cw}(U_n)$

Example 3.3. We refer to Fig. 8 and want to evaluate $T_0^{cw}(U)$ and $T_0^{cw}(V)$. We make a table below to achieve that:

$\mathbf{e}:$	e_1	e_2	$\hat{e}$	f_1	f_2	$\hat{f}$
$\mathbf{R_e}:$	X	X	$\mathbb{C}$	X	X	$\mathbb{C}$

From this, we get

$$\begin{aligned} T_0^{cw}(U) &= R_{e_1} \otimes R_{e_2} \otimes R_{\hat{e}} \\ &= X \otimes X \otimes \mathbb{C} \\ &\cong X \otimes X \end{aligned} \tag{3.6}$$

A similar calculation shows that $T_0^{cw}(V) \cong X \otimes X$.

- **On morphism.** Let $\Sigma : U \to V$ be a bordism in $Bord_2^{def,cw}(\mathcal{D})$, the action of the functor T_0^{cw} on $(\Sigma : U \to V)$ is given by:

$$T_0^{cw}(\Sigma) : T_0^{cw}(U) \xrightarrow{\mathbf{1}_{T_0^{cw}(U)} \otimes \mathcal{P}(\Sigma)} T_0^{cw}(U) \otimes Q(\Sigma) \otimes T_0^{cw}(V) \xrightarrow{\mathcal{E}(\Sigma) \otimes \mathbf{1}_{T_0^{cw}(V)}} T_0^{cw}(V) \tag{3.7}$$

We need to describe all these components: the vector space $Q(\Sigma)$, and the maps $\mathcal{P}(\Sigma)$ (propagator) and $\mathcal{E}(\Sigma)$(*evaluation*). By definition Σ is equipped with a PLCW decomposition. We use this fact to define the space $Q(\Sigma)$ and maps $\mathcal{P}(\Sigma)$, and finally use the basic-gons (see Fig. 7) to define the map $\mathcal{E}(\Sigma)$.

We begin with vector space $Q(\Sigma)$. Let $\partial_{\text{in}}\Sigma$ be the part parameterised by U - the in-boundary of Σ, and $\partial_{\text{out}}\Sigma$ be the part parameterised by V - the out-boundary of Σ. For $P \in C_2(\Sigma)$, we consider triples of the form $(P, e, \mathfrak{O})$ where $e \in C_1(\Sigma)$ is a 1-cell forming an edge of the polygon P, and $\mathfrak{O}$ is an orientation of e. We demand that the triple satisfy the condition that the orientation of P comes from the orientation of Σ, which in turn also orient e as a portion of ∂P. In other words, the pair $(e, \mathfrak{O})$ is a part of ∂P as an orientated edge. We will follow the outward normal first convention. Thus an edge e gets $\mathfrak{O}$ as $+1$ if the orientation on e with the outward normal first convention gives the orientation of P and -1 otherwise. Next, to each such triple $(P, e, \mathfrak{O})$ we assign a vector space:

$$Q_{(P,e,\mathfrak{O})} = \begin{cases} \mathbb{C} & \text{if } \ (e,\mathfrak{O}) \text{ does not intersect } \Sigma_1. \\ X_x & \text{if } \ (e,\mathfrak{O}) \text{ intersects } \Sigma_1 \text{ at a defect with label } x \\ & \text{and is oriiented into the polygon } P, \text{ and} \\ X_x^* & \text{if } \ (e,\mathfrak{O}) \text{ intersects } \Sigma_1 \text{ at a defect with label } x \\ & \text{and is oriented out of the polygon } P \end{cases} \tag{3.8}$$

Finally we define the vector space $Q(\Sigma)$ by:

$$Q(\Sigma) = \bigotimes_{(P,e,\mathfrak{O}),e\notin\partial_{\text{in}}\Sigma} Q_{(P,e,\mathfrak{O})} \tag{3.9}$$

Example 3.4. We again refer to Fig. 8 and want to calculate $Q(\Sigma)$ for the PLCW decomposition given there. We list the data in the following table.

$(\mathbf{P},\mathbf{e},\mathfrak{O}):$	$(P_1,e_1,-)$	$(P_1,g_1,+)$	$(p_1,l_1,-)$	$(P_2,e_2,-)$	$(P_2,l_2,+)$
$\mathbf{Q}_{(\mathbf{P},\mathbf{e},\mathfrak{O})}:$	X	$\mathbb{C}$	X^*	X	X^*

$(\mathbf{P},\mathbf{e},\mathfrak{O}):$	$(P_2,h_1,-)$	$(P_3,l_1,+)$	$(P_3,l_2,-)$	$(P_3,m,+)$	$(P_4,m,-)$
$\mathbf{Q}_{(\mathbf{P},\mathbf{e},\mathfrak{O})}:$	$\mathbb{C}$	X	X	X^*	X

$(\mathbf{P},\mathbf{e},\mathfrak{O}):$	$(P_4,r_1,-)$	$(P_4,r_2,+)$	$(p_5,g_2,+)$	$(P_5,r_1,+)$	$(P_5,f_1,+)$
$\mathbf{Q}_{(\mathbf{P},\mathbf{e},\mathfrak{O})}:$	X^*	X^*	$\mathbb{C}$	X	X^*

$(\mathbf{P},\mathbf{e},\mathfrak{O}):$	$(P_6,r_2,-)$	$(P_6,h_2,-)$	$(p_6,f_2,+)$	$(\hat{P},\hat{e},-)$	$(\hat{P},\hat{f},+)$
$\mathbf{Q}_{(\mathbf{P},\mathbf{e},\mathfrak{O})}:$	X	$\mathbb{C}$	X^*	$\mathbb{C}$	$\mathbb{C}$

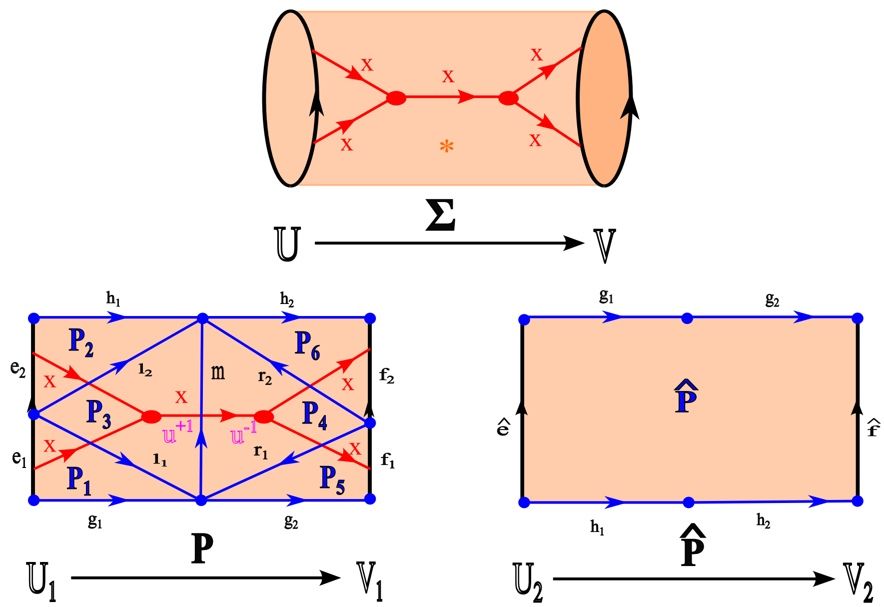


FIGURE 8. U and V both have two 0-defects labelled x. The two bottom pictures shows the PLCW decomposition using basic-gons. It has eight 0-cells (marked as small blue-circles), 15 1-cells, $C_1(M) = \{e_1, e_2, \hat{e}, g_1, g_2, h_1, h_2, l_1, l_2, m, r_1, r_2, f_1, f_2, \hat{f}\}$, and seven 2-cells, $C_2(M) = \{P_1, P_2, P_3, P_4, P_5, P_6, \hat{P}\}$.

$(\mathbf{P}, \mathbf{e}, \mathfrak{O})$:	$(\hat{P}, g_1, -)$	$(\hat{P}, g_2, -)$	$(\hat{P}, h_1, +)$	$(\hat{P}, h_2, +)$
$\mathbf{Q}_{(\mathbf{P},\mathbf{e},\mathfrak{O})}$:	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{C}$

Finally, we drop the contribution from edges e_1, e_2 and $\hat{e}$ to write

$$\begin{aligned} Q(\Sigma) = {} & Q_{(P_1,g_1,+)} \otimes Q_{(P_1,l_1,-)} \otimes Q_{(P_2,l_2,+)} \otimes Q_{(P_2,h_1,-)} \otimes Q_{(P_3,l_1,+)} \\ & \otimes Q_{(P_3,l_2,-)} \otimes Q_{(P_3,m,+)} \otimes Q_{(P_4,m,-)} \otimes Q_{(P_4,r_1,-)} \otimes Q_{(P_4,r_2,+)} \\ & \otimes Q_{(P_5,g_2,+)} \otimes Q_{(P_5,r_1,+)} \otimes Q_{(P_5,f_1,+)} \otimes Q_{(P_6,r_2,-)} \\ & \otimes Q_{(P_6,h_2,-)} \otimes Q_{(P_6,f_2,+)} \otimes Q_{(\hat{P},\hat{f},+)} \otimes Q_{(\hat{P},g_1,-)} \\ & \otimes Q_{(\hat{P},g_2,-)} \otimes Q_{(\hat{P},h_1,+)} \otimes Q_{(\hat{P},h_2,+)} \end{aligned}$$

Which in turn gives

$$\begin{aligned} Q(\Sigma) = {} & \mathbb{C} \otimes X^* \otimes X^* \otimes \mathbb{C} \otimes X \otimes X \otimes X^* \otimes X \otimes X^* \otimes X^* \otimes \mathbb{C} \otimes X \\ & \otimes X^* \otimes X \otimes \mathbb{C} \otimes X^* \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \otimes \mathbb{C} \end{aligned}$$

Where we have intentionally kept copies of $\mathbb{C}$ for now. The reasons for this will be clear soon. We also write the edges from where $\mathbb{C}$ or the vector space X are coming. Since we are only dealing with a single label this does not create any confusion. We will see in subsequent calculations (for evaluation) that it is important to keep track of the edges contributing to $Q(\Sigma)$. So, we conclude this example by just rewriting $Q(\Sigma)$ where the contributions of relevant edges have been marked correctly:

$$\begin{aligned} Q(\Sigma) = {} & \mathbb{C}_{g_1} \otimes X^*_{l_1} \otimes X^*_{l_2} \otimes \mathbb{C}_{h_1} \otimes X_{l_1} \otimes X_{l_2} \otimes X^*_m \otimes X_m \\ & \otimes X^*_{r_1} \otimes X^*_{r_2} \otimes \mathbb{C}_{g_2} \otimes X_{r_1} \otimes X^*_{f_1} \otimes X_{r_2} \otimes \mathbb{C}_{h_2} \\ & \otimes X^*_{f_2} \otimes \mathbb{C}_{\hat{f}} \otimes \mathbb{C}_{g_1} \otimes \mathbb{C}_{g_2} \otimes \mathbb{C}_{h_1} \otimes \mathbb{C}_{h_2} \end{aligned} \tag{3.10}$$

Next, we turn to the propagator $\mathcal{P}(\Sigma) : \mathbb{C} \to Q(\Sigma) \otimes T_0^{cw}(V)$. We note in Eq. (3.10) that each edge appears twice: one with X and other time with X^*.(Same holds for $\mathbb{C}$ but it has been identified with its dual.) This is not a coincidence as we are going to see below that the space $Q(\Sigma)$ has been assembled from the propagator $\mathcal{P}(\Sigma)$. Each edge $e \in C_1(\Sigma)$ appear twice (with opposite orientations): once as in-boundary of some $P \in C_2(M)$ and other time as out-boundary. The construction of the map $\mathcal{P}(\Sigma)$ is based on this fact: defining on each edge e and assembling at the end. Let us denote the two triples involving the edge e by $(P(e)_1, e, \mathfrak{O}_1)$ and $(P(e)_2, e, \mathfrak{O}_2)$. The notation means that e appears as the boundary of $P(e)_1, P(e)_2 \in C_2(\Sigma)$ with orientations $\mathfrak{O}_1$ and $\mathfrak{O}_2$ which are opposite to each other, depending on whether e is an in-boundary or an out-boundary of $P(e)_i$. We have two cases to consider.

- e is an interior edge, that is, $e \notin C_1(\Sigma) \cap \partial\Sigma$. In this case we define the linear the map
$$\mathcal{P}_e : \mathbb{C} \to Q_{(P(e)_1, e, \mathfrak{O}_1)} \otimes Q_{(P(e)_2, e, \mathfrak{O}_2)}$$
according to the following two sub-cases:
  1. If e does not intersect Σ_1 then both $Q_{(P(e)_1, e, \mathfrak{O}_1)}$ and $Q_{(P(e)_2, e, \mathfrak{O}_2)}$ is $\mathbb{C}$. In this case, we take $\mathcal{P}_e = \gamma$ where $\gamma : \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ is the copairing of the Frobenius algebra $\mathbb{C}$, which by Remark 3.3 is the identity map. Thus for $e \notin C_1(\Sigma) \cap \Sigma_1$, $\mathcal{P}_e : \mathbb{C} \to \mathbb{C}$ is the identity $\mathbf{1}_{\mathbb{C}}$.
  2. If e does intersect Σ_1 then it does so in a defect labelled x. In this case one of the $Q_{(P(e)_1, e, \mathfrak{O}_1)}$ and $Q_{(P(e)_2, e, \mathfrak{O}_2)}$ is $X_{x,e}$ and the other is $X^*_{x,e}$. If we choose to write $\mathcal{P}_e : \mathbb{C} \to X_{x,e} \otimes X^*_{x,e}$ then $\mathcal{P}_e$ is given by
$$\mathcal{P}_e(\lambda) = \lambda \sum_i v_i \otimes v_i^* \tag{3.11}$$
where $\{v_i\}$ is a basis of $X_{x,e}$ and $\{v_i^*\}$ be the corresponding dual basis of $X^*_{x,e}$, that is, $v_i^*(v_j) = \delta^i_j$. Here we have denoted the vector space X_x by $X_{x,e}$ to keep track of the edge.

- If e is such that $e \in C_1(\Sigma) \cap \partial_{\text{out}}\Sigma$ then there is exactly one of the triple $(P, e, \mathfrak{O})$ contains e. Let us call such a triple $(P(e), e, \mathfrak{o})$, and define

$$\mathcal{P}_e = Q_{P(e),e,\mathfrak{O}} \otimes R_e \tag{3.12}$$

Note that if one of $Q_{P(e),e,\mathfrak{O}}$ and R_e gets $X_{x,e}$, the other will get its dual $X_{x,e}$.

Altogether, the propagator $\mathcal{P}(\Sigma)$ is defined by

$$\mathcal{P}(\Sigma) = \bigotimes_{e \in C_1(\Sigma), e \notin \partial_{\text{in}}\Sigma} \mathcal{P}_e \tag{3.13}$$

Example 3.5. We again refer to Fig. 8 and Example 3.4 and compute the propagator $\mathcal{P}(\Sigma)$. To do that we list all the individual maps $\mathcal{P}_e$:

$$\begin{array}{ccc} \mathcal{P}_{g_1} : \mathbb{C} \to \mathbb{C}_{g_1} \otimes \mathbb{C}_{g_1} & \mathcal{P}_{l_1} : \mathbb{C} \to X_{l_1} \otimes X_{l_1}^* & \mathcal{P}_{l_2} : \mathbb{C} \to X_{l_2} \otimes X_{l_2}^* \\ \mathcal{P}_{h_1} : \mathbb{C} \to \mathbb{C}_{h_1} \otimes \mathbb{C}_{h_1} & \mathcal{P}_{m} : \mathbb{C} \to X_{m} \otimes X_{m}^* & \mathcal{P}_{g_2} : \mathbb{C} \to \mathbb{C}_{g_2} \otimes \mathbb{C}_{g_2} \\ \mathcal{P}_{r_1} : \mathbb{C} \to X_{r_1} \otimes X_{r_1}^* & \mathcal{P}_{r_2} : \mathbb{C} \to X_{r_2} \otimes X_{r_2}^* & \mathcal{P}_{h_2} : \mathbb{C} \to \mathbb{C}_{h_2} \otimes \mathbb{C}_{h_2} \\ \mathcal{P}_{f_1} : \mathbb{C} \to X_{f_1} \otimes X_{f_1}^* & \mathcal{P}_{f_2} : X_{f_2} \otimes X_{f_2}^* & \mathcal{P}_{\hat{f}} : \mathbb{C} \to \mathbb{C}_{\hat{f}} \otimes \mathbb{C}_{\hat{f}} \end{array} \tag{3.14}$$

We see, after arranging Eq. (3.10) that $\mathcal{P}(\Sigma) : \mathbb{C} \to Q(\Sigma) \otimes T_0^{cw}(V)$.

Finally, we are going to define the evaluation map $\mathcal{E}(\Sigma) : T_0^{cw}(U) \otimes Q(\Sigma) \to \mathbb{C}$. By adjoining $T_0^{cw}(U)$ to $Q(\Sigma)$, we have gathered all the $Q_{(P,e,\mathfrak{o})}$ from the table in Example 3.4 which we dropped when writing the expression of $Q(\Sigma)$, that is,

$$T_0^{cw}(U) \otimes Q(\Sigma) = \bigotimes_{P \in C_2(M), (e,\mathfrak{O}) \in \partial P} Q_{(P,e,\mathfrak{O})}$$

Therefore, for each polygon P we define a $\mathbb{C}$-linear map

$$\mathcal{E}_P : \bigotimes_{(e,\mathfrak{O}) \in P} Q_{(P,e,\mathfrak{O})} \to \mathbb{C} \tag{3.15}$$

according to the following three cases depending on what kind of defects does P contains.

