Chern-Weil Theory

Characteristic Classes

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Conventions

1. In what follows, F is $\mathbb{R}$ or $\mathbb{C}$ and E is an F vector bundle of rank k over a manifold M.
2. M is assumed to be real manifold of dimension m otherwise stated.
3. $\mathcal{F}$ stands for the ring of C^∞ functions on M.

Connection on a vector bundle

Let $\pi : E \to M$ be a C^∞ vector bundle over M. A connection on E is a map

$$\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$$

denoted $(X, e) \mapsto \nabla_X e$ such that $\forall X \in \Gamma(TM)$ and $e \in \Gamma(E)$

1. $\nabla_X e$ is $\mathcal{F}$ linear in X and F linear in e.
2. (Leibniz rule) if f is a c^∞ function on M then

$$\nabla_x(fe) = (Xf)e + f\nabla_x e$$

In other words ∇ is a linear map from $\Gamma(E)$ to $\Gamma(E \otimes T^*M)$
Fact: Every C^∞ vector bundle E over a manifold M has a connection.

Connection and Curvature matrix

We are interested in the local expression for ∇. So, let U be a trivializing neighborhood then:
Fact: ∇ restricts to a connection
$\nabla^U : \Gamma(TU) \times \Gamma(U, E) \to \Gamma(U, E)$ such that for any vector field $X \in \Gamma(TM)$ and section $s \in \Gamma(E)$ we have $\nabla^U_{X|_U}(s|_U) = (\nabla_X s)|_U$.
So, if $e_1, \ldots, e_k$ (denoted simply as e) is a frame for E over U then any section $s \in \Gamma(U, E)$ is a linear combination $s = \Sigma a^j e_j$, and thus by linearity and Leibniz rule, can be computed from $\nabla_X e_j$. As a section of E over U we have

$$\nabla_X e_j = \Sigma \omega^i_j(X) e_i$$

So, $\omega = [\omega^i_j]$ is a matrix of one forms on U.

Similarly, the curvature $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(M, E) \to \Gamma(M, E)$ also has a local expression

$$R(X, Y)e_j = \Sigma \Omega^i_j(X, Y)e_i$$

The connection and curvature matrices (ω and Ω) respectively are related by the second structural equation

$$\Omega = d\omega + \omega \wedge \omega$$

Proposition-1: Suppose $\bar{e}$ is another frame for E over U such that $\bar{e} = ea$ for some $a : U \to GL(k, F)$. If ω and $\bar{\omega}$ (Ω and $\bar{\Omega}$) are the connection (curvature) matrices of ∇ relative to two frames then

1. $\bar{\omega} = a^{-1}\omega a + a^{-1}da$
2. $\bar{\Omega} = a^{-1}\Omega a$

Invariant Polynomials

Let $X = [x^i_j]$ be a $k \times k$ matrix with indeterminate entries x^i_j. A polynomial $P(X)$ on $\mathfrak{gl}(k, F) = F^{k\times k}$ is a polynomial in the entries of X. Such a polynomial is said to be $\mathbf{Ad}GL(k, F)$ - **invariant** or simply **invariant** if

$$P(A^{-1}XA) = P(X) \quad \forall\, A \in GL(k, F)$$

Example

$tr(X)$ and $det(X)$ are two such invariant polynomials.

Example (Characteristic Polynomials)

Consider the expansion

$$det(\lambda I + X) = \lambda^k + \mathfrak{f}_1(X)\lambda^{k-1} + \cdots + \mathfrak{f}_{k-1}(X)\lambda + \mathfrak{f}_k(X)$$

Then it is easy to see that all $\mathfrak{f}_i(X)$ are invariant polynomials. These are coefficients of characteristic polynomials of X with $\mathfrak{f}_1$ and $\mathfrak{f}_k$ trace and determinant respectively.

Example (Trace Polynomials)

For any positive integer r, define the rth **trace polynomial** $\Sigma_r X := tr(X^r)$. Then, $\Sigma_r X$ is an invariant polynomial.

Fact: The set of all $\mathfrak{gl}(k, F)$ invariant polynomials form a ring under usual addition and multiplication of polynomials. We will denote this ring by $\mathfrak{Inv}(\mathfrak{gl}(k, F))$.

