Special Lagrangian Geometry

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Introduction

1. Special Lagrangian m-folds (SL m-folds) are a distinguished class of real m-dimensional minimal submanifolds which may be defined in $\mathbb{C}^m$ or in *Calabi-Yau m-folds*, or more generally in *almost Calabi-Yau m-folds*.

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2. Having a good understanding of the singularities of special Lagrangian submanifolds will be essential in clarifying the SYZ-conjecture on the mirror symmetry of Calabi-Yau 3-folds and also in connection of recent work of Zaslow and Truemann in the paper *Cubic Planar Graph and Legendrian Surface Theory.*

SL submanifolds in $\mathbb{C}^m$

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$$g = |dz_1|^2 + \cdots + |dz_m|^2, \qquad \omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + \cdots + dz_m \wedge d\bar{z}_m)$$

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Then $Re(\theta)$ and $Im(\theta)$, are real m-forms on $\mathbb{C}^m$, both calibrations. **Recall:** ϕ, a closed k-form on a manifold M, is a *calibration* on M if for every oriented k-plane V on M we have $\phi|_V \leq vol_V$. Said differently, $\phi|_V = \alpha.vol_V$ for some $\alpha \in \mathbb{R}$ with $\alpha \leq 1$. For N, an oriented k-submanifold of M, each tangent space T_xN for $x \in N$ is an oriented tangent k-plane. We say that N is a calibrated submanifold or ϕ-submanifold if $\phi|_{T_xN} = vol_{T_xN}$.

Definition: Let L be an oriented real m-submanifold of $\mathbb{C}^m$. We call L a *special Lagrangian Submanifold* of $\mathbb{C}^m$, or *SL* m-fold for short, if L is calibrated w.r.t $Re(\theta)$ in the sense of above.

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Proposition: Let L be a real m-dimensional submanifold of $\mathbb{C}^m$. Then L admits an orientation making it into an SL m-fold if and only if $\omega|_L \equiv 0$ and $Im(\theta)|_L \equiv 0$. More generally, it admits an orientation making it into an SL m-fold with phase $e^{i\psi}$ if and only if $\omega|_L \equiv 0$ and $(cos\psi Im(\theta) - sin\psi Re(\theta))|L \equiv 0$

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Also note that L is an *SL* m-fold with phase $e^{i\psi}$ if and only if $e^{\frac{-i\psi}{m}} L$ is an $SLm - fold$ with phase 1. So, studying *SL* m-fold with phase 1 suffices.

Examples

Let $\mathbb{C}^2$ has complex cordinates (z_0, z_1) with $z_0 = x_0 + ix_1$ and $z_1 = x_2 + ix_3$. Let us denote the corresponding complex structure on $\mathbb{R}^4$ by J_0. Then:

$$g = dx_0{}^2 + \cdots + dx_3{}^2, \qquad \omega = dx_0 \wedge dx_1 + dx_2 \wedge dx_3,$$

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Now define a different set of complex coordinates (w_1, w_2) by $w_1 = x_0 + ix_2$, $w_2 = x_1 - ix_3$, and denote the corresponding complex structure on $\mathbb{R}^4$ by J. We notice, $\omega - iIm(\theta) = dw_1 \wedge dw_2$.

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Thus $L \subset \mathbb{R}^4$ is special Lagrangian if and only if and only if $(dw_1 \wedge dw_2)|_L \equiv 0$. But this holds if and only if L is a *holomorphic curve* w.r.t the complex coordinates (w_1, w_2) or L is a *J-holomorphic curve*. This means that SL 2-folds are already well understood.

SL submanifolds in $\mathbb{C}^m$ as graphs

Let $f : \mathbb{R}^m \to \mathbb{R}$ be smooth, and define

$$\Gamma_f = \{(x_1 + i\frac{\partial f}{\partial x_1}(x_1, \ldots, x_m), \ldots, x_m + i\frac{\partial f}{\partial x_m}(x_1, \ldots, x_m))\}$$

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Then Γ_f is a smooth real m-dimensional submanifold of $\mathbb{C}^m$, with $\omega|_{\Gamma_f} \equiv 0$. Identifying $\mathbb{C}^m \cong \mathbb{R}^{2m} \cong \mathbb{R}^m \times (\mathbb{R}^m)^*$, we may regard Γ_f as the graph of the 1-form df of $\mathbb{R}^m$, so that Γ_f is the graph of a closed 1-form. Locally, but not globally, every Lagrangian submanifold arise in this way.

