Legendrian Weaves and TQFT with decorations

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Motivation

Let

$$K_4 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, c = ab \rangle$$

and

$$A = \mathbb{Z}[K_4]$$

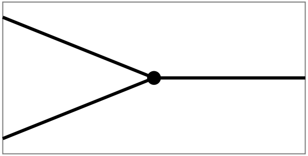
equipped with the multiplication

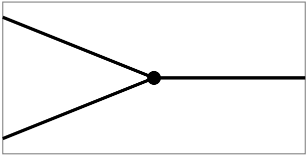
$$\mu : A \otimes A \to A$$

given by

$$\begin{array}{llll} 1 \otimes 1 \mapsto 0 & 1 \otimes a \mapsto a & 1 \otimes b \mapsto b & 1 \otimes c \mapsto c \\ a \otimes 1 \mapsto a & a \otimes a \mapsto 0 & a \otimes b \mapsto c & a \otimes c \mapsto b \\ b \otimes 1 \mapsto b & b \otimes a \mapsto c & b \otimes b \mapsto 0 & b \otimes c \mapsto a \\ c \otimes 1 \mapsto c & c \otimes a \mapsto b & c \otimes b \mapsto a & c \otimes c \mapsto 0 \end{array}$$

Diagrammatically μ is represented as



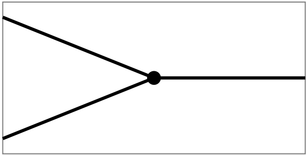


We define a pairing

$$\beta : A \otimes A \to \mathbb{Z}$$

$$1 \otimes 1, a \otimes a, b \otimes b, c \otimes c \mapsto 1_{\mathbb{Z}} \ ; 0 \text{ otherwise}$$

and extended linearly.



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and a co-pairing

$$\gamma : \mathbb{Z} \to A \otimes A \quad ; \quad 1_{\mathbb{Z}} \mapsto a \otimes a + b \otimes b + c \otimes c$$

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Remark: β is associative but NOT non-degenerate!

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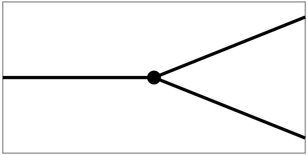
Nonetheless, we get a co-multiplication

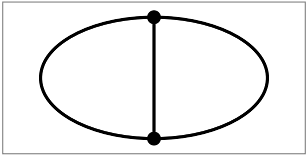
$$\Delta : A \to A \otimes A \quad ; \quad 1 \mapsto a \otimes a + b \otimes b + c \otimes c$$

$$a \mapsto b \otimes c + c \otimes b, \quad b \mapsto c \otimes a + a \otimes c, \quad c \mapsto a \otimes b + b \otimes a$$

represented diagrammatically as



Now, consider the following example



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So, there exists an integer n such that $\phi(v) = nv$.

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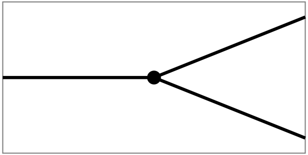

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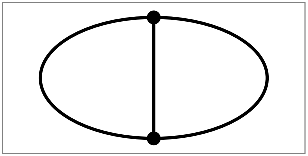

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Question: How is this n related to the above graph?

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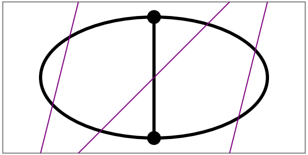

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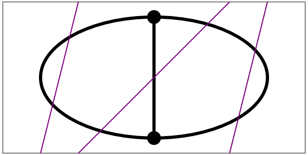

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This gives

$$1_{\mathbb{Z}} \xrightarrow{\gamma} a \otimes a + b \otimes b + c \otimes c$$

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$$a \otimes a \xrightarrow{id \otimes \Delta} a \otimes (b \otimes c + c \otimes b) = a \otimes b \otimes c + a \otimes c \otimes b \xrightarrow{\mu \otimes id} c \otimes c + b \otimes b$$

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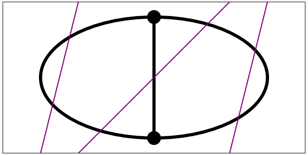

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and using the symmetry we see that $\chi = 6$ which is the number of 3-edge coloring for the theta graph.