Proposition-2: The ring $\mathfrak{Inv}(\mathfrak{gl}(k, F))$ is generated (as a ring) either by the coefficients $f_i(X)$ of the characteristic polynomial or by the trace polynomials Σ_r:

$$\mathfrak{Inv}(\mathfrak{gl}(k, F)) = F[f_1, \ldots, f_k] = F[\Sigma_1, \ldots, \Sigma_k]$$

Chern-Weil Homomorphism

Now consider a homogeneous invariant polynomial $P(X)$ of degree n on $\mathfrak{gl}(k, F)$. If $\Omega = [\Omega^i_j]$ is the curvature matrix of a connection ∇ on an open set U with respect to a frame e around some (fix) point $q \in U$, then $P(\Omega)$ defines a $2n$ form on U. By Proposition-1, if $\bar{e}$ is another frame around q and $\bar{\Omega}$ is the corresponding curvature matrix then

$$\bar{\Omega} = a^{-1}\Omega a \quad \text{for some } \ a : U \to GL(k, F)$$

The fact that $P(X)$ is invariant forces $P(\bar{\Omega} = P(\Omega)$. Thus if $\{U_\alpha\}_{\alpha \in A}$ is a trivializing open cover for E with curvature matrices $\{\Omega_\alpha\}_{\alpha \in A}$ then the collection of $2n$ forms $\{P(\Omega_\alpha)\}_{\alpha \in A}$ patches together to give rise to a global $2n$ form.

Now, we are ready to formulate our main result.

Theorem (Chern-Weil)

Let E be a vector bundle of rank k over a manifold M, ∇ a connection on E, and P a homogeneous invariant polynomial of degree n on $\mathfrak{gl}(k, f)$. Then

1. *the global $2n$ form $P(\Omega)$ on M is closed;*
2. *the cohomology class of the closed form $P(\Omega)$ is independent of the connection.*

Thus for any vector bundle E of rank k over M and any connection ∇ on E, the map

$$C_E : \mathfrak{Inv}(\mathfrak{gl}(k, f) \to H^* M$$

$$P(X) \mapsto [P(\Omega)]$$

is an F-algebra homomorphism. It is called Chern-Weil homomorphism.

We don't have time for the proof but just to mention, proof uses Proposition-2 and Bianchi identity.
Corollary: The cohomology class $[P(\Omega)]$ satisfies the naturality property and therefore defines a **characterisitic class**, i.e.

$$[P(f^*\Omega)] = f^*[P(\Omega)]$$

where $f : N \to M$ is a map of C^∞ manifolds.

Consequences of Chern-Weil homomorphism

1. By part-2 of the theorem, the cohomology class doesn't depend on connection. So, we can always choose a nice connection (e.g. a Riemannian connection on a Riemannian manifold) to deduce our results. For instance, since Ω of a Riemannian connection is skew symmetric, we get

$$\mathfrak{f}_1(\Omega) = tr(\Omega) = 0$$

Similarly

$$\Sigma_r = tr(\Omega^r) = 0 \text{ for } r \text{ odd}$$

Therefore, it follows that if P is a homogeneous polynomial of odd degree n then the cohomology class $P[\Omega]$ is zero in $H^{2n}(M)$.

2. Similarly a special connection can be used to prove the **Whitney product formula**, namely, as we obtain direct sum of two linear transformations.

Various Characteristic Classes

1. Take $F = \mathbb{R}$, then $[\mathfrak{f}_{2n}(\frac{i\Omega}{2\pi})] \in H^{4n}(M)$ is the **k-th Pontrjagin class**.
2. With $F = \mathbb{R}$ as before, if Ω is a curvature matrix corresponding to a positively oriented orthonormal frame e over a Riemannian bundle E over a manifold of dimension $2m$, then the characteristic class $[Pf(\frac{\Omega}{2\pi})] \in H^{2m}(M)$ is called the **Euler Class** of the oriented Riemannian bundle E.
3. Take $F = \mathbb{C}$ and let Q be an invariant complex polynomial on $\mathfrak{gl}(k, \mathbb{C})$ then the image of Q under the Chern-Weil homomorphism defines a characteristic class. In particular when $Q(X) = det(I + \frac{iX}{2\pi})$ we obtain **Chern Classes** $c_i(E)$ from

$$det(\lambda I + \frac{i\Omega}{2\pi}) = 1 + c_1(E) + \ldots c_k(E)$$

Alternatively (John Baez) the **n-th Chern class** is the image of the trace polynomial Σ_n under the Chern-Weil homomorphism.

Vistas

1. The L-genus in the Hirzebruch signature formula is associated to L-polynomials evaluated at Pontrjagin classes.
2. The Hirzebruch-Riemann-Roch theorem uses two other distinct classes of polynomials, namely, Chern characters and Todd polynomials evaluated at Chern classes of the bundle and that of manifold respectively.