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Now, Γ_f is special Lagrangian m-fold if and only if $Im(\theta)|_{\Gamma_f} \equiv 0$. This is not difficult to see that $Im(\theta)|_{\Gamma_f} \equiv 0$ if and only if

$$\text{Im } det_{\mathbb{C}}(I_m + i \ \text{ Hess } f) \equiv 0 \quad \text{on } \ \mathbb{R}^m$$

where Hess f of f is the $m \times m$ matrix $(\frac{\partial^2 f}{\partial x_i \partial x_j})_{i,j=1}^m$ of real functions on $\mathbb{R}^n$.

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It is known that if f is a global solution to this p.d.e. satisfying one of several extra conditions, related to convexity or order of growth, then f must be a quadratic polynomial, so that Γ_f is a real affine m-plane in $\mathbb{C}^m$.

Deformations of SL m-fold

We begin with investigating deformations of SL m-fold in $\mathbb{C}^m$. So, let L_0 be a special Lagrangian submanifold in $\mathbb{C}^m$. We are interested in the family of special Lagrangian deformations of L_0 i. e. special Lagrangian submanifolds L that are close to L_0 in a suitable sense.

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Locally, every special Lagrangian m-folds look like $\mathbb{R}^m$ in $\mathbb{C}^m$. Therefore, their deformations should also locally look like the deformation of $\mathbb{R}^m$ in $\mathbb{C}^m$. So, we would like to know what SL m-folds L in $\mathbb{C}^m$ close to $\mathbb{R}^m$ look like.

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Since $\mathbb{R}^m$ in $\mathbb{C}^m$ is the graph Γ_f associated to the function $f \equiv 0$, a graph Γ_f will be close to $\mathbb{R}^m$ if the function f and its derivatives are small. But, then Hess f is small and

$$\text{Im } det_{\mathbb{C}}(I_m + i \text{ Hess } f) = \text{ Tr Hess } f + \text{ higher order terms}$$

by approximating the LHS by its linearization.

Thus when second derivatives of f are small, Γ_f is Lagrangian if and only if $\mathrm{Tr\ Hess}\, f \equiv 0$. But, $\mathrm{Tr\ Hess}\, f = \frac{\partial^2 f}{(\partial x_1)^2} + \cdots + \frac{\partial^2 f}{(\partial x_m)^2}$, which is nothing but $-\Delta f$, where Δ is the Laplacian on $\mathbb{R}^m$.

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Hence small Lagrangian deformations of $\mathbb{R}^m$ in $\mathbb{C}^m$ are parametrized by suitable harmonic functions on $\mathbb{R}^m$. Now, since Γ_f can also be viewed as the graph of df in $\mathbb{R}^m \times (\mathbb{R}^m)^*$ we could have instead parametrised by df for corresponding f and f being harmonic implies $d^* df = \Delta f = 0$. So we state:

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Small special Lagrangian deformations of a special Lagrangian m-fold L are approximately parametrized by closed and coclosed 1-forms α on L.
This is the basic idea behind McLean's theorem that we are going to discuss now, but first let us define SL m-fold in the general setting.

SL m-fold in (almost) Calabi-Yau m-fold

Let $m \geq 2$. An *almost Calabi-Yau m-fold* is a quadraple (M, J, ω, θ) such that (M, J) is a compact m-dimensional complex manifold, is the Kahler form of a Kahler metric g on M, and θ is a non-vanishing holomorphic $(m, 0)$ form on M.

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We call (M, J, ω, θ) a *Calabi-Yau m-fold* if in addition ω and θ satisfy

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \theta \wedge \bar{\theta}$$

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Then for every $x \in M$ there exists an isomorphism $T_x M \cong \mathbb{C}^m$ that identifies g_x, ω_x and θ_x with that of $\mathbb{C}^m$ discussed above. We then define

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Definition: With M as above, let N be a real m-dimensional submanifold of M. We call N a *special Lagrangian submanifold* if $\omega|_N \equiv \theta|_N \equiv 0$. It then follows that $Re(\theta)|N$ is a nonvanishing m-form on N, making it orientable with a unique orientation in which $Re(\theta)|_N$ is positive.

