Legendrian Knots

Contact Structures

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Introduction

Legendrian Knots lie at the crossroad of *knot theory* and *contact geometry*.

In other words, one can say that Legendrian Knots arise when contact topology imposes extra structure on knot theory.

Knot

A Knot is a smooth embedding of S^1 in $\mathbb{R}^3$, or into any 3-manifold.

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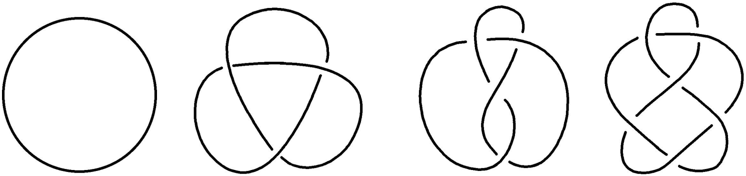

Figure: Knots

Contact Geometry

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The standard contact structure on $\mathbb{R}^3$ is the plane field ξ_{std} where the plane at (x, y, z) has normal vector equal to $[y, 0, -1]$ or equivalently, is spanned by the pair of vectors $\{[0, 1, 0], [1, 0, y]\}$.

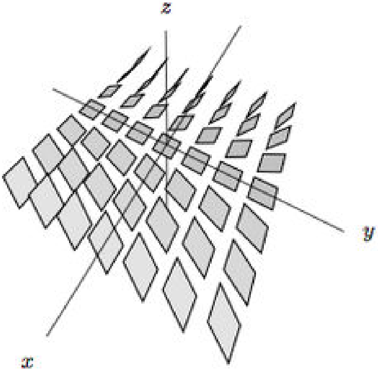

Figure: $(\mathbb{R}^3, \xi_{std} = ker(dz - ydx))$

However, there do exist curves whose tangent vectors do lie in the contact structure; such curves are called Legendrian, and a knot that is also a Legendrian curves is called ... you can guess.

Standard Legendrian Knot

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$$K : S^1 \to \mathbb{R}^3 : t \mapsto (x(t), y(t), z(t))$$

subject to:

$$z'(t) - y(t)x'(t) = 0$$

Shape of Legendrian Knot

There are two ways to picture Legendrian knot K, via front projection and Lagrangian projection.

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The projection to the xz plane is called the front projection.

Front Projection

The equation $z'(t) - y(t)x'(t) = 0$ implies that the y coordinate of K is determined by the slope of it's front projection

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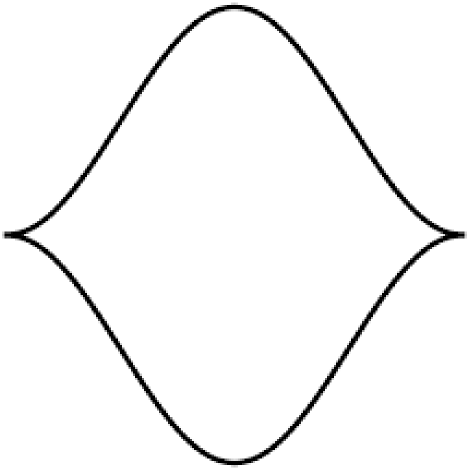

Figure: An example of a front projection

Lagrangian Projection

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The Lagrangian projection of a Legendrian knot must bound the zero signed area. In particular, a circle cannot be the Lagrangian projection of a Legendrian Knot.

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Invariants

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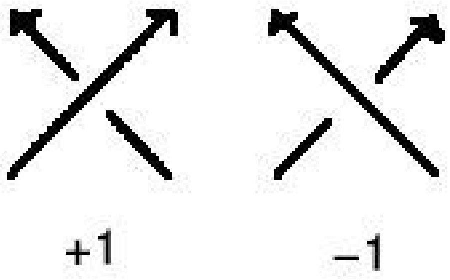

is defined as:

$$tb(D) = w(D) - \frac{1}{2}(\text{number of cusps})$$

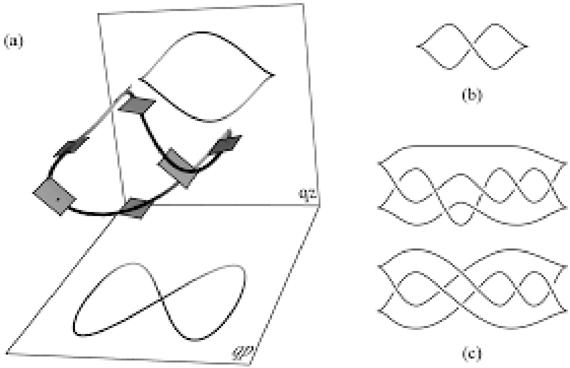

Figure: The Legendrian knots corresponding to (a) and (b) are not equivalent as $tb(a) = -1$ while $tb(b) = -2$.