1. P does not intersect Σ_0 or Σ_1. In this case we get $\mathcal{E} : \otimes^n \mathbb{C} \to \mathbb{C}$ which is given by $\mathcal{E}_P(c_1 \otimes \cdots \otimes c_n) = \epsilon_{\mathbb{C}}(c_1 \ldots c_m)$ where $\epsilon_{\mathbb{C}}$ is the counit from Remark 3.3 which we saw to be identity. So, under the identification of the space $\otimes^n \mathbb{C}$ with $\mathbb{C}$ we see that $\mathcal{E}_P : \mathbb{C} \to \mathbb{C}$ is simply the identity map $\mathbf{1}_{\mathbb{C}}$.
2. P intersects Σ_1 but not Σ_0. By the definition of $Bord_2^{def,\, cw}(\mathcal{D})$, it must resemble Fig. 7(iii), that is, there is only one such component of Σ_1. Let it be x. There is one oriented edge where x leaves P. If we choose to denote this (oriented) edge by $(e_1, \mathfrak{O}_1)$ then $Q_{(P,e_1,\mathfrak{O}_1)}$ equals X_x^*. Starting from this

edge we traverse the edges of ∂P in anti-clockwise manner. The linear map $\mathcal{E}_P$ then takes the form

$$\mathcal{E}_P : X_x^* \otimes (\otimes^{n_1}\mathbb{C}) \otimes X_x \otimes (\otimes^{n_2}\mathbb{C}) \to \mathbb{C}$$

We set

$$\begin{aligned}\mathcal{E}_P(v_x^* \otimes c_1 \otimes \cdots \otimes c_{n_1} \otimes w_x \otimes c_{n_1+1} \otimes \cdots \otimes c_{n_1+n_2}) \\ = v_x^*((c_1 \dots c_{n_1}) w_x (c_{n_1+1} \dots c_{n_1+n_2}))\end{aligned} \tag{3.16}$$

By linearity it becomes $(c_1 \dots c_{n_1}) v_x^*(w_x)(c_{n_1+1} \dots c_{n_1+n_2})$, which also shows that in case of a trivial theory, we could have picked-up any edge of P.

(3) As the last case suppose P does contains a component of Σ_0. Then by the definition of $Bord_2^{def,\ cw}(\mathcal{D})$, it must look like Fig. 7 (iv). Explicitly, each oriented edge $(e_i, \mathfrak{O}_i) \in \partial P$ intersects Σ_1. We choose an arbitrary edge $(e_1, \mathfrak{O}_1)$ and order the remaining edges in an anti-clockwise manner. Let $u^\epsilon \in X_0$ be the label at the only element of $\Sigma_0 \cap P$, and $\psi_{0,1}(u^\epsilon) = [(x_1^{\epsilon_1}, \dots, x_n^{\epsilon_n})]$. If $\epsilon = +1$ then the TFT T_0^{cw} assigns u^{+1} an element $\mu_u \in \circlearrowleft_{\text{Inv}} \text{Hom}_{\mathbb{K}}(X_{x_1}^{\epsilon_1} \otimes \cdots \otimes X_{x_n}^{\epsilon_n}, \mathbb{K})$. We set $\mathcal{E}_P(v_1 \otimes \cdots \otimes v_n) = \mu_u(v_1 \otimes \cdots \otimes v_n)$. Note that this is independent of the choice of $(e_1, \mathfrak{O}_1)$ since$\circlearrowleft_{\text{Inv}} \text{Hom}_{\mathbb{K}}(X_{x_1}^{\epsilon_1} \otimes \cdots \otimes X_{x_n}^{\epsilon_n}, \mathbb{K})$is defined that way. See Definition 3.3 for detail.

If $\epsilon = -1$ then we repeat the same argument but with the class $[(x_n^{-\epsilon_n}, \dots, x_1^{-\epsilon_1})]$.

Example 3.6. We continue to evaluate the TFT assigned to Fig. 8. Moving ahead of Example 3.5 we now write $\mathcal{E}_P$ for polygons in $C_2(\Sigma)$. The only polygon of type-(1) is $\hat{P}$, for which

$$\begin{aligned}\mathcal{E}_{\hat{P}} :\quad & Q_{(\hat{P},\hat{e},-)} \otimes Q_{(\hat{P},h_1,+)} \otimes Q_{(\hat{P},h_2,+)} \otimes Q_{(\hat{P},g_1,-)} \otimes Q_{(\hat{P},g_2,-)} \otimes Q_{(\hat{P},\hat{f},+)} \to \mathbb{C} \\ & \mathbb{C}_{\hat{e}} \otimes \mathbb{C}_{h_1} \otimes \mathbb{C}_{h_2} \otimes \mathbb{C}_{g_1} \otimes \mathbb{C}_{g_2} \otimes \mathbb{C}_{\hat{f}} \longrightarrow \mathbb{C} \\ & \lambda_{\hat{e}} \otimes \lambda_{h_1} \otimes \lambda_{h_2} \otimes \lambda_{g_1} \otimes \lambda_{g_2} \otimes \lambda_{\hat{f}} \longmapsto \lambda_{\hat{e}}\lambda_{h_1}\lambda_{h_2}\lambda_{g_1}\lambda_{g_2}\lambda_{\hat{f}}\end{aligned} \tag{3.17}$$

P_1, P_2, P_5 and P_6 is of type-(2):

$$\begin{aligned}\mathcal{E}_{P_1} :\quad & Q_{(P_1,l_1,-)} \otimes Q_{(P_1,e_1,-)} \otimes Q_{(P_1,g_1,+)} \to \mathbb{C} \\ & X_{l_1}^* \otimes X_{e_1} \otimes \mathbb{C}_{g_1} \longrightarrow \mathbb{C} \\ & v^* \otimes w \otimes \lambda_{g_1} \longmapsto \lambda_{g_1} v^*(w)\end{aligned} \tag{3.18}$$

and similarly for P_2, P_3 and P_6. In this case all of these maps are same except for the indexing. Finally, P_3 and P_4 is of type-(3).Since $\psi_{0,1}(u) = [(x, x, x^{-1})]$ T_0^{cw} assigns u some $\mu_1 \in \circlearrowleft_{\text{Inv}} \text{Hom}_{\mathbb{C}}(X_{l_1} \otimes X_{l_2} \otimes X_m^*, \mathbb{C})$ where all of X_m, X_{l_1} and X_{l_2} are X_x.

$$\begin{aligned} \mathcal{E}_{P_3}: \quad Q_{(P_3,l_1,+)} \otimes Q_{(P_3,l_2,-)} \otimes Q_{(P_3,m,+)} &\to \mathbb{C} \\ X_{l_1} \otimes X_{l_2} \otimes X_m^{ast} &\longrightarrow \mathbb{C} \\ v_1 \otimes v_2 \otimes v_3 &\longmapsto \mu_1(v_1 \otimes v_2 \otimes v_3) \end{aligned} \tag{3.19}$$

Next we turn to P_4, the TFT T_0^{cw} assigns u^{-1} an element $\mu_2 \in \circlearrowleft_{\mathrm{Inv}} \mathrm{Hom}_{\mathbb{C}}(X_{r_1}^* \otimes X_m \otimes X_{r_2}^*, \mathbb{C})$. Again, all of X_m, X_{r_1} and X_{r_2} are X_x.

$$\begin{aligned} \mathcal{E}_{P_4}: \quad Q_{(P_4,r_1,-)} \otimes Q_{(P_4,m,-)} \otimes Q_{(P_4,r_2,+)} &\to \mathbb{C} \\ X_{r_1}^* \otimes X_m \otimes X_{r_2}^{ast} &\longrightarrow \mathbb{C} \\ v_1 \otimes v_2 \otimes v_3 &\longmapsto \mu_2(v_1 \otimes v_2 \otimes v_3) \end{aligned} \tag{3.20}$$

We conclude this example by pointing out that the map μ from Example 3.2 can take place of both μ_1 and μ_2. This map will be useful in next sections.

This finalises the construction of a lattice TFT with a trivial surrounding theory. All these conditions can be deduced from the lattice TFT construction in [DKR11] with the special case when the trivial Frobenius algebra, namely $\mathbb{K}$ is assigned to all the two dimensional strata. Therefore the trivial surrounding theory enjoys all facilities as its more general counterpart - lattice TFT. In particular, the trivial surrounding theory is independent of the choice of a PLCW decomposition. Two such cell-decomposition is forced to look similar in the vicinity of a defect by Fig. 7 but it can always be refined, and altered in many ways. We refer to [DKR11] and [KJ12] for details. Alternatively, one could simply define the map $\mathcal{P}$ and $\mathcal{E}$ by declaring identity on regions with no lower-dimensional defects (that is without resorting to the Frobenius algebra property of $\mathbb{C}$) and using just the property of vector space and then proving the independence on cell-decomposition from scratch, following section-7 and 8 of [KJ12]. However, we will not take this approach here.

3.3. **Some useful results.** Now, we are going to state and prove results that is going to simply the calculation in the case of a trivial surrounding theory. All the proofs in this section relies on two facts. First, that the TFT is independent of a PLCW decomposition. Second, a trivial surrounding theory assigns $\mathbb{C}$ to two-dimensional strata.

let $C_{uv} : U \to V$ be the morphism from U to V such that underlying surface with boundary is a cylinder. Since, it has to respect the distinguished point, there exists a PLCW decomposition decomposing C_{uv} into two polygons M and $\hat{M}$, where $\hat{M}$ is such that it contains no 0 or 1 dimensional strata.(See Fig. 9) for a visual demonstration.) Under this decomposition U (respectively V) decomposes as $U_1 \cup U_2$ (respectively $V_1 \cup V_2$) such that U_1 (respectively V_1) is the restriction of U (respectively V) to M, and U_2 (respectively V_2) is the restriction of U (respectively V) to $\hat{M}$. Let $T_0^{cw}(U_1) = \otimes_{e \in C_1(U_1)} R_e$ and $T_0^{cw}(V_1) = \otimes_{f \in C_1(V_1)} R_f$ and we define $\mathcal{P}(M)$ by keeping

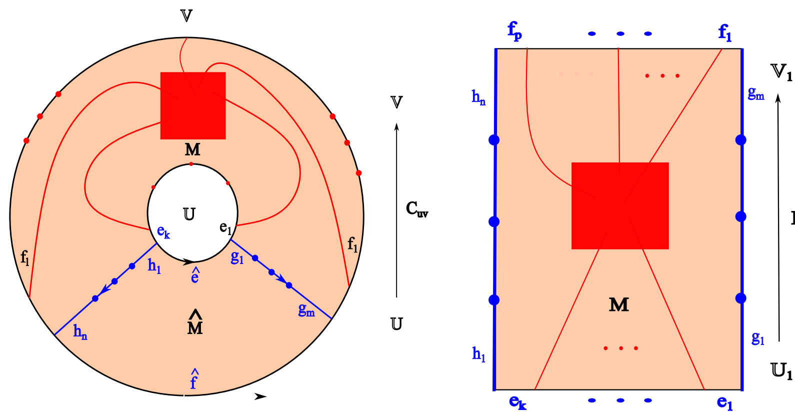


FIGURE 9. We have the cylinder as a morphism in the left, and on right is the polygon containing all the defects of C_{uv} $C_1(U_1) = \{e_1, \ldots, e_k\}$ which forms $C_1(U)$ together with $\hat{e}$. Similarly, $C_1(V_1) = \{f_1, \ldots, f_p\}$ which with $\hat{f}$ forms $C_1(V)$. The red dots $(\ldots)$ depicts the continuation of 1-dimensional defects in that region. The cells $\{h_1, \ldots, h_n\}$ and $\{g_1, \ldots, g_m\}$ are the 1-cells of $C_1(C_{uv})$ that forms the two common boundaries of M and $\hat{M}$

only contributions from M, that is, if $e \in C_1(\partial M) \cap C_1(\partial \hat{M})$, we define a truncated propagator $\mathcal{P}'_e : \mathbb{C} \to Q_{(P(e),e,\mathfrak{D})}$ where $P(e) \in C_2(M)$ and define

(3.21)
$$\mathcal{P}(M) := \Big(\bigotimes_{e \notin \hat{M}, e \notin \partial_{\text{in}} C_{uv}, e \in C_1(C_{uv})} \mathcal{P}_e \Big) \otimes_{e \in C_1(\partial \hat{M})} \mathcal{P}'_e \quad , \quad \mathcal{E}(M) := \bigotimes_{P \in C_2(C_{uv}), P \neq \hat{M}} \mathcal{E}_P$$

and $Q(M)$ be the restriction of $Q(C_{uv})$ to the codomain of $P(M)$.

The following theorem says that under such conditions, the calculation can be done on planar polygon M.

Proposition 3.2. If T_0^{cw} is a trivial surrounding theory then $T_0^{cw}(C_{uv}) : T_0^{cw}(U) \to T_0^{cw}(V)$ equals $T_0^{cw}(M) : T_0^{cw}(U_1) \to T_0^{cw}(V_1)$. Where $T_0^{cw}(M)$ is defined is the composite:

$$T_0^{cw}(U_1) \xrightarrow{\mathbf{1}_{T_0^{cw}(U_1)} \otimes P(M)} T_0^{cw}(U_1) \otimes Q(M) \otimes T_0^{cw}(V_1) \xrightarrow{E(M) \otimes \mathbf{1}_{T_0^{cw}(V_1)}} T_0^{cw}(V_1) \tag{3.22}$$

Proof. Let's denote the single edge covering U_2 (respectively V_2) by $\hat{e}$ (respectively $\hat{f}$). The map $T^{cw}(C_{uv}) : T^{cw}(U) \to T^{cw}(V)$ is given by

$$T^{cw}(C_{uv}) : T^{cw}(U) \xrightarrow{\mathbf{1}_{T^{cw}(U)} \otimes P(C_{uv})} T^{cw}(U) \otimes Q(C_{uv}) \otimes T^{cw}(V) \xrightarrow{E(C_{uv}) \otimes \mathbf{1}_{T^{cw}(V)}} T^{cw}(V) \tag{3.23}$$

The key step is to write

$$\mathcal{E}(C_{uv}) = (\bigotimes_{P \in C_2(C_{uv}), P \neq \hat{M}} \mathcal{E}_p) \otimes \mathcal{E}_{\hat{M}}$$

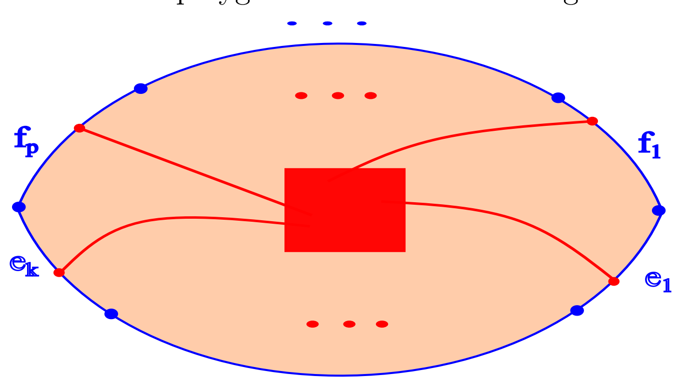
and arrange the middle term in the Eq. (3.23) according to it. We get $T_0^{cw}(U) = T_0^{cw}(U_1) \otimes R_{\hat{e}}$ and $T_0^{cw}(V) = T_0^{cw}(V_1) \otimes R_{\hat{f}}$. The propagators $\mathcal{P}_{g_i} : \mathbb{C} \to \mathbb{C}_{g_i} \otimes \mathbb{C}_{g_i}$ has the form $1 \mapsto 1_{g_i} \otimes 1_{g_i}$ and similarly for h_i for every i. Let P_{g_i} (respectively P_{h_i}) be the polygons containing g_i (respectively h_i). One of the two factors from $\mathcal{P}_{g_i}(1)$ (respectively $\mathcal{P}_{h_i}(1)$) goes with $\mathcal{E}_{P_{g_i}}$ (respectively $\mathcal{E}_{P_{h_i}}$) and the other with $\mathcal{E}_{\hat{P}}$ reducing Eq. (3.23) to:

$$\begin{aligned} T_0^{cw}(U_1) \otimes \mathbb{C}_{\hat{e}} \xrightarrow{\mathbf{1}_{T_0^{cw}(U_1)} \otimes P(M)} T_0^{cw}(U_1) \otimes Q(M) \otimes \mathbb{C}_{\hat{e}} \otimes (\otimes_{g_i} \mathbb{C}_{g_i}) \otimes (\otimes_{h_i} \mathbb{C}_{h_i}) \\ \otimes T_0^{cw}(V_1) \otimes \mathbb{C}_{\hat{f}} \xrightarrow{E(M) \otimes \mathbf{1}_{T_0^{cw}(V_1)}} T_0^{cw}(V_1) \otimes \mathbb{C}_{\hat{f}} \end{aligned} \tag{3.24}$$

Which gives the desired result since for a $\mathbb{C}$-vector space X, $X \otimes \mathbb{C} \cong \mathbb{C}$.

□

Theorem 3.1. The calculation of a trivial surrounding theory $T_0^{cw}(C_{uv})$ as in Proposition 3.2 can be done on the polygon of the kind M in Fig. 9 as shown below:



(3.25)

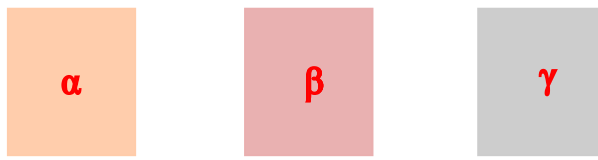
Proof. This is simple application of the fact that the TFT is trivial, it assigns a $\mathbb{C}$ vector space to co-dimension 1 strata, and the definition of $\mathcal{E}_P$. First, note that 3.25 is obtained from M in Fig. 9 by identifying all the zero cells on non-object sides $(g_1, \ldots, g_m)$ and $(h_1, \ldots, h_n)$ to (two distinct) single 0-cells. Then, since $\mathcal{P}$ assigns $\mathbb{C}$ to these edges and both v^* and μ_v are $\mathbb{C}$-linear, one can take the contribution from these edges out, for instance, Eq. (3.16) can be written as $(c_1 \ldots c_{n_1}) V_x^*(w_x)(c_{n_1+1} \ldots c_{n_1+n_2})$, which is same as if the polygon had only two sides, one where the defect enters and the other where the defect leaves, since c_i are scalars. If (say) $g_1, \ldots, g_m$ are vertices of one of the basic-gons with a 0-defects, then each cell containing g_i and g_{i+1} is of the kind discussed in Eq. (3.16) and thus can be identified. Repeating this process identifies each g_i and the same holds for h_i. □

To state our next proposition we need to use the fact that the category of 2-defect TFT is equivalent to a *Pivotal 2-category*. We do not make this construction explicit here and refer to [Car16] (2.2) and (2.3). In what follows, we only highlight the key features of this construction.