TQFT with decoration

This is going to be the case provided one could make sense of the diagrammatic representation above as a tensor category: the boundary of a graph only makes sense when it is embedded as a 1-skeleton of a CW complex.

(Meta)Theorem: Under the above representation of this tensor category, a planar trivalent graph Γ correspond to a linear map $\psi : \mathbb{Z} \to \mathbb{Z}$ with the property

$$\psi(1) = \#\{3 - \text{edge coloring of } \Gamma\}$$

N-graphs

Let $N \in \mathbb{N}$. An N-graph G on Σ is a collection of N objects, a surface Σ - the zeroth object in the collection, and $N-1$ trivent graphs $G_1, \ldots G_{n-1}$ embedded on Σ but subject to the following conditions:

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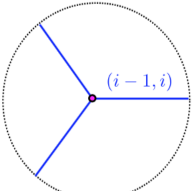

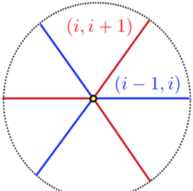

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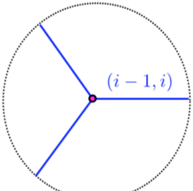

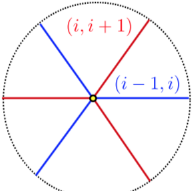

Example: A 4-graph with $\Sigma = \mathbb{S}^2$.

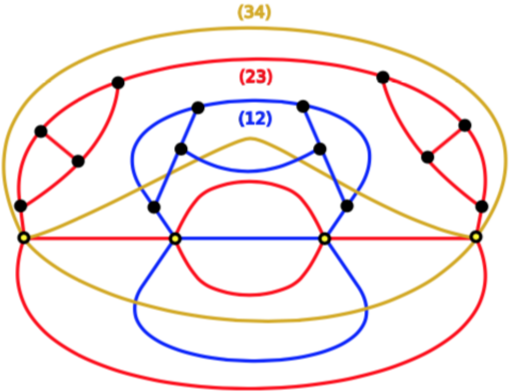

Figure: Notice that G_1(blue) and G_3 (yellow) intersect **virtually**. **Source:** 'Legendrian Weaves' Casals-Zaslow

Legendrian Weaves: Construction of the wavefront

Let G be an N-graph on Σ. It allows us to assign a Legendrian surface $\Lambda(G)$ in $(J^1\Sigma, \alpha)$ by associating singularities A_1^2 to the edges of the graph, and using the fact both D_4^- and A_1^3 are uniquely determined by their A_1^2 front singularities. The local germs of these singularities and the Legendrian front projection $\Pi : J^1\Sigma \to \Sigma \times \mathbb{R}$ allow us to assign this Legendrian to the wavefront $W(G) \subset \Sigma \times \mathbb{R}$ coming from G as follows, and then gluing these local wavefronts:

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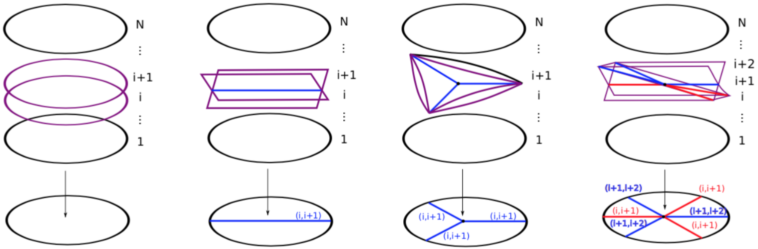