For a defect TFT $T : Bord_2^{def,\ cw}(\mathcal{D}) \to \mathrm{Vect}_F(\mathbb{K})$, the data of the 2-category $\mathcal{B}_T$ consists of:

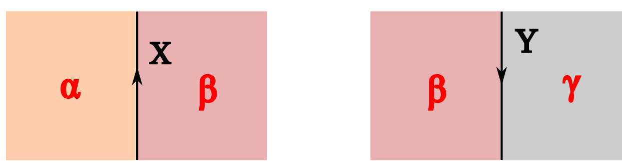
- (level-0) the class $\mathrm{Obj}(\mathcal{B}_T) = D_2$. The string diagram is shown below:

(3.26)



- (level-1) Given two objects $\alpha, \beta \in D_2$, a $\mathbb{K}$-linear category $\mathcal{B}_T(\alpha, \beta)$ whose objects are 1-morphisms $X : \alpha \to \beta$ and as a category it is a free category generated by the pre-category given by maps $s, t : D_1 \to D_2$. (See Definition 2.2). As string diagram:

(3.27)

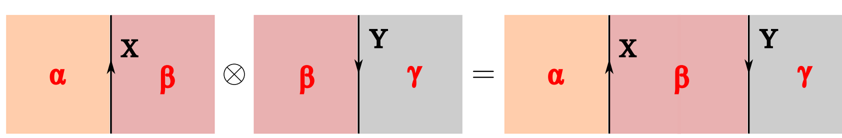


We have morphism in this category, a $\mathbb{K}$-linear map

$$\mathcal{B}_T(\beta, \gamma) \otimes \mathcal{B}_T(\alpha, \beta) \to \mathcal{B}_T(\alpha, \gamma)$$

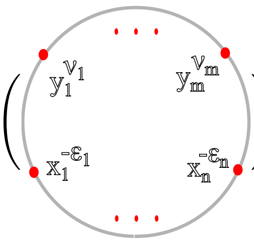
called 'fusion' and represented as the string diagram:

(3.28)

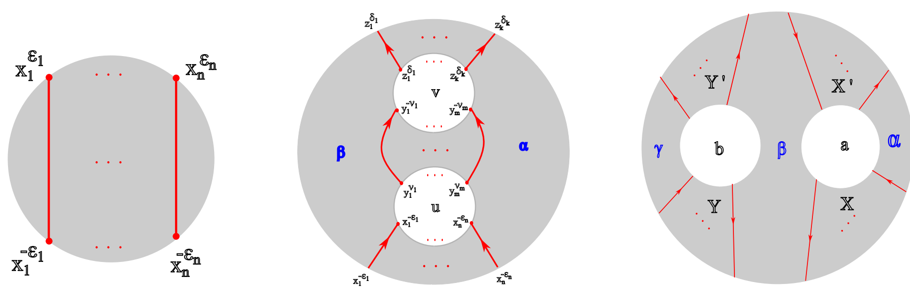


The output in Eq. (3.28) is usually written as $X \otimes Y^{-1} : \alpha \to \gamma$.

- (level-2) The $\mathbb{K}$-linear space of 2-morphism between $X := (x_1^{\epsilon_1}, \dots x_n^{\epsilon_n}) : \alpha \to \beta$ and $Y := (y_1^{\nu_1}, \dots, y_m^{\nu_m}) : \alpha \to \beta$, $\mathrm{Hom}(X, Y)$ is given by $T(Y \otimes X^{-1})$:

(3.29) $Hom(X, Y) = T\Big($  $\Big)$

This does actually correspond to the local operators inserted at defect junctions. We refer to [Car16] (2.17) for details on how to build the set D_0 from this data. For now, we only mention that one should think this space as the space of discs with at most one labelled vertex available to fill-in. For example when $Y = X$ the identity 2-morphism $\mathbf{1}_X$, which is an element of $\mathrm{Hom}(X, X)$ is given by (i) below.



(i) (ii) (iii)

(ii) shows the *vertical composition* $v \circ u$ of two 2-morphisms $u \in \mathrm{Hom}(X, Y)$ and $v \in \mathrm{Hom}(Y, Z)$, while (iii) shows the *horizontal composition* $b \otimes a$ of $b \in \mathrm{Hom}(Y, Y')$ and $a \in \mathrm{Hom}(X, X')$. Note that we have used the same symbol $\otimes$ for both horizontal composition and fusion as they coincide.

Remark 3.4. A consequence of the functoriality of fusion $\otimes$ is that the horizontal and vertical composition satisfies the *interchange law*: For $\psi \in \mathrm{Hom}(Y, Y'), \phi \in \mathrm{Hom}(X, X')$

$$\psi \otimes \phi = (\psi \otimes \mathbf{1}_X) \circ (\mathbf{1}_{Y'} \otimes \phi) = (\mathbf{1}_Y \otimes \phi) \circ (\psi \otimes \mathbf{1}_{X'}) \tag{3.30}$$

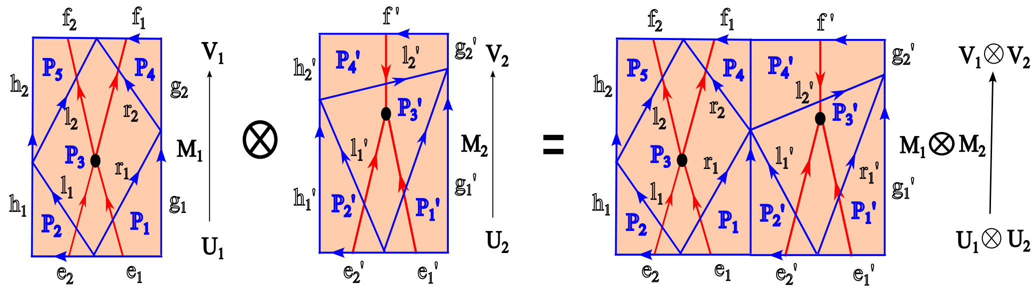
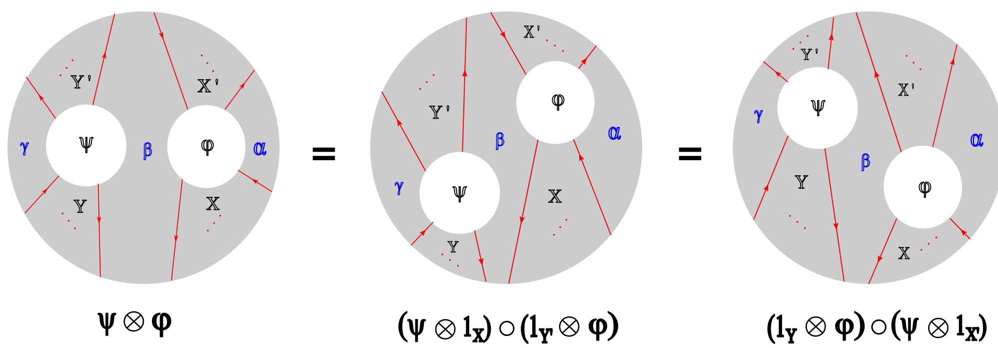


FIGURE 10. Despite being an element of the same category their cell-decomposition may produce extra cells when fusing. One can subdivide to get a cell decomposition on $M_1 \otimes M_2$ that restricts to the given cell-decomposition on both M_1 and M_2



Now, we return to the case of lattice TFT and trivial surrounding theory and can state the following proposition

Proposition 3.3. Let $M_1 : U_1 \to V_1$ and $M_2 : U_2 \to V_2$ be two bordism restricted to respective polygons. With the definition of $T_o^{cw}(M)$ as in Proposition 3.2, for the fusion

$$M_1 \otimes M_2 : U_1 \otimes U_2 \to V_1 \otimes V_2$$

we get

$$T_0^{cw}(M_1 \otimes M_2) = T_o^{cw}(M_1) \otimes T_o^{cw}(M_2)$$

Proof. Choose a cell decomposition of $M_1 \otimes M_2$ such that it gives a cell-decomposition of both M_1 and M_2 and such that the same holds for U with respect to U_1 and U_2, and for V with respect to V_1 and V_2. This is always possible after suitable refinements of cell decompostions of M_1 and M_2. See Fig. 10 for an example. With this we get the data as follows:

$$
\begin{aligned}
T_0^{cw}(U) &= \bigotimes_{e\in C_1(U)} R_e \\
&= \bigotimes_{e\in C_1(U_1)} R_e \bigotimes_{e'\in C_1(U_2)} (R_{e'}) \qquad (3.31)\\
&= T_0^{cw}(U_1)\otimes T_0^{cw}(U_2)
\end{aligned}
$$

Similarly, we get $T_0^{cw}(V) = T_0^{cw}(V_1)\otimes T_0^{cw}(V_2)$. Furthermore,

(3.32)

$$
\begin{aligned}
\mathcal{P}(M_1\otimes M_2) &= \bigotimes_{\substack{e\in C_1(M_1\otimes M_2)\\ e\notin\partial_{in}(M_1\otimes M_2)}} \mathcal{P}_e \\
&= \bigotimes_{\substack{e\notin\partial_{in}M_1\\ e\in C_1(M_1)}} \mathcal{P}_e \bigotimes_{\substack{e'\notin\partial_{in}M_2\\ e'\in C_1(M_2)}} \mathcal{P}_{e'} \\
&= \mathcal{P}_{M_1}\otimes\mathcal{P}_{M_2}
\end{aligned}
\qquad
\begin{aligned}
\mathcal{E}(M_1\otimes M_2) &= \bigotimes_{P\in C_2(M_1\otimes M_2)} \mathcal{E}_P \\
&= \bigotimes_{P\in C_2(M_1)} \mathcal{E}_p \bigotimes_{P'\in C_2(M_2)} \mathcal{E}_{P'} \\
&= \mathcal{E}(M_1)\otimes\mathcal{E}(M_2)
\end{aligned}
$$

Note that while defining the propagator $\mathcal{P}(M)$ in Eq. (3.21) we kept only one copy of $Q_{(P(e'),e',\mathfrak{O})}$ for an externel edge e' of the polygon M. We get two copies of $Q_{(P(e'),e',\mathfrak{O})}$ with opposite signs $\mathfrak{O}$ from this edge in $M_1\otimes M_2$ - one from each of M_1 and M_2; as one would expect from en internal edge.

Next, $T_0^{cw}(M_1\otimes M_2)$ is given by the composition

$$
\begin{aligned}
T_0^{cw}(U_1\otimes U_2) \xrightarrow{\mathbf{1}\otimes\mathcal{P}(M_1\otimes M_2)} T_0^{cw}(U_1\otimes U_2)\otimes Q(M_1\otimes M_2)\otimes T_0^{cw}(V_1\otimes V_2)\\
\xrightarrow{E(M_1\otimes M_2)\otimes\mathbf{1}} T_0^{cw}(V_1\otimes V_2)
\end{aligned}
$$

Using Eq. (3.31), Eq. (3.32) and $Q(M_1\otimes M_2) = Q(M_1)\otimes Q(M_2)$ we get

$$
\begin{aligned}
T_0^{cw}(U_1)\otimes T_0^{cw}(U_2) \xrightarrow{\mathbf{1}\otimes\mathcal{P}(M_1)\otimes\mathcal{P}(M_2)} T_0^{cw}(U_1)\otimes T_0^{cw}(U_2)\otimes Q(M_1)\otimes Q(M_2)\\
\otimes T_0^{cw}(V_1)\otimes T_0^{cw}(V_2) \xrightarrow{\mathcal{E}(M_1)\otimes\mathcal{E}(M_2)\otimes\mathbf{1}} T_0^{cw}(V_1)\otimes T_0^{cw}(V_2)
\end{aligned}
\tag{3.33}
$$

After arranging it gives

$$
\begin{aligned}
T_0^{cw}(U_1)\otimes T_0^{cw}(U_2) \xrightarrow{\mathbf{1}\otimes\mathcal{P}(M_1)\otimes\mathcal{P}(M_2)} T_0^{cw}(U_1)\otimes Q(M_1)\otimes T_0^{cw}(V_1)\otimes T_0^{cw}(U_2)\\
\otimes Q(M_2)\otimes\otimes T_0^{cw}(V_2) \xrightarrow{\mathcal{E}(M_1)\otimes\mathcal{E}(M_2)\otimes\mathbf{1}} T_0^{cw}(V_1)\otimes T_0^{cw}(V_2)
\end{aligned}
\tag{3.34}
$$

Now, look at the following composition of maps:

$$\begin{aligned}(3.35)\qquad & T_0^{cw}(U_1)\otimes T_0^{cw}(U_2)\xrightarrow{\mathbf{1}\otimes\mathcal{P}(M_1)\otimes\mathbf{1}} T_0^{cw}(U_1)\otimes Q(M_1)\otimes T_0^{cw}(V_1)\\ & \otimes T_0^{cw}(U_2)\xrightarrow{\mathcal{E}(M_1)\otimes\mathbf{1}\otimes\mathbf{1}} T_0^{cw}(V_1)\otimes T_0^{cw}(U_2)\xrightarrow{\mathbf{1}\otimes\mathbf{1}\otimes\mathcal{P}(M_2)} T_0^{cw}(V_1)\\ & \otimes T_0^{cw}(U_2)\otimes Q(M_2)\otimes T_0^{cw}(V_2)\xrightarrow{\mathbf{1}\otimes\mathcal{E}(M_2)\otimes\mathbf{1}} T_0^{cw}(V_1)\otimes T_0^{cw}(V_2)\end{aligned}$$

This is the map $(\mathbf{1}\otimes T_0^{cw})\circ(T_0^{cw}\otimes\mathbf{1})$, but this also equals:

$$(\mathbf{1}\otimes\mathcal{E}(M_2)\otimes\mathbf{1})\circ(\mathbf{1}\otimes\mathbf{1}\otimes\mathcal{P}(M_2))\circ(\mathcal{E}(M_1)\otimes\mathbf{1}\otimes\mathbf{1})\circ(\mathbf{1}\otimes\mathcal{P}(M_1)\otimes\mathbf{1})$$

Two terms in the middle can be interchanged as a consequence of the interchange law in the monoidal category $\mathrm{Vect}_F(\mathbb{C})$. This gives:

$$(\mathbf{1}\otimes\mathcal{E}(M_2)\otimes\mathbf{1})\circ(\mathcal{E}(M_1)\otimes\mathbf{1}\otimes\mathbf{1})\circ(\mathbf{1}\otimes\mathbf{1}\otimes\mathcal{P}(M_2))\circ(\mathbf{1}\otimes\mathcal{P}(M_1)\otimes\mathbf{1})$$

, but that equals by functoriality of $\otimes$ to:

$$(\mathcal{E}(M_1)\otimes\mathcal{E}(M_2)\otimes\mathbf{1})\circ(\mathbf{1}\otimes\mathcal{P}(M_1)\otimes\mathcal{P}(M_2))$$

Comparing this with Eq. (3.33) we get the desired result.

□

Alternatively, one could have first proven (again using a clever PLCW decomposition) $T_0^{cw}(1\otimes M)=\mathbf{1}\otimes T_0^{cw}(M)$ and then functoriality and interchange law to prove Proposition 3.3

$$\begin{aligned}T_0^{cw}(M_1\otimes M_2) &= T_0^{cw}((M_1\otimes 1)\circ(1\otimes M_2))\\ &= T_0^{cw}((M_1\otimes 1))\circ T_0^{cw}((1\otimes M_2))\ \text{[functoriality of } T_0^{cw}]\\ &= (T_0^{cw}(M_1)\otimes\mathbf{1})\circ(\mathbf{1}\otimes T_0^{cw}(M_2))\\ &= T_0^{cw}(M_1)\otimes T_0^{cw}(M_2)\ \text{[functoriality of } \otimes]\end{aligned}$$

4. Surface of defects from a Group presentation

In reference to Fig. 7 one can easily see that one can read off all the ingredients of the defect conditions $\mathcal{D}$ of the category $Bord_2^{\text{def}}(\mathcal{D})$ by looking at all the basic-gons in the category $Bord_2^{\text{def, cw}}(\mathcal{D})$. The basic-gons of type (iii) gives the map $\psi_{1,2}$, while the map $\psi_{0,1}$ is obtained from the basic gons of type (iv) as explained below Fig. 7. Moreover, if $t(x)=s(x)$ for some $x\in D_1$, then x can not be distinguished from x^{-1} and we can get rid of direction (orientation; note that this condition is trivially satisfied if D_2 is a singleton.) In other words, the category $Bord_2^{\text{def, cw}}(\mathcal{D})$ can be fully specified by specifying all the basic-gons that can appear in this category. Hence,

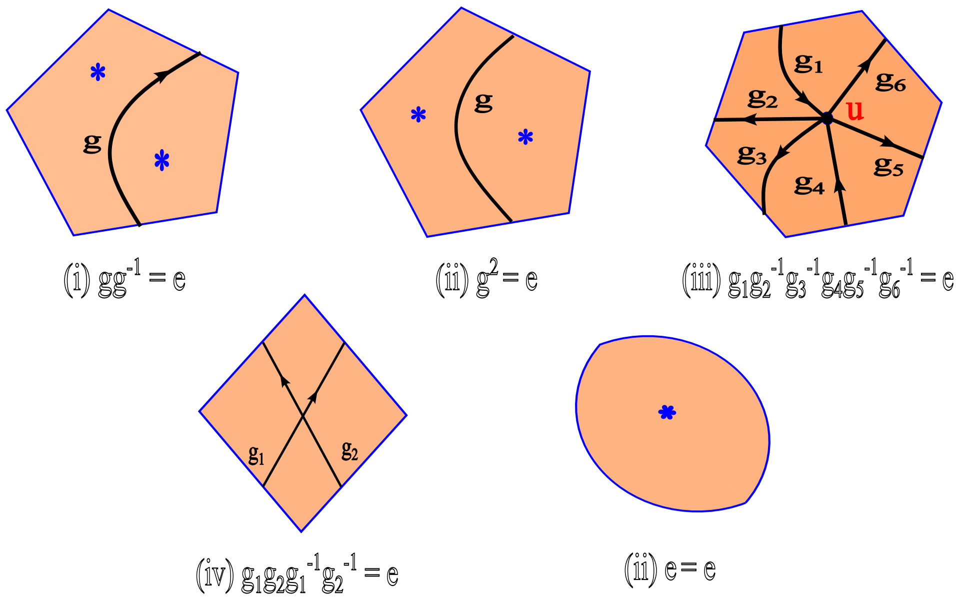


Figure 11. The condition for (ii) is trivially satisfied when D_2 is a singleton. (iv) is a depiction of permutation of defects which is a part of morphism. There is a special vertex there which is characterised by the property that it is idempotent under the operation of vertical-composition. See the caption below Fig. 12.

using the forgetful functor in Eq. (4.3) the category $Bord_2^{def}(\mathcal{D})$ can also be fully specified in this manner.

With that in mind, given a group G and a presentation $P_G \coloneqq \langle B_G \mid R_G \rangle$ we define a collection of basic-gons for each (trivial or non-trivial) relations in R_G as shown in Fig. 11.

Proposition 4.1. The basic-gons defined in Fig. 11 defines a set of defect conditions.

Proof. We need to check that the maps $\psi_{1,2}$ and $\psi_{0,1}$ are well-defined and satisfies the orientation consistency conditions of Definition 2.2. The set D_2 is singleton. Thus the map $\psi_{1,2}$ well defined and trivially satisfies the orientation consistency condition. Well-definition of $\psi_{0,1}$ follows from the following easy fact:

- If a word $g_{i_1}^{\epsilon_1} \dots g_{i_n}^{\epsilon_1}$ is in R_G then so is any cyclic permutation of it.

For the orientation consistency condition we use Eq. (2.2). $\psi_{0,1}(u^{-1})$ is given by inverting the word $\psi_{0,1}$, which is also in R_G.

□

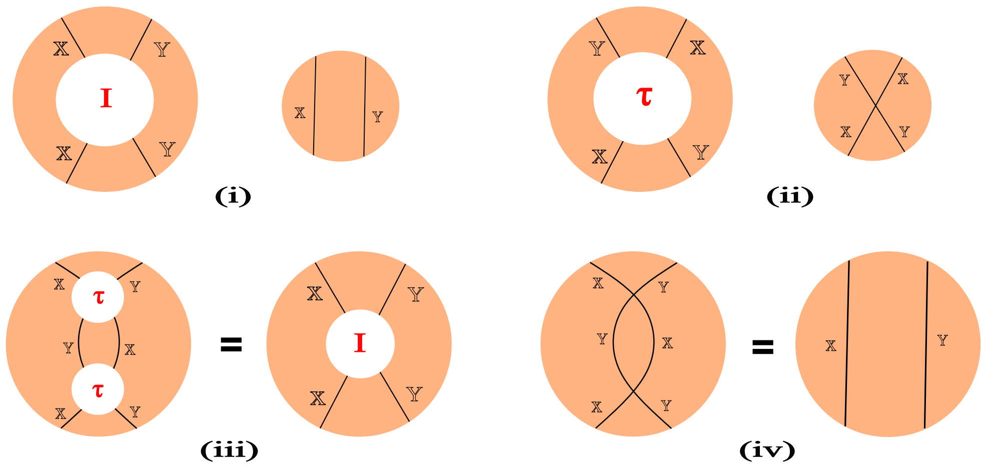


FIGURE 12. (ii) represent a 2-morphism $\tau : X \otimes Y \to Y \otimes X$ in the same manner as (i) represent the identity 2-morphism $I : X \otimes Y \to X \otimes Y$. (iv) shows the interpretation of (iii) in terms of filled-in discs analogy of 2-morphism mentioned in the paragraph below 3.29.

The basic-gon (iv) in Fig. 11 can be better understood in terms of the interpretation of the data of D_0 given in 3.29. The basic-gon (iv) should be thought as a 2-morphism $\tau : X \otimes Y \to Y \otimes X$ with the property: $\tau \circ \tau = I$. Where '$\circ$' is the vertical composition of 2-morphisms. Pictures in Fig. 12 explains it better by drawing analogy with the identity map.

Remark 4.1. Eq. (2.2) reveal the hidden group structure for defect conditions. The proof of Proposition 4.1 reveals the basic procedure: form the basic-gons for every word in R_G; the value of the map $\psi_{0,1}$ on the inverse will be given by the inverse words.

Remark 4.2. D_2 does not have to be a singleton. Any suitable set for which the map $\psi_{1,2}$ is well-defined and satisfies orientation consistency condition can be taken as D_2.

Definition 4.1. Given a group G and a presentation $P_G := \langle B_G \mid R_G \rangle$ we define the category $Bord_2^{\text{def, cw}}(\mathcal{P}_\mathcal{G})$ as follows:

(1) as a category it is $Bord_2^{\text{def}}(\mathcal{D})$ where $D_2 = \{*\}, D_1 = B_G$, and D_0 and $\psi_{0,1}$ is determined by basic-gons of type (iii) in Fig. 11. The map $\psi_{1,2}$ is trivial.
(2) The basic-gons corresponds to words in R_G as in Fig. 11.

In other words, the category $Bord_2^{\text{def, cw}}(\mathcal{P}_\mathcal{G})$ has morphisms as surfaces with defects with a PLCW decomposition such that each generalized cell looks like one of the basic-gons in Fig. 11.

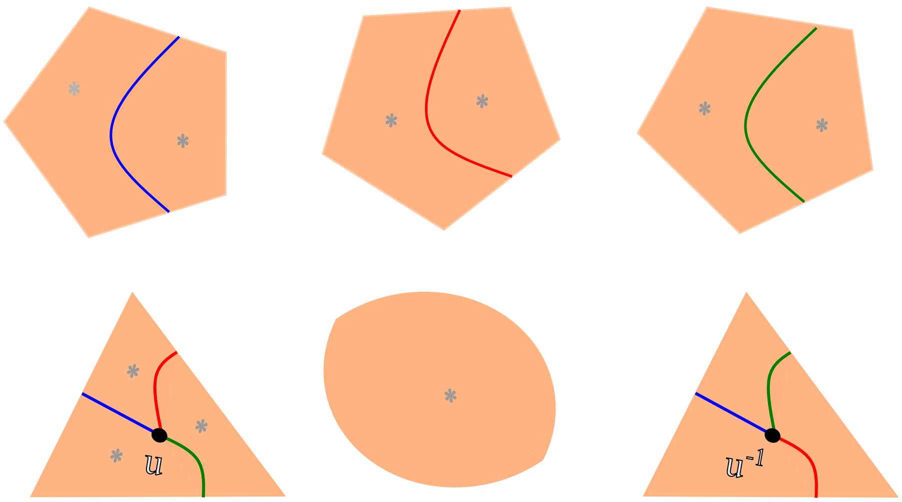


FIGURE 13. We have denoted a, b, c by colors blue, red, green respectively.

.

Example 4.1. Let K_4 be the *Klien four-group* with the presentation $P_{K_4} := \langle a, b, c \mid a^2 = b^2 = c^2 = abc = 1\rangle$ be a presentation of Klein-four group. The basic-gons for the category $Bord_2^{\text{def, cw}}(\mathcal{P}_{K_4})$ is shown in Fig. 13.

Although, we have not defined *coloring* yet (we will do it in the next section) but relying on pictures Fig. 13 for now, we note that a surface with defect $\hat{\Sigma}$ in $\mathbf{Mor}(Bord_2^{\text{def, cw}}(\mathcal{P}_{K_4}))$ is precisely a pair (Σ, Γ) where Γ is a trivalent 3-edge colorable graph embedded in Σ. Here by a coloring of an edge e of Γ we mean the image of e under d in B_{K_4}, which is the set $\{a, b, c\}$.

Example 4.2. Consider the symmetric group S_n with the presentation

$$S_n = \left\langle \tau_1, \dots, \tau_{n-1} \middle| \begin{array}{ccc} \tau_i\tau_j = \tau_j\tau_i & & \mid i-j \mid > 1 \\ \tau_i\tau_{i+1}\tau_i & = & \tau_{i+1}\tau_i\tau_{i+1} \\ \tau_i^2 & = & 1 \end{array} \right\rangle$$

The basic-gons for the category $Bord_2^{\text{def, cw}}(\mathcal{S}_4)$ is given in Fig. 14.

The class $\mathbf{Mor}(Bord_2^{\text{def, cw}}(\mathcal{S}_n))$ consists of n-graphs with only hexagonal vertices. It is worth mentioning that a general n-graph also have trivalent vertices. n-graphs were introduced in [CZ23] where it was used to construct Legendrian surfaces in the first jet space of the underlying surface of the n-graph. We do not make this construction explicit here.

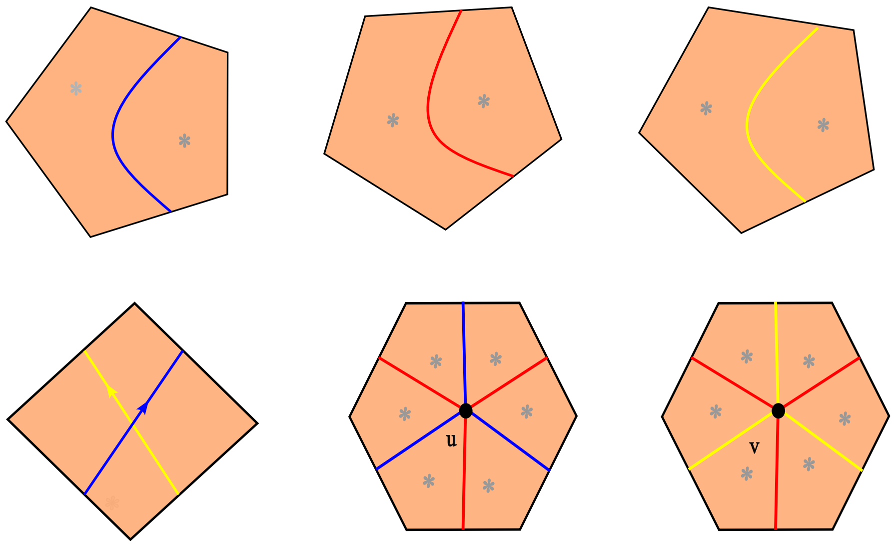


FIGURE 14. Caption

Definition 4.2. Given $D_2 = \{*\}$ and $D_1 = \{\bullet\}$, both singletons, we define a category $Bord_2^{def,\ cw}(\mathcal{D}^{\aleph})$ with the property that it has a basic-gon of type-(iii) with n sides for every $n \geq 2$. We define the subcategory $Bord_2^{def,\ cw}(\mathcal{D}^{\mathbf{n}})$ with the property that the only basic-gon of type-(iii) it has are those with n sides. Similarly, the category $Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{n}})$ is the category with n-regular undirected graphs.

Definition 4.3. We define a forgetful functor

$$\Pi^{cw} : Bord_2^{def,\ cw}(\mathcal{P}_G) \to Bord_2^{def,\ cw}(\mathcal{D}^{\aleph})$$

as follows:

- On **objects** it changes the labels on a (disjoint union of) circles from $g^{\epsilon} \in B_G$ to $\bullet^{\epsilon}$;
- on **morphism** it is defined via its action on basic-gons, where it changes a label of 1-strata from $g^{\epsilon} \in B_G$ to $\bullet^{\epsilon}$

Again, referring only to pictures, the map Π^{cw} can be thought as *bleaching* that forgets all the colors on 1-dimensional stratum (or replace all of them by a $\bullet$ without forgetting the signs.)

Given two objects O_1 and O_2 in $Bord_2^{def,cw}(\mathcal{P}_{K_4})$, the assignment $\Sigma \mapsto \Pi^{cw}(\Sigma)$ induces a function

$$\pi^{cw} : \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{P}_{K_4}))(O_1, O_2) \to \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}^{\aleph})(\Pi^{cw}(O_1), \Pi^{cw}(O_2)) \tag{4.1}$$

Since a surface Σ in the set $\mathrm{Mor}(Bord_2^{def,cw}(\mathcal{P}_{K_4}))$ includes the information about the source and target objects in its boundaries via the cobordism $O_1 \nearrow \hat{\Sigma} \searrow O_2$, we simply write Eq. (4.1) as:

$$\pi^{cw} : \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{P}_{K_4})) \to \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}^{\aleph}) \tag{4.2}$$

We conclude this section with the following remarks:

Remark 4.3. The category $Bord_2^{def}(\mathcal{D}^{\aleph})$ which is obtained from $Bord_2^{def,\ cw}(\mathcal{D}^{\aleph})$ using the forgetful functor F in Eq. (4.3) is the category one gets by using single defects for both D_2, D_1 and adjusting D_0 accordingly.

Remark 4.4. The forgetful functor 'bleach' induces a forgetful functor

$$\Pi : Bord_2^{def}(\mathcal{P}_{\mathcal{G}})) \to Bord_2^{def}(\mathcal{D}^{\aleph}))$$

in a canonical way, namely the following diagram commute on the level of functor.

$$\begin{array}{ccc} Bord_2^{\text{def, cw}}(\mathcal{P}_{\mathcal{G}}) & \xrightarrow{\Pi^{\hat{cw}}} & Bord_2^{\text{def,cw}}(\mathcal{D}^{\aleph}) \\ \downarrow F & & \downarrow F \\ Bord_2^{\text{def}}(\mathcal{P}_{\mathcal{G}}) & \xrightarrow{\Pi} & Bord_2^{\text{def}}(\mathcal{D}^{\aleph}) \end{array} \tag{4.3}$$

and there is a function

$$\pi : \mathrm{Mor}(Bord_2^{def}(\mathcal{P}_{K_4})) \to \mathrm{Mor}(Bord_2^{def}(\mathcal{D}^{\aleph}) \tag{4.4}$$

analogous to the functor π^{cw} in Eq. (4.2)

We refer to both π^{cw} and π as *bleach*.

5. A special trivial surrounding theory

This sections aims to give an example of a lattice TFT which is a trivial surrounding theory as introduced in Construction 1. In what follows, let X be a $\mathbb{C}$-vector space generated by a, b and c. Further let X^* be the dual vector space with corresponding dual basis a^*, b^* and c^*. We assume $X \cong X^*$ via the induced inner-product. A choice of such an X gives meaning to the fact that on undirected graph, one can choose any direction when calculating a TFT.

Definition 5.1. Let X be the vector space $\mathbb{C}a, b, c\rangle$ as in Example 3.2. We define the trivial surrounding theory $\chi^{cw} : Bord_2^{def,cw}(\mathcal{D}^{\mathbf{3}}) \to \mathrm{Vect}_F(\mathbb{C})$, with the properties that

- it assigns to a 1-cell e containing the single defect, the vector space $R_e = X$, and
- to a trivalent vertex u, the map $\mu : X \otimes X \otimes X \to \mathbb{C}$ as defined in Example 3.2.

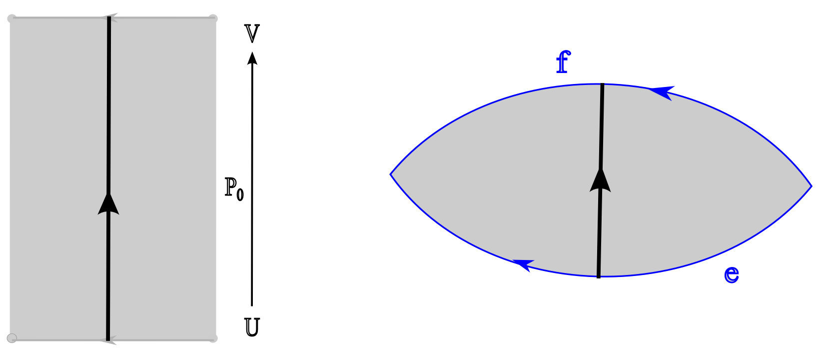
Remark 5.1. A consequence of Definition 5.1 is that χ^{cw} assigns to a circle with n-defects, the vector space $X^{\otimes n}$.

Remark 5.2. Since $X \cong X^*$, χ^{cw} is passes to a functor, which we also write as χ^{cw} by the abuse of notation,

$$\chi^{cw} : Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}) \to \mathrm{Vect}_F(\mathbb{C}) \tag{5.1}$$

where the category $Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}})$ is defined in Definition 4.2. To do that, we choose any orientation on the edges of a 1-strata.