Braid closure and Casals-Zaslow construction

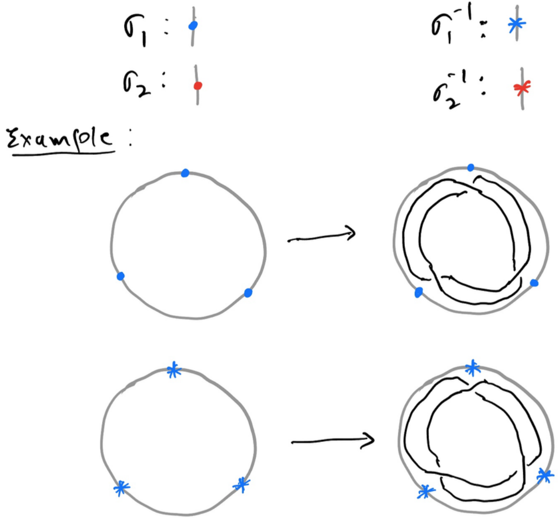


Figure: Positive(top) and Negative (bottom) Trefoil

Hexagonal vertex, Yang-Baxter and RD-III

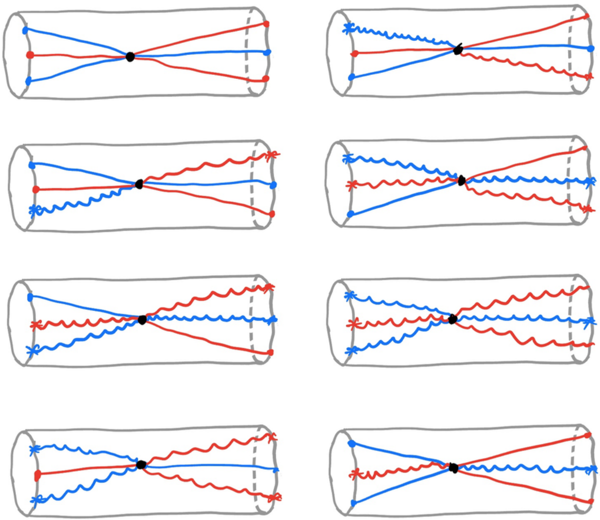
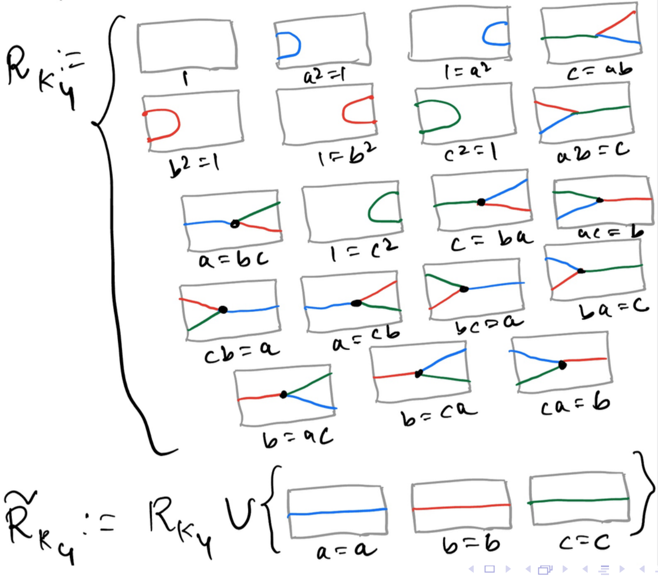


Figure: together with their reflections

Decoration with K_4 and graph coloring

$$K_4 := \langle a, b, c \mid a^2 = b^2 = c^2 = 1,\ c = ab \rangle$$



TQFT with decoration: Prototypical Example

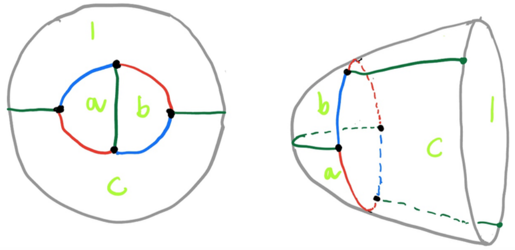


Figure: coloring of graph induces coloring of region

Thank You!

Jet Space

Given a manifold M, let λ be the Louville form on T^*M, then the 1 - Jet spaces, denoted, $J^1M := T^*M \times \mathbb{R}$ has a contact structure given by the 1-form $\alpha = dz - \lambda$. Where z is the co-ordinate on $\mathbb{R}$

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Convention: For our purpose M will be a surface Σ.

Example: For $M = \mathbb{R}^2_{(x_1,x_2)}$, $J^1M \cong \mathbb{R}^5_{(x_1,x_2,y_1,y_2,z)}$ with the contact structure $\xi_{std} = \ker\,(dz - y_1dx_1 - y_2dx_2)$.