Now, we compute the value of χ^{cw} on some simple patterns and basic-gons. We begin with $\chi^{cw}(P_0)$ for the polygon $P_0 : U \to V$ as shown below:



(5.2) (i) (ii)

In the light of Theorem 3.1 we use the cell-decomposition (ii) to calculate $\chi^{cw}(P_0)$. First, we see that both of $\chi^{cw}(U)$ and $\chi^{cw}(V)$ is X, which gives $\chi^{cw}(P_0) : X \to X$. What is this map? Well, we calculate using the composition:

$$\chi^{cw}(P_0) : \chi^{cw}(U) \xrightarrow{\mathbf{1}\otimes\mathcal{P}(P_0)} \chi^{cw}(U) \otimes Q(P_0) \otimes \chi^{cw}(V) \xrightarrow{\mathcal{E}(P_0)\otimes\mathbf{1}} \chi^{cw}(V)$$

We need the data:

$(P_0, e, \mathfrak{D})$	$(P_0, e, -)$	$(P_0, f, +)$
$Q_{(P_0,e,\mathfrak{D})}$	X_e	X_f^*

From this we get $Q(P_0) = Q_{(P_0,f,+)}$ and $\mathcal{P}(\mathcal{P}) = \mathcal{P}_f$ which is the co-pairing map $\mathbb{C} \to X_f^* \otimes X_f$. This leads to

$$\chi^{cw}(P_0) : X_e \xrightarrow{1\otimes\mathcal{P}(P_0)} X_e \otimes X_f^* \otimes X_f \xrightarrow{\mathcal{E}(P_0)\otimes\mathbf{1}} X_f$$

which is explicitly given by:

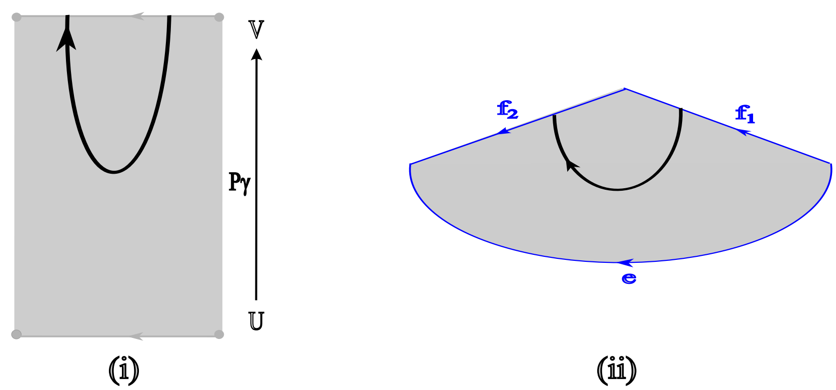
$$v \xrightarrow{1\otimes\mathcal{P}(P_0)} v\otimes(a_f^*\otimes a_f+b_f^*\otimes b_f+c_f^*\otimes c_f) \xrightarrow{\mathcal{E}(P_0)\otimes\mathbf{1}} a_f^*(x)a_f+b_f^*(x)b_f+c_f^*(x)c_f$$

Therefore the map

$$\chi^{cw}(P_0):\begin{cases} a\mapsto a\\ b\mapsto b\\ c\mapsto c\end{cases}$$

Hence $\chi^{cw}(P_0)=\mathbf{1}$.

Next, we consider $P_\gamma : U\to V$ shown below:

(5.3) 

Again, we use the cell-decomposition (ii) to calculate $\chi^{cw}(P_\gamma)$. In this case we have $\chi^{cw}(U)=\mathbb{C}_e$ and $\chi^{cw}(V)=X_{f_2}\otimes X_{f_1}^*$, which gives $\chi^{cw}(P_\gamma):\mathbb{C}\to X\otimes X^*$. We want to find out what is this map? We need the following data:

$(P_\gamma,e,\mathfrak{O})$	$(P_\gamma,e,-)$	$(P_\gamma,f_1,+)$	$(P_\gamma,f_2,+)$
$Q_{(P_\gamma,e,\mathfrak{O})}$	$\mathbb{C}_e$	X_{f_1}	$X_{f_2}^*$

From this we get $Q(P_\gamma)=Q_{(P_\gamma,f_1,+)}\otimes Q_{(P_\gamma,f_2,+)}$ and $\mathcal{P}(\mathcal{P}_\gamma)=\mathcal{P}_{f_1}\otimes\mathcal{P}_{f_2}$ with

$$\mathcal{P}_{f_1}:\ \ \mathbb{C}\to X_{f_1}\otimes X_{f_1}^*\quad \mathcal{P}_{f_2}\ \ :\mathbb{C}\to X_{f_2}^*\otimes X_{f_2}$$

Therefore we get

$$\chi^{cw}(P_\gamma):\mathbb{C}_e\xrightarrow{1\otimes\mathcal{P}(P_\gamma)}\mathbb{C}_e\otimes X_{f_1}\otimes X_{f_2}^*\otimes X_{f_2}\otimes X_{f_1}^*\xrightarrow{\mathcal{E}(P_0)\otimes\mathbf{1}} X_{f_2}\otimes X_{f_1}^*$$

among which $\mathbf{1}\otimes\mathcal{P}(P_\gamma)$:

$$1_e\mapsto 1_e\otimes(a_{f_2}^*\otimes a_{f_2}+b_{f_2}^*\otimes b_{f_2}+c_{f_2}^*\otimes c_{f_2})\otimes(a_{f_1}^*\otimes a_{f_1}+b_{f_1}^*\otimes b_{f_1}+c_{f_1}^*\otimes c_{f_1})$$

Thus the image of $1 \in \mathbb{C}_e$ under $\mathbf{1} \otimes \mathcal{P}(P_\gamma)$ equals

$$\begin{aligned} &1_e \otimes a^*_{f_2} \otimes a_{f_2} \otimes a_{f_1} \otimes a^*_{f_1} + 1_e \otimes a^*_{f_2} \otimes a_{f_2} \otimes b_{f_1} \otimes b^*_{f_1} + 1_e \otimes a^*_{f_2} \otimes a_{f_2} \otimes c_{f_1} \otimes c^*_{f_1} \\ &+1_e \otimes b^*_{f_2} \otimes b_{f_2} \otimes a_{f_1} \otimes a^*_{f_1} + 1_e \otimes b^*_{f_2} \otimes b_{f_2} \otimes b_{f_1} \otimes b^*_{f_1} + 1_e \otimes b^*_{f_2} \otimes b_{f_2} \otimes c_{f_1} \otimes c^*_{f_1} \\ &+1_e \otimes c^*_{f_2} \otimes c_{f_2} \otimes a_{f_1} \otimes a^*_{f_1} + 1_e \otimes c^*_{f_2} \otimes c_{f_2} \otimes b_{f_1} \otimes b^*_{f_1} + 1_e \otimes c^*_{f_2} \otimes c_{f_2} \otimes c_{f_1} \otimes c^*_{f_1} \end{aligned}$$

The action of $\mathcal{E}(P_\gamma) \otimes \mathbf{1}$ on these is given by

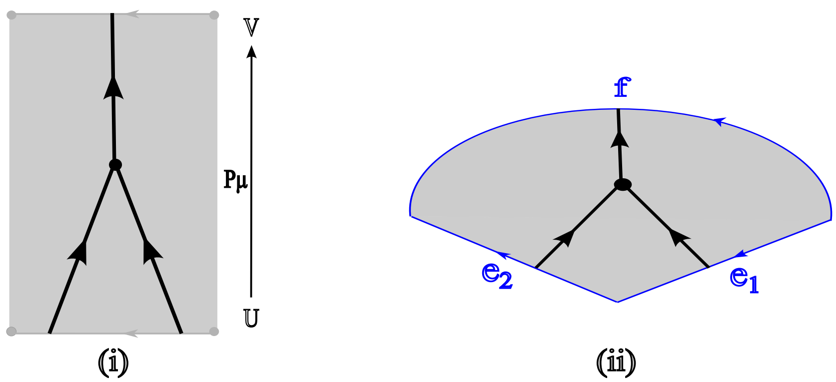
$$\begin{array}{ccccc} a^*_{f_2}(1_e a_{f_1}) a_{f_2} \otimes a^*_{f_1} & + & a^*_{f_2}(1_e b_{f_1}) a_{f_2} \otimes b^*_{f_1} & + & a^*_{f_2}(1_e c_{f_1}) a_{f_2} \otimes c^*_{f_1} \\ +b^*_{f_2}(1_e a_{f_1}) a_{f_2} \otimes a^*_{f_1} & + & b^*_{f_2}(1_e b_{f_1}) b_{f_2} \otimes b^*_{f_1} & + & b^*_{f_2}(1_e c_{f_1}) b_{f_2} \otimes c^*_{f_1} \\ +c^*_{f_2}(1_e a_{f_1}) c_{f_2} \otimes a^*_{f_1} & + & c^*_{f_2}(1_e b_{f_1}) c_{f_2} \otimes b^*_{f_1} & + & c^*_{f_2}(1_e c_{f_1}) a_{f_2} \otimes c^*_{f_1} \end{array}$$

Therefore the map

$$\chi^{cw}(P_\gamma) : 1 \mapsto a \otimes a^* + b \otimes b^* + c \otimes c^* \tag{5.4}$$

is the very co-evaluation map.

Now, we turn to the map $P_\mu : U \to V$ shown below:



(5.5) (i) (ii)

Like earlier, we use the cell-decomposition (ii) to calculate $\chi^{cw}(P_\mu)$. In this case we have

$$\begin{aligned} \chi^{cw}(U) &= R_{e_1} \otimes R_{e_2} \quad , \quad & \chi^{cw}(V) &= R_f \\ &= X_{e_1} \otimes X_{e_2} & &= X_f \end{aligned}$$

Thus we get the map

$$\chi^{cw}(P_\mu) : X_{e_1} \otimes X_{e_2} \to X_f$$

. We need the following data to know this map explicitly:

$(P_\mu, e, \mathfrak{D})$	$(P_\mu, e_1, -)$	$(P_\mu, e_2, -)$	$(P_\mu, f, +)$
$Q_{(P_\mu, e, \mathfrak{D})}$	X_{e_1}	X_{e_2}	X^*_f

From this we get $Q(P_\mu) = Q_{(P_\mu,f,+)}$ and $\mathcal{P}(\mathcal{P}_\mu) = \mathcal{P}_f$ defined as $\mathcal{P}_f : \mathbb{C} \to X_f^* \otimes X_f$ we get

$$\chi^{cw}(P_\mu) : X_{e_1} \otimes X_{e_2} \xrightarrow{1\otimes\mathcal{P}(P_\mu)} X_{e_1} \otimes X_{e_2} \otimes X_f^* \otimes X_f \xrightarrow{\mathcal{E}(P_\mu)\otimes\mathbf{1}} X_f$$

which, for $x \in X_{e_1}$ and $y \in X_{e_2}$, is given by

$$\begin{aligned} x \otimes y \ &\mapsto \ x \otimes y \otimes (a_f^* \otimes a_f + b_f^* \otimes b_f + c_f^* \otimes c_f) \\ &= \ x \otimes y \otimes a_f^* \otimes a_f + x \otimes y \otimes b_f^* \otimes b_f + x \otimes y \otimes c_f^* \otimes c_f \end{aligned} \tag{5.6}$$

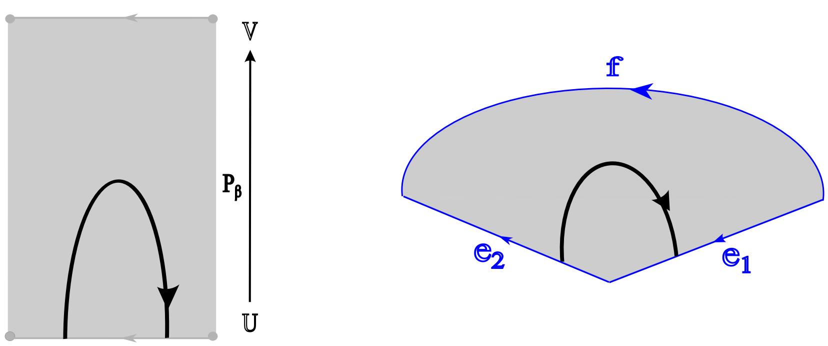
Recall the map $\mu : X \otimes X \otimes X \to \mathbb{C}$ from Example 3.2. The action of $(\mathcal{E}(P_\mu) \otimes \mathbf{1})$ on 5.6 is given by

$$\begin{aligned} &x \otimes y \otimes a_f^* \otimes a_f + x \otimes y \otimes b_f^* \otimes b_f + x \otimes y \otimes c_f^* \otimes c_f \\ &\qquad \mapsto \mu(a_f^* \otimes y \otimes x)a_f + \mu(b_f^* \otimes y \otimes x)b_f + \mu(c_f^* \otimes y \otimes x)c_f \end{aligned} \tag{5.7}$$

Therefore the action of the map $\chi^{cw}(P_\mu)$ on the basis elements are given by:

$$\chi^{cw}(P_\mu) : \begin{cases} a \otimes a, b \otimes b, c \otimes c & \mapsto 0 \\ a \otimes b, b \otimes a & \mapsto c \\ c \otimes a, a \otimes c & \mapsto b \\ b \otimes c, c \otimes b & \mapsto a \end{cases} \tag{5.8}$$

Finally, we want to know the action of the TFT χ^{cw} on $(P_\beta) : U \to V$ shown below:



(5.9) (i) (ii)

We see from (ii) that $C_(U) = \{e_1, e_2\}$ and $C_(V) = \{f\}$, which gives $\chi^{cw}(U) \coloneqq R_{e_1} \otimes R_{e_2} = X_{e_1}^* \otimes X_{e_2}$ and similarly $\chi^{cw}(V) \coloneqq R_f = \mathbb{C}_f$. The data for $Q_{(P_\beta)}$ is given by the table:

$(P_\beta, e, \mathfrak{O})$	$(P_\beta, e_1, -)$	$(P_\beta, e_2, -)$	$(P_\beta, f, +)$
$Q_{(P_\beta, e, \mathfrak{O})}$	$X^*_{e_1}$	X_{e_2}	$\mathbb{C}_f$

and $\mathcal{P}(P_\beta) = \mathcal{P}_f$, where $\mathcal{P}_f : \mathbb{C} \to \mathbb{C}_f \otimes \mathbb{C}_f$. Therefore, we get the composition:

$$\chi^{cw}(P_\beta) : X_{e_1} \otimes X^*_{e_2} \xrightarrow{1 \otimes \mathcal{P}(P_\beta)} X_{e_1} \otimes X^*_{e_2} \otimes \mathbb{C}_f \otimes \mathbb{C}_f \xrightarrow{\mathcal{E}(P_\mu) \otimes \mathbf{1}} \mathbb{C}_f$$

given explicitly by:

$$x \otimes y^* \xrightarrow{\mathbf{1} \otimes \mathcal{P}(P_\beta)} x \otimes y^* \otimes 1 \otimes 1 \xrightarrow{\mathcal{E}(P_\beta) \otimes \mathbf{1}} y^*(x) \tag{5.10}$$

Thus $\chi^{cw}(P_\beta)$ is given on the basis elements by

$$\chi^{cw}(P_\mu) : \begin{cases} a \otimes a^*, b \otimes b^*, c \otimes c^* & \mapsto 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.11}$$

which is nothing but the evaluation map.

We will return to χ^{cw} in the next-section and will interpret computations of this section in-terms of graph-coloring.

6. Applications to graph coloring

We met the category $Bord_2^{\text{def, cw}}(\mathcal{P}_{K_4})$ in Example 4.1, and mentioned that an element of the set $\mathbf{Mor}(Bord_2^{\text{def, cw}}(\mathcal{P}_{K_4})$ is precisely a pair (Σ, Γ), where Γ is a trivalent, 3-edge colorable, graph embedded in Σ. In this section, we give the proper definition of 3-edge coloring, a 3-edge colorable graph, and construct a trivial surrounding theory $\chi^{cw} : Bord_2^{\text{def, cw}}(\mathcal{D}^{\mathbf{3}})) \to \mathrm{Vect}_F(\mathbb{C})$ which counts the number of Tait-coloring of a trivalent planar graph.

Definition 6.1. Let X be a directed set, that is, an element of X is in the form of an ordered pair (x, ϵ) with $\epsilon = \pm 1$. Let (Σ, Γ) be a pair such that Γ is a un-directed graph embedded in Σ, and is such that each of its vertex has valency greater than or equal to 2. An *admissible coloring* of Γ, with values in X, is characterised by the following features:

(1) Every edge of Γ gets assigned an elements of X,
(2) each edge, sharing a vertex, gets assigned different elements of X, and
(3) an equivalence relation that decides when two such assignments for Γ are equivalent.

When $\Sigma = \mathbb{S}^2$ and Γ is trivalent, that is, it is 3-regular, we define an equivalence relation on the assignments, which is generated by identifying (x, ϵ) with $(x, -\epsilon)$. The corresponding admissible coloring is called the *Tait-coloring* of the graph Γ. The total number of such assignments (modulo the equivalence relation) is called the number of Tait-coloring or 3-edge coloring of the graph Γ. A theorem due to Tait (see [Tai80]) establishes correspondence between 4-color theorem and number of Tait-colorings of a planar trivalent graphs.

Example 6.1. Consider the pair $(\mathbb{S}^2, \Theta)$. Let R be the subgroup of rotations of $\mathrm{Diff}(\Sigma, \Gamma)$. Declare two assignments of Θ R-equivalent, if there exists an element of group R taking one to another. Then, the number of Tait-coloring of Θ is six, but the number of admissible coloring mod R is three.

Conjecture 6.1. For every such equivalence relation, there exists a group (at least a groupoid) whose orbit is this equivalence class.

Remark 6.1. The definition of admissible coloring given in Definition 6.1 is more general, but reduces to the usual definition of Tait-coloring used by several authors, including the work of Penrose from 70's in [P+71] and recent works like [Bal18] and [BM23]. The name *admissible coloring* is inspired from an analogous concept in [KR21]

Next, we are going to use the tools we have developed so far to give the definition of coloring. Recall the forgetful functor $\Pi^{cw} : Bord_2^{def,cw}(\mathcal{P}_{\mathcal{G}}) \to Bord_2^{def,cw}(\mathcal{D}^{\aleph})$ from Definition 4.3. In the case, when $G = \mathbb{K}_4$, and $\mathcal{P}_{\mathcal{G}}$ is as in Example 4.1, the target category is much smaller, and we have, by abuse of notation,

$$\Pi^{cw} : Bord_2^{def,cw}(\mathcal{P}_{K_4})) \to Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))$$

mapping into the subcategory $Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}})$. A natural question that can be asked is: **"Is the functor Π^{cw} full?"** In terms of 4.1, this amounts to asking whether the function π^{cw}, of Eq. (4.1), surjective for every two objects O_1 and O_2 in $Bord_2^{def,cw}(\mathcal{P}_{K_4})$. Theorem 6.1 answers it negatively, and gives a way to construct many counter-examples, but first, we give the definition of coloring by $\mathbb{K}_4$ in terms of the category $Bord_2^{def,cw}(\mathbb{K}_4)$ and the map π^{cw}.