This example is important since because of *Darboux's Theorem* that's how our $J^1\Sigma$ looks like locally (in a Darboux's chart.)

Definition: A surface $\Lambda(C)$ in $J^1\Sigma$ (i .e. a 2 - dimensional submanifold of $J^1\Sigma$) is called *Legendrian* if for all $p \in \lambda(C)$, $T_p\Lambda(C) \subset \xi_p$.

Arnold's Theorem

The Legendrian fibration $\Pi : \mathbb{R}^5 \to \mathbb{R}^3$,
$\Pi(x_1, x_2, y_1, y_2, z) \to (x_1, x_2, z)$ allows us to assign a legendrian surface $\Lambda(C)$ in the domain of Π to a singular surface C in $\mathbb{R}^3$. Where

$$y_1 = x_1 - \text{slope of the tangent plane} - T_{(x_1,x_2,z)}C, \text{ and}$$

$$y_2 = x_2 - \text{slope of the tangent plane} - T_{(x_1,x_2,z)}C$$

i.e. in a local parametrisation $(u, v, z(u, v))$ of the surface C, $y_1 = \partial_u z(u, v)$ and $y_2 = \partial_v z(u, v)$.

Theorem(Arnold): **(Singularities of spatial fronts)** A generic front in 3 - space has the following singularities:

1. (semicubic) Cuspidal edge (A_2).
2. swallowtails (A_3).
3. points of transversal self-intersection (A_1^2, A_1A_2, A_1^3).

Other than these three there are two unstable singularities: D_4^+ and (very important for our purpose) D_4^-.