Definition 6.2. For an object O in $Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}})$, a coloring of O is an object $\hat{O}$ in $Bord_2^{def,cw}(\mathcal{P}_{K_4})$ such that $\Pi^{cw}(\hat{O}) = O$.

Definition 6.3. Given (Σ, Γ) in $\mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))$, let $\pi^{-1}(\Sigma, \Gamma) \coloneqq \{(\Sigma_i, \Gamma_i, \mathbb{K}_4) \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{P}_{K_4})) \mid \pi^{cw}(\Sigma_i, \Gamma_i, \mathbb{K}_4) = (\Sigma, \Gamma)\, \forall\, i\}$. A coloring is a map

$$s : \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}) \to \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{P}_{K_4}))$$

in the opposite direction of map π^{cw} in 4.2 such that $(\pi^{cw} \circ s)(\Sigma, \Gamma) = (\Sigma, \Gamma)$ if $\pi^{-1}(\Sigma, \Gamma)$ is non-empty, and $s(\Sigma, \Gamma) = \emptyset_2$ (the empty morphism), if $\pi^{-1}(\Sigma, \Gamma)$ is empty.

The value of s at a surface with defects (Σ, Γ) is called a coloring of (Σ, Γ). A trivalent graph Γ embedded in a surface Σ is said to be 3-edge colorable if $s(\Sigma, \Gamma) \neq \emptyset_2$, or equivalently $\pi^{-1}(\Sigma, \Gamma)$ is non-empty

I do not like this definition. Covering space perspective is more appealing to me. Also, that is more mathematically elegant. It is covered in another comment box, together with few gaps.

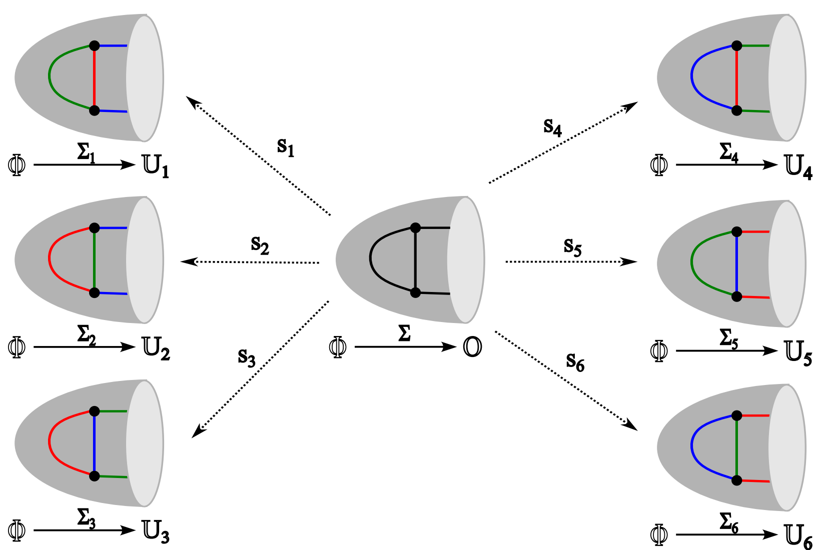
— Amit

From the point of view of this defintion, one can think the coloring process Definition 6.6 as a step to show that such a function does exist.

— Amit

Remark 6.2. It follows from the definition of Π^{cw} that if $\hat{\Sigma} \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{P}_{K_4}))$ is such that $\pi^{cw}(\hat{\Sigma}) = (\Sigma, \Gamma)$, then they have identical (isotopic) underlying stratified space, namely given as in Example 2.1 (6). Thus each individual element in $\pi^{-1}(\Sigma, \Gamma)$ is a copy of (Σ, Γ) as a stratified space. Writing $\pi^{-1}(\Sigma, \Gamma) = \sqcup_i(\Sigma_i, \Gamma_i, \mathbb{K}_4)$, we can define the map $\mathfrak{p} : \pi^{-1}((\Sigma, \Gamma)) \to (\Sigma, \Gamma)$ by $\mathfrak{p}(\hat{\Sigma}) = \pi^{cw}(\hat{\Sigma})$, which satisfies $\mathfrak{p} \circ s = \mathbf{1}$, thus s can be viewed as a section of $\mathfrak{p} : \pi^{-1}((\Sigma, \Gamma)) \to (\Sigma, \Gamma)$.

Example 6.2. We take $O_1 = \emptyset, O_2$ as a single circle O with two 0-defects, and $(\Sigma, \Gamma) : \phi \to O$ as shown below in Eq. (6.1) (middle). Note that orientation on the 1-strata does not matter and both 0-defects has got the same label '$\bullet$'. The following figure shows some maps $s : (\Sigma, \Gamma) \to \pi^{-1}((\Sigma, \Gamma))$ such that $\mathfrak{p} \circ s = \mathbf{1}$

(6.1) 

Note, that an object with two 0-defects labelled by any two (but different) of a, b or c also lies in $\Pi^{-1}(O)$ in the category $Bord_2^{def}(\mathcal{P}_{K_4})$, but they do not appear as an out-boundary of a morphism in the Figure 6.1. This is not a coincidence, and is a consequence of Theorem 6.1. We will return to it.

We see using Definition 6.3 that the question, whether Π^{cw} is full can be rephrased as: whether every trivalent graph, embedded in some surface, is 3-edge colorable. Theorem 6.1 gives a necessary condition on a planar trivalent graph to be 3-edge colorable using the group structure of $\mathbb{K}_4$, and Corollary 6.1 answers the question about the fullness of Π^{cw} negatively.

Fix $O_1 = \emptyset$ in 4.1, a morphism $(D, \Gamma_D, \mathbb{K}_4)$ from $\emptyset$ to a single marked circle $\hat{\mathbb{S}}^1$, in $Bord_2^{def,cw}(\mathcal{P}_{K_4})$, has a stratified disc underneath the defect data, with stratification as in Example 2.1 (6), given by Γ_D. The bleach map π^{cw} sends it to some morphism in $Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}})(\emptyset, \Pi^{cw}(\hat{\mathbb{S}}^1))$. It follows from the definition of Π^{cw} (see Definition 4.3) that the underlying space is again a stratified disc. In fact, it is (D, Γ_D). However, there are more such discs in $Bord_2^{def,cw}(\mathcal{P}_{K_4})(\emptyset, \hat{\mathbb{S}}^1)$ that are mapped by π^{cw} to (D, Γ) in $Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}})(\emptyset, \Pi^{cw}(\hat{\mathbb{S}}^1))$. Let $\pi^{-1}((D,\Gamma)) = \sqcup_i (D_i, \Gamma_{D_i}, \mathbb{K}_4)$ where the disjoint union is over all discs $(D_i, \Gamma_{D_i}, \mathbb{K}_4)$ with the property that $\pi^{cw}((D_i, \Gamma_{D_i}, \mathbb{K}_4)) = (D, \Gamma_D)$. Moreover, π^{cw} is isotopy of underlying stratified spaces. Give $\pi^{-1}((D, \Gamma_D))$ a discrete topology and define $\mathfrak{p} : \pi^{-1}((D, \Gamma_D)) \to (D, \Gamma_D)$ by $\mathfrak{p} \mid_{(D_i, \Gamma_{D_i}, \mathbb{K}_4)} = \pi^{cw}$. With this in hand, we define:

Definition 6.4. For a basic-gon $P \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))$ viewed as a disc (P, Γ_P), a coloring of it is a section of $\mathfrak{p} : \pi^{-1}((P, \Gamma_P)) \to (P, \gamma_P)$, that is, a map $s : (P, \Gamma_P) \to \pi^{-1}((P, \Gamma_P))$ of stratified spaces satisfying $\pi^{cw} \circ s = \mathbf{1}$

This situation is that of a stratified covering. Each (stratified) open disc in (Σ, Γ) is well covered by a number of copies (isotopy replaces homeomorphism) of it. However, we are not interested in an arbitrary neighborhood at this point (will be useful for sheaf perspective as in the picture on the top) but only the 2-cells.
For the coloring perspective, s is even simpler. It is an isotopy.

— Amit

Next, we want formulate the idea of *coloring process* for a given trivalent graph Γ, embedded in a surface Σ. We do so by considering the pair (Σ, Γ) in the set $\mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))$. The first definition in the line is:

Definition 6.5. For a surface with defect (Σ, Γ) in $\mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))$, let $P \in C_2(\Sigma)$ be a basic-gon considered as a morphism from $\emptyset$ to ∂P , a choice of a 3-*edge-coloring* localised at P is a the value S_P under the function s of Definition 6.3. Alternatively, it is a choice of a section s_P of $\mathfrak{p} : \pi^{-1}(P) \to P$ in the sense of Remark 6.2.

Here, by $P \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))$ we really mean the pair (P, Γ_P) but we have suppressed Γ_P for convenience of notation. We will follow this convention throughout this manuscript.

Remark 6.3. It follows from the definition of Π and the identity $\pi \circ s_P = P$ that the image $s(P)$ has an isotopic underlying stratified spaces as P. Thus $s(P)$ is isomorphic to one of the discs in $\pi^{-1}(P)$ as surfaces with defects. The color assigned to the graph Γ in P is the label, (in $\mathcal{P}_{K_4}$), that edges of Γ gets under this s.

By definition, a surface $(\Sigma, \Gamma) \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}^{\mathbf{3}}_{+}))$ comes equipped with a PLCW decomposition into cells $C_0(\Sigma), C_1(\Sigma)$ and $C_2(\Sigma)$ such that each 2-cell is isomorphic (as a surface with defects) to one of basic-gons as in Fig. 7 (ii), (iii) and (iv).

Convention 6.1. Referring to Fig. 7,

(1) for two basic-gons P_i, P_j of type (ii) or (iii), we denote by $P_i \otimes P_j$ the gluing of P_i and P_j along $P_{ij} := P_i \cap P_j \in C_1(\Sigma)$. Although, this seems like an abuse of notation, but it is indeed the fusion (horizontal composition, see 3.28) when we consider the defect data of P_i and P_j.
(2) For a basic-gon P_μ of type (iii) we use the vertical composition $P_\mu \circ (P_{i_1} \otimes \cdots \otimes P_{i_k})$ to denote its gluing along the 1-cell formed by the intersection of $P_\mu, P_{i_1}, \ldots, P_{i_k}$. Again, this is indeed the vertical composition of the underlying defect data.

We are ready to give the definition of a coloring process:

Definition 6.6. Given an un-directed trivalent graph Γ, embedded in a surface Σ, a *coloring process* is the following data assigned to the surface with defects $(\Sigma, \Gamma) \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}^{\mathbf{3}}_{+}))$:

- A coloring s_P for every $P \in C_2(\Sigma)$, as defined in Definition 6.5,
- A coloring $s_{(P_i \otimes P_j)}$ for every fused cells $P_i \otimes \cdots \otimes P_j$ given by $s_{(P_i \otimes P_j)} := s_{P_i} \otimes s_{P_j}$, and
- A coloring $s_{(P_\mu \circ P_\nu)}$ for each vertical composition $P_\mu \circ P_\nu$ given by $s_{(P_\mu \circ P_\nu)} := s_{P_\mu} \circ s_{P_\nu}$.

In short, a coloring process is an assignment of coloring to each 2-cells and a schema to glue them together with the aim to produce a coloring of the entire surface (Σ, Γ) as (2) and (3) facilitate gluing of coloring of an arbitrary (finite) number of cells by repeated application.

Do I need to show that Definition 6.6 is well-defined? That is, $s_{P_i \otimes P_j}$ defined in the second bullet is indeed a coloring?

— Amit

Definition 6.7. A surface with defects $(\Sigma, \Gamma) \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}^{\mathbf{3}}_{+}))$ admits a 3-edge coloring or, is 3-edge colorable, if these exist an $s : (\Sigma, \Gamma) \to \pi^{-1}((\Sigma, \Gamma))$ extending all s_P for $P \in C_2(\Sigma)$, and satisfies the condition of Definition 6.6 when restricted to a sub-complex formed by fusing and composing a number of 2-cells.

What is the definition of extending in this context? What does it mean to be extending all s_P?

— Amit

We see that this map $\mathfrak{p}$ coincide with the one defined in Definition 6.5 on 2-cells. Therefore, it is right to say that the map s in Definition 6.7 is a global section, or a graph Γ, embedded in Σ, is 3-edge colorable if a section s, as in Definition 6.6, exists globally on the surface with defects (Σ, Γ). Note that, given a surface $(\Sigma, \Gamma) \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}^{\mathbf{3}}))(O_1, O_2)$, local sections always exists at every basic-gons. Definition 6.7 says that the graph Γ is 3-edge colorable if all these local-sections can be patched together to give a global-section. In that case, a coloring of Γ is given by such a global section.

Theorem 6.1. Consider the surface with defect $(\mathbb{S}^2, \Gamma, \mathbb{K}_4)$ as an element in the set $\mathrm{Mor}(Bord_2^{def,\ cw}(\mathbb{K}_4))$ Let $\bar{\mathbb{S}^1}$ be a generic cross-section of $(\mathbb{S}^2, \Gamma, \mathbb{K}_4)$ then the product of defects on $\bar{\mathbb{S}^1}$ is 1.

For the rest of this section, we only consider graphs with single component. Given a graph Γ, a *bridge* is an edge of Γ whose deletion disconnects the graph into two components. (See [Bol98] for more detail and general, as well as, alternative definitions.)

Using Definition 6.7, we deduce the following famous result from Theorem 6.1, which has been known to people since Tait:

Corollary 6.1. A planar trivalent graph Γ with bridge is not 3-edge colorable.

Put differently, it means that a pair $(\mathbb{S}^2, \Gamma)$ with Γ having a bridge never lies in the image of π^{cw}. Setting $O_1 = O_2 = \emptyset$ we see that π^{cw} is not surjective. Hence Π^{cw} is not full.

We prove the corollary first using Theorem 6.1

Proof. If the trivaelnt graph Γ with bridge e is 3-edge colorable then the edge e gets a, b or c as the color. If $\bar{\mathbb{S}}^1_e$ be a generic cross-section, then it contains a single defect labelled by a, b or c. A contradiction to 6.1. □

Proof of Theorem 6.1

Proof. Let γ_{xy} denote the union of all the edges of the graph Γ with color x and y. Then all of γ_{ab}, γ_{bc} and γ_{ac} are piecewise linear simple (Jordan) curve embedded in $\mathbb{S}^2$ and thus intersects any generic cross section even number of times. Let S_t be a generic cross section and $2n_1, 2n_2$ and $2n_3$ be the number of intersection points of it with γ_{ab}, γ_{bc} and γ_{ac} respectively. Note that it is enough to consider only two of them, say γ_{ab} and γ_{bc}. The contribution from γ_{ab} will be of the form $a^k b^{2n_1-k}$ for some positive integer k. The share of c comes from the curve γ_{bc} and is equal to

$c^{2n_2-(2n_1-k)}$. Therefore the product of defects of S_t equals $a^k b^{2n_1-k} c^{2n_2-2n_1+k}$. This product simplifies to $(ab^{-1})^k c^k$ or c^{2k}, which equals 1. □

Theorem 6.1 is more general than the classical statement of Corollary 6.1. We return to the comment made below Example 6.2 in connection with it. Now, we see that there can not be a morphism between $\emptyset$ and a single circle labelled with two distinct defects that projects to Σ. For, if there is such a morphism, take its dual and vertically compose along the common circle to produce a pair $(\mathbb{S}^2, \Gamma, \mathbb{K}_4)$. We see that, it contradicts Theorem 6.1. However, one could deduce the same from Corollary 6.1 by a suitable clever construction. In fact, this make us to conjecture:

Conjecture 6.2. The statement of Theorem 6.1 and Corollary 6.1 are equivalent, that is, one could deduce Theorem 6.1 from the validity of Corollary 6.1.

I think, I do have a proof.

— Amit

6.1. **Planar trivalent graphs.** Finally, we restrict our attention to un-directed, trivalent, planar graphs. In the language of surface with defects, it is a pair $(\mathbb{S}^2, \Gamma) \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))(\emptyset, \emptyset)$, with admissible decomposition as discussed in Example 2.1 (6). The goal of this section is to address the question of coloring of such a graph. In other words, whether a given surface with defects $(\mathbb{S}^2, \Gamma)$ lies in the image of π^{cw} in Eq. (4.1). We saw in Corollary 6.1 that it is not always possible to find a global section $s : (\Sigma, \Gamma) \to \pi^{-1}((\Sigma, \Gamma))$ such that $\mathfrak{p} \circ s = \mathbf{1}$. Note that the cardinality of $\pi^{-1}((\mathbb{S}^2, \Gamma))$ is precisely the number of Tait-coloring of the planar graph Γ. By definition of s, it is also the total number of such global sections s.

We begin with an example demonstrating the coloring process for planar trivalent graphs:

Example 6.3. We see that for the dumbbell graph below, there are three choices of sections for each of P_1, P_2 and P_4 and six choices for P_3 but no such choice of

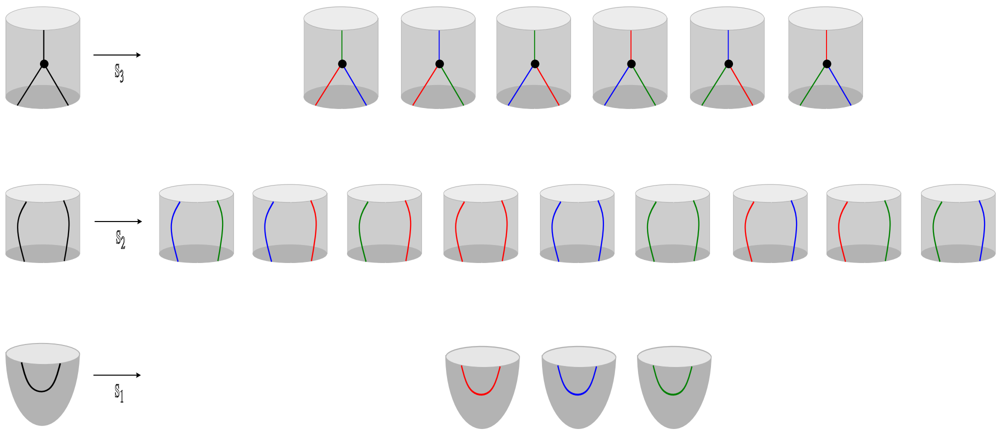


Figure 15. shows the existance of local sections on the southern hemisphere made by fusing P_1 and P_1' (bottom), cylinder made by fusing P_2, P_3 and P_2'(middle), and finally cylinder made by fusing P_4, P_5, P_6 and P_3'(top). They do not glue in any manner to produce a section on the southern hemisphere of 6.2 that restricts to individual sections.

$s_{P_1}, s_{P_2}, s_{P_3}$ and s_{P_4} extends to a global-section s as this will contradict Theorem 6.1.

(6.2)

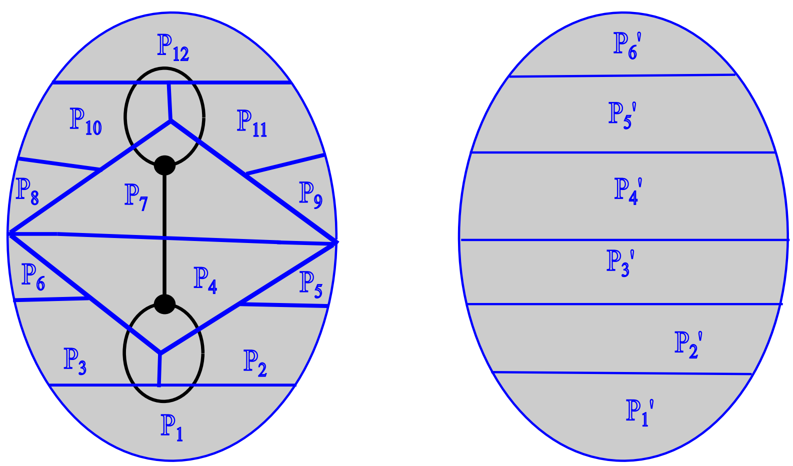


Fig. 15 demonstrate the coloring process for the dumbbell graph in 6.2: extending the sections to $P_3 \otimes P_2 \otimes P_2'$ under horizontal composition by the rule $s_{P_3 \otimes P_2 \otimes P_2'} = s_{P_3} \otimes s_{P_2} \otimes s_{P_2'}$, and to the vertical composition $(P_3 \otimes P_2 \otimes P_2') \circ (P_1 \otimes P_1')$ by $(s_{P_3} \otimes s_{P_2} \otimes s_{P_2'}) \circ (s_{P_1} \otimes s_{P_1'})$. Note that for no choice of $s_{P_1}, \ldots, s_{P_6}$, these individual sections can be extended to $(P_6 \otimes P_4 \otimes P_5 \otimes P_3') \circ (P_3 \otimes P_2 \otimes P_2') \circ (P_1 \otimes P_1')$. The caption below Fig. 15 delve deeper.

It is the vertical composition that is the real deal. Note, for a cylinder C in $\mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))(O_1, O_2)$, if $s(C)$ exists then it is a cylinder $\hat{C}$ in the category $\mathrm{Mor}(Bord_2^{def,cw}(\mathcal{P}_{K_4}))(U_1, U_2)$ for some circle with defects U_1 and U_2 with the property that $\Pi^{cw}(U_1) = O_1$ and $\Pi^{cw}(U_2) = O_2$. Therefore, if C_1 and C_2 are two such cylinders in $\mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))$ such that $C_2 \circ C_1$ is defined then s_{C_1} and s_{C_2} extends to a section $s_{C_2 \circ C_1}$ if and only if the composition $s_{C_2} \circ s_{C_1}$ exists in $Bord_2^{def,cw}(\mathcal{P}_{K_4})$, in which case a section $s_{C_2 \circ C_1}$ is given by the composition $s_{C_2} \circ s_{C_1}$, as suggested by the coloring process.

Next, recall the trivial surrounding theory $\chi^{cw} : Bord_2^{def,cw}(\mathcal{D}^{\mathbf{3}}) \to \mathrm{Vect}_F(\mathbb{C})$ from Section 5. Under the isomorphism $X \cong X^*$, it is independent of the orientation on the edges of the graph Γ and thus we can talk about the correlator of a surface with defects in $Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}})$ by choosing an arbitrary orientation of 1-strata. Thus we formulate the main result:

Theorem 6.2. Let Γ be a trivalent graph embedded in $\mathbb{S}^2$. Consider the surface with defect $(\mathbb{S}^2, \Gamma)$ in $\mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))(\emptyset, \emptyset)$. The action of the functor χ^{cw} on $(\mathbb{S}^2, \Gamma)$ is the assignment

$$\begin{aligned} \chi^{cw}(\mathbb{S}^2, \Gamma) : \mathbb{C} &\longrightarrow \mathbb{C} \\ \lambda &\mapsto \#\mathrm{Tait}(\Gamma)\lambda \end{aligned} \tag{6.3}$$

In other words the number $\chi^{cw}(\mathbb{S}^2, \Gamma)(1)$ is the number of Tait-coloring of the planar trivalent graph Γ.

Proof of Theorem 6.2 will take us a while. First thing on this line is the *planar trivalent decomposition theorem* stated and proved below:

Theorem 6.3. Every planar trivalent graph, when seen as a surface with defects $(\mathbb{S}^2, \Gamma) \in \mathrm{Mor}(\mathrm{Bord}_2^{\mathrm{def}}(\mathcal{D}_+^{\mathbf{3}}))$ can be written as the composite $\rho_{i_1} \circ \cdots \circ \rho_{i_m}$ where each ρ_{i_j} is one of the four patterns shown in the figure - 16.

Proof. Because Γ has only a finite number of vertices, it can isotoped so that the handle decomposition of $\mathbb{S}^2$ in terms of cylinder contains at most one vertex. Now, the portion of the cylinder far from this unique vertex is planar and thus generated by $U_i \coloneqq \gamma_i \circ \beta_i$. (See [DP03] or [Kau90] for a proof of this fact.) On the other hand the trivalent vertex will look either like μ_i or one of the three patterns in the bottom of figure - 17. However, all of these can be obtained by a combination I, μ_i, β_i and γ_i as shown in figure- 18. □

We also prove the following analogue for the category $\mathrm{Bord}_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}})$

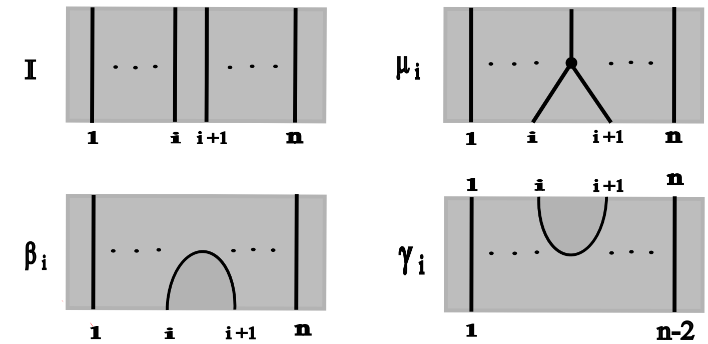


FIGURE 16. The four patterns I, μ_i, β_i, and γ_i. The value of n can be 2, in which case i equals 1 and $i+1$ equals 2. We have presented only the rectangle part of the cylinder. The part of the cylinder not shown is the region on the sphere without defect.

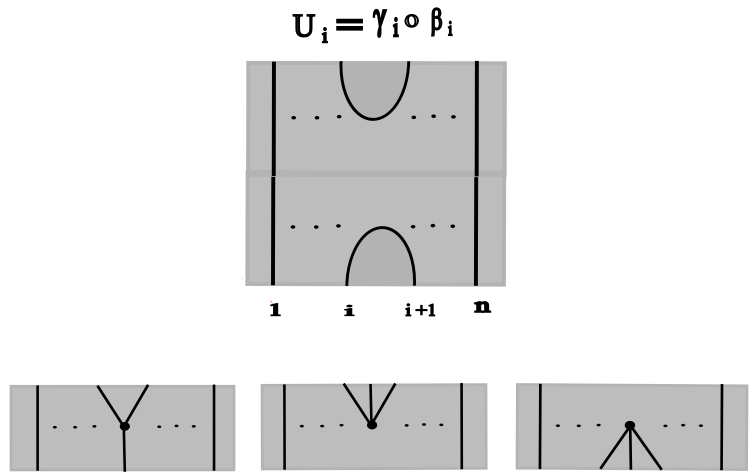


FIGURE 17. The three possible configurations of trivalent vertices, other than μ_i, is displayed at the bottom. The top shows how to get U_i - the generators of planar diagrams - using γ_i and β_i.

Proposition 6.1. Given a planar graph $(\mathbb{S}^2, \Gamma) \in \mathrm{Mor}(\mathrm{Bord}_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))$, there is a PLCW decomposition of it making ρ_{i_j} of Theorem 6.3. More precisely, each ρ_{i_j} can be written as $\rho_{i_j} = P_1^{i_j} \otimes \cdots \otimes P_k^{i_j}$ for some $P_1^{i_j}, \ldots P_k^{i_j} \in C_2(\mathbb{S}^2)$.

Proof. Figure- 6.2 gives an idea about how to do it. First, choose a height function on $\mathbb{S}^2$ and obtain generic sections containing the cylinders ρ_{i_j}. Since, each ρ_{i_j} has finitely many 1-defects, insert a 0-cell between any two consecutive defects. Referring to Fig. 16, we see that there is a bijection between all such 0-cells inserted on either

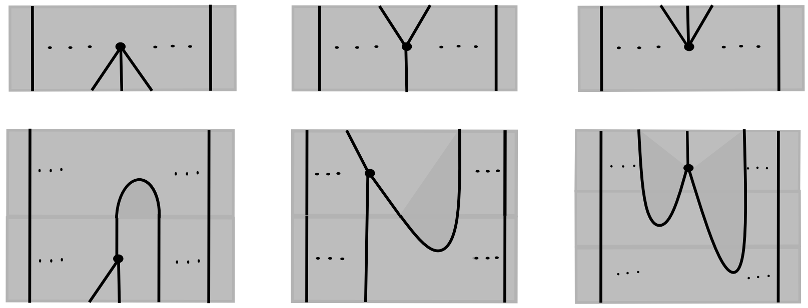
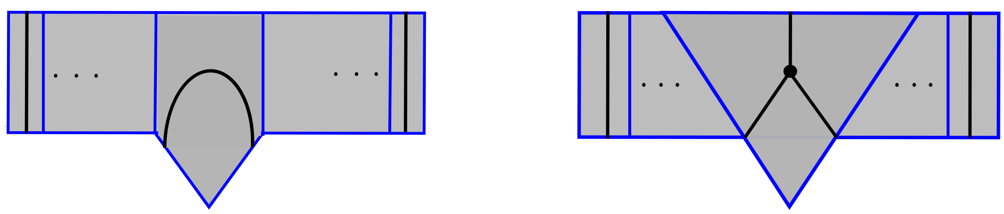


FIGURE 18. The picture is arranged in "top-bottom" pairs. Rectangles at bottom shows how to write the trivalent vertex above it as a word involving I, μ, γ and β from Figure- 16. Where we have dropped the subscript i for notational convenience.

side of rectangles except for those between i and $i+1$. Join these two to form 1-cells. For I, this will immediately give a decomposition into basic-gons. For β_i, μ_i, we join the two neighboring 0-cells of the 0-cell between i and $i+1$ as shown in the picture below. γ_i is done in a similar fashion as β_i.



□

Next, we prove the following important property of 3-edge coloring which is analogous to the *sum over intermediate states* property in QFT. (See [CR17], Section-2.1.)

Lemma 6.1. Let $(C_{12}, \Gamma_{12}) \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}^{\mathbf{3}}_+))(O_1, \partial O_2)$ and O_t be a generic cross-section of (C_{12}, Γ_{12}) that fits into the composite bordism

$$O_1 \xrightarrow{\iota_1} (C_{1t}, \Gamma_{1t}) \xleftarrow{\mathfrak{o}_1} O_t \xrightarrow{\iota_2} (C_{t2}, \Gamma t2) \xleftarrow{\mathfrak{o}_2} O_2$$

If $\mathrm{Tait}_{\hat{x},\hat{y}}\Gamma_{xy}$ stands for coloring of the cylinder (C_{xy}, Γ_{xy}), with given (fixed) colors $\hat{x}$ on the in-boundary x and $\hat{y}$ on the out-boundary y, then

$$\#\mathrm{Tait}_{\hat{O}_1,\hat{O}_2}\Gamma_{12} = \sum_{\hat{O}_t} (\#\mathrm{Tait}_{\hat{O}_1,\hat{O}_t}\Gamma_{1t})(\#\mathrm{Tait}_{\hat{O}_t,\hat{O}_2}\Gamma_{12}) \tag{6.4}$$

Proof. The lemma says that if we choose a coloring $\hat{O}_1$ of O_1 and $\hat{O}_2$ of O_2, then the number of 3-edge coloring of Γ_{12}, such that the in-boundary O_1 receives the color $\hat{O}_1$ and the out-boundary receives the color $\hat{O}_2$, is the sum of the product of number of 3-edge coloring of the graph Γ_{1t} with in-boundary $\hat{O}_1$ and out-boundary $\hat{O}_t$, and Γ_{t2} with in-boundary $\hat{O}_t$ and out-boundary $\hat{O}_2$ over all the coloring $\hat{O}_t$ of an (given) intermediate cross-section O_t. We will prove this by establishing equality between two sets A:

$$\{s \mid s \text{ is a Tait-coloring of}(C_{12}, \Gamma_{12}) \text{ with in-boundary} \hat{O}_1 \text{ and out-boundary } \hat{O}_2\}$$

and B:

$$\{s | \ s \text{ is obtained by gluing } s_1 \text{ and } s_2 \text{ along the common boundary}$$
$$\hat{O}_t \text{ where } s_1 \text{ is a Tait-coloring of } \Gamma_{1t} \text{ with in-boundary } \hat{O}_1$$
$$\text{and } s_2 \text{ is a Tait-coloring of } \Gamma_{t2} \text{ with out-boundary } \hat{O}_2\}$$

First, $B \subset A$ is obvious. Conversely, if $s \in A$, then s restricts to two sections s_1 and s_2 that can be glued (composed) along the common boundary, namely the color that O_t receives to give a Tait-coloring of Γ_{12}; proving $A \subset B$. Eq. (6.4) is then a statement about the cardinality of A (left) and B (right). To find the cardinality of B, notice that for a coloring $\hat{O}_t$ of O_t, if there are m distinct coloring of Γ_{1t} with out-boundary $\hat{O}_t$, and n distinct coloring of Γ_{t2} with in-boundary $\hat{O}_t$, then they can be combined in mn ways to give a Tait-coloring of Γ_{12}. The cardinality of B is obtained by summing over all such coloring $\hat{O}_t$ of O_t.

□

Compare Eq. (6.4) with the matrix product formula. Although, this is formally stated below.

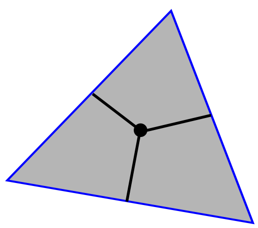
— Amit

Lemma 6.1 says that the sum can be taken over arbitrary coloring of O_t, that is, it may or may not lead to a Tait-coloring on any of Γ_{1t} or Γ_{t2}. The contribution from a color $\hat{O}_t$, which can not be extended to Tait-coloring of Γ_{12}, is zero because of the relation $A \subset B$. It also means that for such a coloring either $\#\text{Tait}_{\hat{O}_1,\hat{O}_t}\Gamma_{1t}$ is zero or $\#\text{Tait}_{\hat{O}_t,\hat{O}_2}\Gamma_{12}$ is zero.

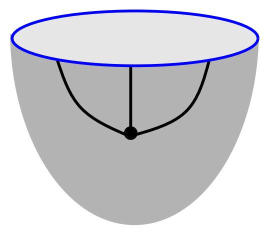
Let $\mathcal{B}(V)$ denotes the set of bases of the $\mathbb{C}$-vector space V. For $V = X^{\otimes n}$, this set is the set of colors or states that χ^{cw} assignes to a circle with n-defects. We state the following interpretation of the calculations of $\chi^{cw}(P)$ where P is a polygon as in 5.2, 5.3, 5.5, and 5.9.

Proposition 6.2. To the basic-gons of the category $Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}})$, when viewed as a cup $\mathbb{D}_P \in \mathrm{Mor}(Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}))(\emptyset, \partial\mathbb{D}_P)$, χ^{cw} assigns a vector $v \in \chi^{cw}(\partial\mathbb{D}_P)$ whose component in the direction of a basis vector $w_i \in \mathcal{D}(\chi^{cw}(\partial\mathbb{D}_P))$ is the number of ways the embedded graph Γ_P can be 3-edge colored so that the out-boundary $\partial\mathbb{D}_P$ receives a color w_i.