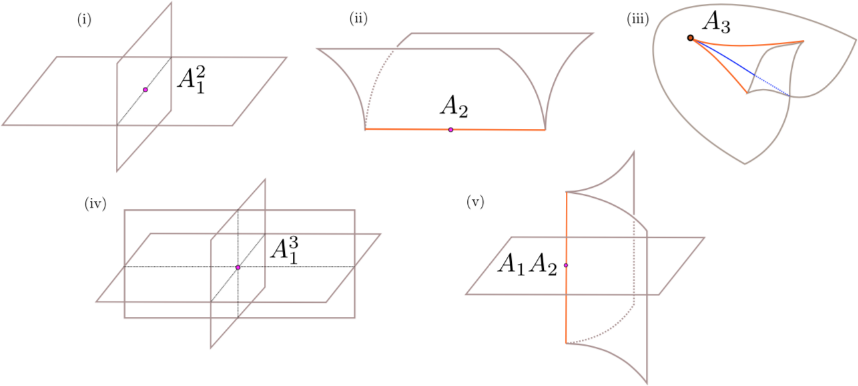

Figure: Generic Singularities

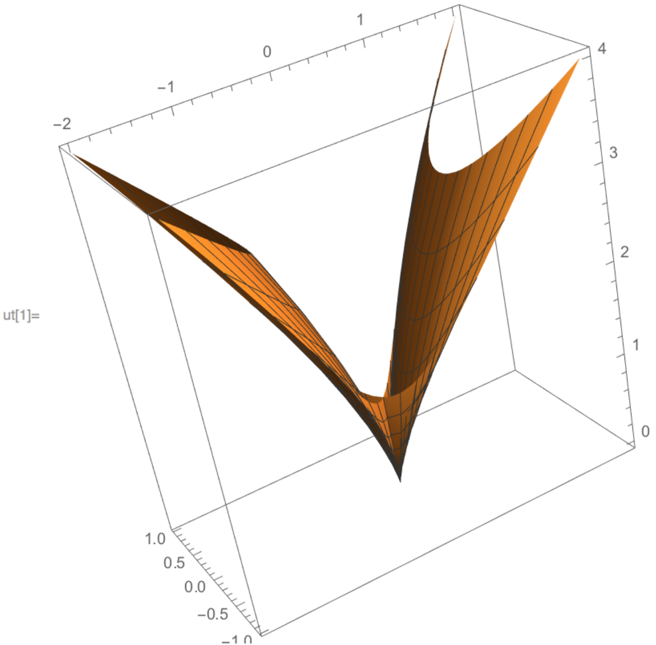

Figure: D_4^+

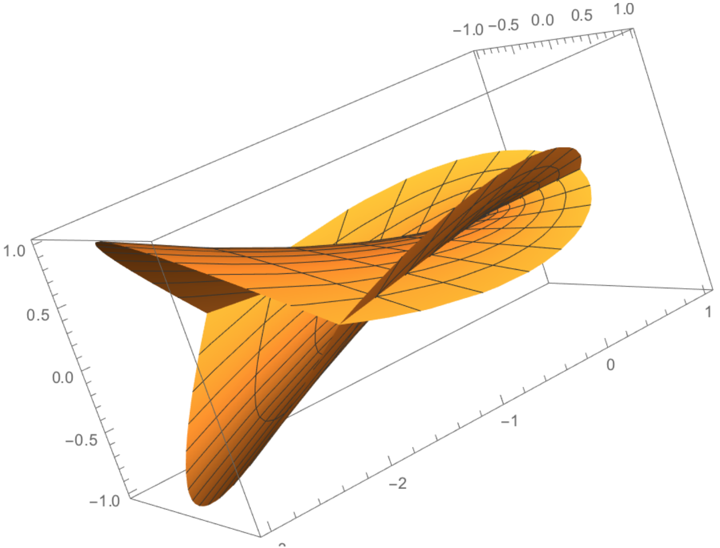

Figure: D_4^-, wild potato chips

and then gluing these local wavefronts. More precisely for an open cover $\{U_i\}$ of Σ, the local wavefront is constructed in $U_i \times \mathbb{R}$ as above (note that $U_i \cong \mathbb{D}^2$) which are targents of front projections in the Darboux chart $(J^1\Sigma, \alpha) \cong (J^1\mathbb{D}^2, \xi_{std})$.

Definition: Let G be an N-graph on Σ, the Legendrian weave

$$\Lambda(G) \subset (J^1\Sigma, \alpha)$$

is the embedded Legendrian surface whose wavefront $W(G) \subset \Sigma \times \mathbb{R}$ is obtained by weaving the wavefronts $\Sigma \times \{1\} \cup \cdots \cup \Sigma \times \{N\} \subset \Sigma \times \mathbb{R}$ according to the local patterns.

Here are examples of Reidemeister moves - 0, I, II, and III:

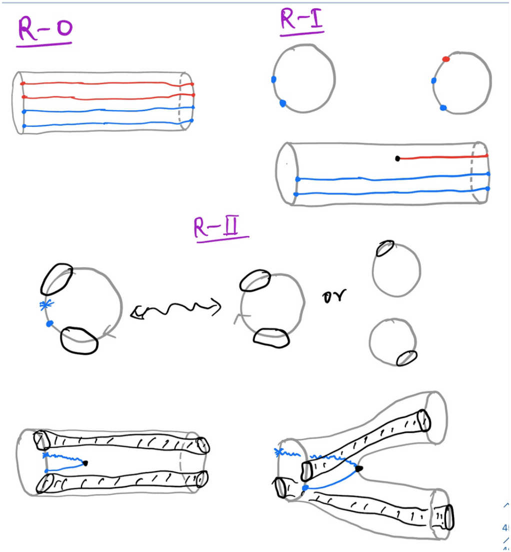

Figure: Together with their reflections

Decoration with general group : $\mathbb{Z}_2$

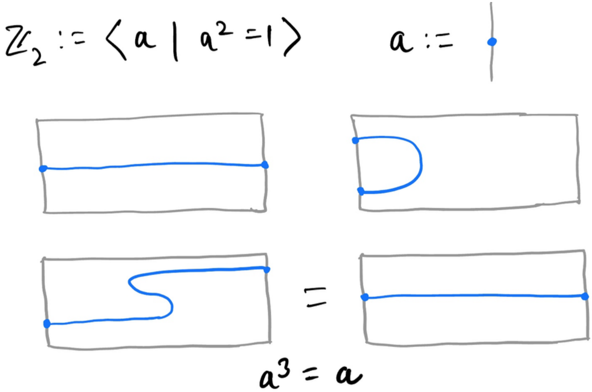

Figure: relations

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