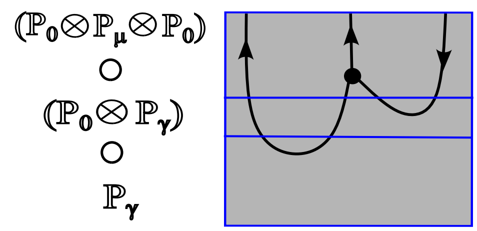
Proof. First, note that patterns 5.2 and 5.9 are both Pattern P_γ from 5.9 as a basic-gon. So, in this case the statement of the proposition is verified by Eq. (5.4). (See the map S_1 in Fig. 15.) For $\mathbb{D}_\mu$ it follows from Proposition 3.3 and the decomposition (iii) in the picture below.
(6.5)



(i)



(ii)



(iii)

It follows from the vertical composition shown in (iii), Proposition 3.2, and functoriality of χ^{cw} that

$$\chi^{cw}(\mathbb{D}_\mu) = \chi^{cw}(P_0 \otimes P_\mu \otimes P_0) \circ \chi^{cw}(P_o \otimes P_\gamma) \circ x^{cw}(P_\gamma)$$

which gives

$$\begin{aligned} 1_{\mathbb{C}} &\xrightarrow{\chi^{cw}(P_\gamma)} a\otimes a + b\otimes b + c\otimes C \xrightarrow{\chi^{cw}(P_0)\otimes\chi^{cw}(P_\gamma)} a\otimes a\otimes a\otimes a + a\otimes a\otimes b\otimes b \\ &+a\otimes a\otimes c\otimes c + b\otimes b\otimes a\otimes a + b\otimes b\otimes b\otimes b + b\otimes b\otimes c\otimes c + c\otimes c\otimes a\otimes a + \\ &c\otimes c\otimes b\otimes b + c\otimes c\otimes c\otimes c \xrightarrow{\chi^{cw}(P_0)\otimes\chi^{cw}(P_\mu)\otimes\chi^{cw}(P_0)} a\otimes c\otimes b + a\otimes b\otimes c + \\ &b\otimes c\otimes a + b\otimes a\otimes c + c\otimes b\otimes a + c\otimes a\otimes b \end{aligned}$$

but these are the words from the boundary of all the sections $s : \mathbb{D}_\mu \to \pi^{-1}(\mathbb{D}_\mu)$ □

The following lemma generalises Proposition 6.2:

Lemma 6.2. Let $\mathbb{S}_s$ and $\mathbb{S}_t$ be two objects in the category $Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}})$ comprising of single marked circles with n_s and n_t number of markings (0-defects) respectively. The TFT $\chi^{cw} : Bord_2^{def,cw}(\mathcal{D}_+^{\mathbf{3}}) \to \mathrm{Vect}_F(\mathbb{C})$ assigns to a cylinder $(C_{st}, \Gamma_{st}) : \mathbb{S}_s \to \mathbb{S}_t$ a linear map $\chi^{cw}(C_{st}, \Gamma_{st}) : X^{\otimes n_s} \to X^{\otimes n_t}$ that sends a basis vector $v_j \in \mathcal{B}(X^{\otimes n_s})$ to a vector $B \in X^{\otimes n_t}$ such that the component of B in the direction of a vector $w_i \in \mathcal{B}(X^{\otimes n_t})$ is the number of ways Γ_{st} can be 3-edge colored so that the in-boundary $\mathbb{S}_s$ receives the color v_j and the out-boundary S_t receives the color w_i.

If we denote the linear map $\chi^{cw}(C_{st}, \Gamma_{st}) : X^{\otimes n_s} \to X^{\otimes n_t}$ by a $3^{n_t} \times 3^{n_s}$ matrix $A = (a_{ij})$, then with the notation of Lemma 6.1:

$$a_{ij} = \#\text{Tait}_{v_j, w_i} \Gamma_{st} \tag{6.6}$$

for $v_j \in \mathcal{B}(X^{\otimes n_s})$ and $w_i \in \mathcal{B}(X^{\otimes n_t})$. In simple words, a_{ij} is the number of ways Γ_{st} can be 3-edge colored so that the in-boundary $\mathbb{S}_s$ receives the color v_j and the out-boundary $\mathbb{S}_t$ receives the color w_i. In this language Eq. (6.4) is the familiar matrix product $a_{ij} = \sum_k b_{ik} c_{kj}$. Something, which is expected from the composition of linear maps in Lemma 6.2.

The following corollary to Lemma 6.2 is immediate:

Corollary 6.2. The TFT χ^{cw} assigns

(1) to a cup $(D_t, \Gamma_t) : \emptyset \to \mathbb{S}_t$ a vector $w \in X^{\otimes n_t}$ whose component in the direction of $w_i \in \mathcal{B}(X^{\otimes n_t})$ is the number of ways one can 3-edge color the graph Γ_t so that w_i is the color received by the boundary circle $\mathbb{S}_t$.

(2) For a cap $(U_s, \Gamma_s) : \mathbb{S}_s \to \mathbb{C}$, $\chi^{cw}(U_s, \Gamma_s)$ assigns a covector v which evaluates to κ_i on $v_i \in \mathcal{B}(X^{\otimes n_s})$ with the property that there are κ_i ways to 3-edge color Γ_s so that the in-boundary $\mathbb{S}_s$ receives the color v_i.

First, we give a proof of Theorem 6.2 from Lemma 6.2 and Corollary 6.2:

Proof. For a given surface with defects $(\mathbb{S}^2, \Gamma)$, choose a generic cross-section $\mathbb{S}_t$. By Corollary 6.2 (1), $\chi^{cw}(D_t, \Gamma_t)$ is a vector of the form $\sum_i \lambda_i w_i$ where λ_i is the number of ways one can color Γ_t so that the cross-section $\mathbb{S}_t$ gets the color w_i. Now, consider the cap (U_t, Γ'_t), where Γ'_t is the portion of Γ embedded in the cap U_t. By Corollary 6.2 (2), $\chi^{cw}(U_t, \Gamma'_t)$ is a covector that maps $w_i \in \mathcal{B}(X^{\otimes n_t})$ to $\kappa_i \in \mathbb{C}$ with the property that the graph Γ'_t can be colored in κ_i ways so that $\mathbb{S}_t$ receives a color w_i. Composing the two we get:

$$\begin{aligned} \chi^{cw}(\mathbb{S}^2, \Gamma)(1) &= \chi^{cw}((U_t, \Gamma'_t) \circ (D_t, \Gamma_t))(1) \\ &= \chi^{cw}((U_t, \Gamma'_t)) \circ \chi^{cw}((D_t, \Gamma_t))(1) \\ &= \sum_i \kappa_i \lambda_i \end{aligned} \tag{6.7}$$

Which is the number of Tait-coloring of $(\mathbb{S}^2, \Gamma)$ by Lemma 6.1. So, once we have shown that this number is independent of the choice of the generic cross-section, we are done. For that, let $\mathbb{S}_s$ be another generic cross-section. Without loss of generality, we can assume that it fits into the following composition

$$\emptyset \xrightarrow{(D_s, \Gamma_s)} \mathbb{S}_s \xrightarrow{(C_{st}, \Gamma_{st})} \mathbb{S}_t \xrightarrow{(U_t, \Gamma'_t)} \emptyset \tag{6.8}$$

The action of χ^{cw} on it gives:

$$\mathbb{C} \xrightarrow{\chi^{cw}((D_s,\Gamma_s))} X^{\otimes n_s} \xrightarrow{\chi^{cw}((C_{st},\Gamma_{st}))} X^{\otimes n_t} \xrightarrow{\chi^{cw}((U_t,\Gamma'_t))} \mathbb{C} \tag{6.9}$$

Let $\chi^{cw}(\mathbb{S}^2,\Gamma)(1) = \sum_j \kappa'_j\lambda'_j$ along $\mathbb{S}_s$, which means $\chi^{cw}(D_s,\Gamma_s) = \sum_j \lambda'_j v_j$ for $v_j \in \mathcal{B}(X^{\otimes n_s})$, and $\chi^{cw}(U_s,\Gamma'_s)$ maps $v_j \in \mathcal{B}(X^{\otimes n_s})$ to κ'_j. Now, suppose $\chi^{cw}(C_{st},\Gamma_{st}) = (a_{ij})$ in the same bases $\{v_j\}$ of $X^{\otimes n_s}$ and $\{w_i\}$ of $X^{\otimes n_t}$. The functoriality of χ^{cw} applied on the identities $(C_{st},\Gamma_{st}) \circ (D_s,\Gamma_s) = (D_t,\Gamma_t)$ and $(U_t,\Gamma'_t) \circ (C_{st},\Gamma_{st}) = (U_s,\Gamma'_s)$ gives

$$\lambda_i = \sum_j a_{ij}\lambda'_j \quad , \quad \kappa'_j = \sum_i \kappa_i a_{ij}$$

respectively. This leads to

$$\sum_j \kappa'_j\lambda'_j = \sum_j\sum_i \kappa_i a_{ij}\lambda'_j = \sum_i\sum_j \kappa_i a_{ij}\lambda'_j = \sum_i \kappa_i \sum_j a_{ij}\lambda'_j = \sum_i \kappa_i\lambda_i$$

□

Proof of Lemma 6.2:

Proof. By Theorem 6.3 every cylinder (C_{st},Γ_{st}) can be written as the composition of basic cylinders as in Fig. 16. Let n be the length of such decomposition, that is, the minimum number of basic cylinders ρ_{i_j} required to make a given cylinder (C_{st},Γ_{st}). We prove Lemma 6.2 by induction on n. The base case is $n=1$. In this case, (C_{st},Γ_{st}) is one of the four basic cylinders in Fig. 16. Use Proposition 6.1 to obtain a cell-decomposition and write each basic cylinders as horizontal composition of basic-gons. Then the statement of the Lemma 6.2 follows from Proposition 3.3 and Proposition 6.2. (Since, other than $(i,i+1)$, everything else is the identity, $(i,i+1)$ is one of the four patterns appearing in Proposition 6.2.) Now, for the induction step, assume the statement of Lemma 6.2 is true for all $k < n$. Choose a generic cross-section $\mathbb{S}_o$ of the cylinder (C_{st},Γ_{st}) and obtain the composite

$$\begin{array}{ccccccccc} & & (C_{so},\Gamma_{so}) & & & & (C_{ot},\Gamma ot) & & \\ & \overset{\iota_1}{\nearrow} & & \overset{\mathfrak{o}_1}{\searrow} & & \overset{\iota_2}{\nearrow} & & \overset{\mathfrak{o}_2}{\searrow} & \\ \mathbb{S}_s & & & & \mathbb{S}_o & & & & \mathbb{S}_t \end{array}$$

Each of the cylinders (C_{so},Γ_{so}) and (C_{ot},Γ_{ot}) has lengths less than n, so the statement of Lemma 6.2 is true for them by the induction hypothesis. By the functoriality of χ^{cw}, we get the composite

$$X^{\otimes n_s} \xrightarrow{\chi^{cw}(C_{so},\Gamma_{so})} X^{\otimes n_o} \xrightarrow{\chi^{cw}(C_{ot},\Gamma_{ot})} X^{\otimes n_t} \tag{6.10}$$

which equals $\chi^{cw}(C_{st},\Gamma_{st})$. Let $\mathcal{B}(X^{\otimes n_s}) = \{v_j\}$, $\mathcal{B}(X^{\otimes n_o}) = \{z_k\}$ and $\mathcal{B}(X^{\otimes n_t}) = \{w_i\}$, and in these bases, the matrices of $\chi^{cw}(C_{st},\Gamma_{st})$, $\chi^{cw}(C_{so},\Gamma_{so})$, and $\chi^{cw}(C_{ot},\Gamma_{ot})$

are given by $A := (a_{ij})$, $B := (b_{kj})$, and the matrix of $C := (c_{ik})$ respectively. By linearity we get

$$a_{ij} = \sum_k c_{ik} b_{kj} \tag{6.11}$$

By Lemma 6.2 c_{ik} is the number of coloring of (C_{ot}, Γ_{ot}) with in-boundary color z_k and out-boundary color w_i. Similarly, b_{kj} is the number of coloring of (C_{so}, Γ_{so}) with in-boundary color v_j and out-boundary color z_k. Now, Lemma 6.1 implies that Eq. (6.11) is nothing but the number of 3-edge coloring of (C_{st}, Γ_{st}) with an in-boundary color v_j and out-boundary color w_i, but by the definition of a matrix, a_{ij} is the component of $\chi^{cw}(C_{st}, \Gamma_{st})(v_j)$ in the direction of w_i.

□

Not sure, if the proof based on induction is the best someone can do. I am not satisfied and still looking for another proof.

— Amit

Remark 6.4. We chose a PLCW decomposition to define the Tait-coloring, but Theorem 6.2 also shows that the number of Tait-coloring is independent of this choice as the functor χ^{cw} is. See 4.3 and [DKR11], Section-3.6.

In fact, it is not difficult to show using 'Kirillov-moves' ([KJ12], Section-6,7) that the definition of a 3-edge coloring is independent of a choice of a PLCW decomposition, but we only need the number of such coloring, so we are going to content ourselves with Remark 6.4.

We conclude this section with a conjecture, which is a reformulation of 4-color theorem in the language we have developed so far:

Conjecture 6.3. If Γ is a planar trivalent graph with no bridge then $\chi^{cw}(\mathbb{S}^2, \Gamma)(1) \neq 0$.

It is immediate from Corollary 6.1 that if Γ has a bridge, then $\chi^{cw}(\mathbb{S}^2, \Gamma)(1)$ equals 0. Conjecture 6.3 is the converse of it. Equivalence with the 4-color theorem is easily established from the statement of Theorem 6.2 and a result due to Tait, see [Tai80] and [Bal18] for details. Since, we are only working with one-components graph, a bridge on a planar graph is equivalent to the existence of a generic cross-section with a single defect. Therefore, Conjecture 6.3 can be reformulated as:

Conjecture 6.4. If the linear map $\chi^{cw}(\mathbb{S}^2, \Gamma) : \mathbb{C} \to \mathbb{C}$ can be written as the composition

$$\mathbb{C} \xrightarrow{\chi^{cw}(\emptyset, \mathbb{S}_t)} X \xrightarrow{\chi^{cw}(\mathbb{S}_t, \emptyset)} \mathbb{C}$$

then it is the zero map.

7. Conclusion and future direction

We conclude this manuscript with the mention of future projects of potential interests.

7.1. Reformulation using constructible sheaf and infinity category. There are two immediate projects related to my PhD work that I have already started to pursue. These are:

- Redefining everything in terms of the language of $(\infty, 2)$ category. The (∞, n) perspective of a topological field theory with defects was already touched by Lurie in [[Lur08], section-4.3]. I have two reason to pursue this. First, having a potential project in hands is a golden chance to learn infinity category. Second, the work of Khovanov-Robert [KR21] on Foams appears very similar to work done by me, but one dimensional higher. Foam find its place in defect TFT in the subject of *orbifold-completion* as discussed in [Car23] and [Car16].
- Defects were formulated using the notion of constructible sheaf in [[FMT22], section-2.4, 2.5]. I myself saw the possibility to introduce constructible sheaf, but it has already been developed by Freed, et-al. So, I am looking forward to translate my work and in the process learn about the connection of defects with topological symmetries in QFT.

These two may very well be related. At this point, I am not sure but have a strong gut feeling. This is why I have listed them as two bullet-points of same project.

7.2. A generalisation of the universal construction. Referring to [Kho20] [section-1] if we choose α to be the number of Tait-coloring of a planar trivalent graph then it satisfies the property $\alpha(\Gamma_1 \sqcup \Gamma_2) = \alpha(\Gamma_1)\alpha(\Gamma_2)$. So, the question is can one generalise this construction for defect TFTs? In fact, my initial plan was to generalise this and to prove that the two functors χ^{cw} and χ are naturally equivalent. Given the correspondance between extended and defect TFT [[Kap10], section-2.3] and the fact that there is already a version for 2-extended TFT given in [Kho02], this should not be very difficult.

7.3. Tait's correspondence, defects, and obstruction. Both [DKR11], footnote-7 and [CDZR23] Introduction, paragraph two mentions how defects generalises groups. This action of 1-defects is exactly the procedure Tait describes to establish correspondence between 4-face coloring and 3-edge coloring in the case of planar trivalent graphs. Can this pursuit, with the tools given by defect TFT, and possibly with projects mentioned in Section 7.1, lead to an obstruction theory that proves Tait's conjecture?

7.4. **Word problem and n-deformation.** We promised in the introduction that the word problem gets interpreted as a local to global problem. Here is the precise statement:

Theorem 7.1. Given a group G and a presentation $\mathcal{P}_G$, form the category $Bord_2^{def,cw}(\mathcal{P}_G)$. Two words w_1 and w_2 represent the same element of the group G if and only of the two circles with defects are cobordant in $Bord_2^{def,cw}(\mathcal{P}_G)$.

So, now the question is: can we produce a TFT that captures the obstruction? It may even be vague to state at this point. I do not think that it is going to be that straight forward. Same construction also gives the category $Bord_2^{def,cw}(\mathcal{G})$, which has the property that only those elements of the group meets around a junction whose product is the identity. The definition of $Bord_2^{def,cw}(\mathcal{P}_G)$ has uncanny resemblance with some of the conditions for 2-deformation given in [Wri75]. It is better to work with the category $Bord_2^{def,cw}(\mathcal{G})$ defining two surfaces that differs by a presentation as weekly equivalent (as described in [Gro15]) so that the overall mathematics does not depend on a specific presentation. Again, this problem might be strongly connected with the future direction discussed in ‘Section 7.1.

7.5. **Graph connection and trivalent vertices of N-graphs.** It is time to reveal that the result of my PhD work is actually a chance discovery, and I was really working on something different: ribbon graph formulation of N-graphs. There was more than one problem in my mind coming from the work of [CZ23] and [TZ16]. The main reason to introduce ribbon graphs was to connect with special Legendrians in $\mathbb{S}^5$ (see [Wan02]). However, one would need to allow to glue more than just disks if they want to work with legendrian weaves. There could be ways as discussed in [Bar21] but there might be other ways, which I refrain to discuss here. The chance discovery came while exploring the connection given in Appendix-A in [CZ23]. Indeed, an N-graph with only hexagonal vertices lives in the set $\mathrm{Mor}(Bord_2^{def,cw}(\mathcal{P}_{S_n}))$ where

$$S_n = \left\langle \tau_1, \dots, \tau_{n-1} \middle| \begin{array}{c} \tau_i\tau_j = \tau_j\tau_i \quad |\, i-j\,| > 1 \\ \tau_i\tau_{i+1}\tau_i \;=\; \tau_{i+1}\tau_i\tau_{i+1} \\ \tau_i^2 \;=\; 1 \end{array} \right\rangle$$

We still do not know how to incorporate trivalent vertices. I suspect that this problem is closely related to connect to the theory of *graph-connections* [see [BZ23], section-3], to which my work has some uncanny connections.

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