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Theorem: Let N be a compact SL m-fold in an almost Calabi-Yau m-fold (M, J, ω, θ). Then the moduli space $\mathcal{M}_N$ of special Lagrangian deformations of N is a smooth manifold of dimension $b^1(N)$, the first Betti number of N.

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The proof follows from the discussion in the flat case above and the *Lagrangian neighborhood theorem* after a bit of modification since (M, ω) is also a symplectic manifold.

Singularities of SL m-folds

General singularities of SL m-folds can be very bad and difficult to study. Therefore we would like to restrict ourselves to a class of SL m-folds with well behaved singularities i.e. those with finite codimension in the space of all SL m-folds. Sl m-folds with *isolated conical singularities* are one such candidate. That is we consider an SL m-fold X in an almost Calabi-Yau m-fold M with singularities at $x_1, \ldots, x_n$ in M such that there exists special Lagrangian cones C_i in $T_{x_i}M \cong \mathbb{C}^m$ with $C_i - \{0\}$ non-singular and X approaches C_i near x_i in some appropriate sense (asymptotic C^1).

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So, for that reason, we are now going to discuss **special Lagrangian cones** in $\mathbb{C}^m$. These are SL m-folds C in $\mathbb{C}^m$ which is invariant under the dilation $C \mapsto tC$ for $t > 0$ and is generally singular at their vertex 0. Where $tC = \{t\mathbf{x} : \mathbf{x} \in C\}$.

Special Lagrangian Cones

Let C be a closed SL cone in $\mathbb{C}^m$ with an isolated singularity at 0. Consider $\Sigma = C \cap \mathcal{S}^{2m-1}$. Then Σ is a compact, nonsingular (m-1)-submanifold of $\mathcal{S}^{2m-1}$. In fact, Σ is Legendrian w.r.t standard contact form on $\mathcal{S}^{2m-1}$ coming from ω.

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Let g' be the restriction of g to Σ, and set $C' = C - \{0\}$. Define $\mathfrak{i} : \Sigma \times (0, \infty) \to \mathbb{C}^m$ by $i(\sigma, r) = r\sigma$. Then $\mathfrak{I}$ has image C'. By an abuse of notation, we identify C' by $\Sigma \times (0, \infty)$ using $\mathfrak{i}$. The cone metric on $C' \cong \Sigma \times (0, \infty)$ is $g = \mathfrak{i}^*(g) = dr^2 + r^2 g'$.

Harvey-Lawson Cone

Here is a family of special Lagrangian cone constructed by Harvey and Lawson. Form $m \geq 3$ define

$$C^m_{HL} = \{(z_1, \ldots, z_m) : i^{m+1} z_1 \ldots z_m \in [0, \infty), |z_1| = \cdots = |z_m|\}$$

Then C^m_{HL} is a special Lagrangian cone in $\mathbb{C}^m$ with an isolated singularity at 0, and $\Sigma = C^m_{HL} \cap \mathcal{S}^{2m-1}$ is an $(m-1)$ torus T^{m-1}.

One central problem in *Singularity Theory* is to study the effect on the singularity under deformation. We saw above that such deformations are parametrised in terms of harmonic functions. However, to be well defined on C we need a *homogenous harmonic functions* on $\mathbb{C}^n$, C' in particular. There has been plenty of works in this direction, exploring questions concerning *stability* and *obstruction* to existence of such a singularity, both separate and well-developed sub-branches in itself. However, we are going to wrap-up this talk with a brief about the SYZ conjecture, and Zaslow's and Treumann's work mentioned in the introduction.

SYZ Conjecture

Let X, $\widehat{X}$ be mirror Calabi-Yau 3-folds. There is a compact 3 manifold B and continuous, surjective $f : X \to B$ and $\widehat{f} : \widehat{X} \to B$ such that

1. for b in a dense $B_0 \subset B$, the fibres $f^{-1}(b)$ and $\widehat{f}^{-1}(b)$ are dual SL 3-Tori T^3 in X, $\widehat{X}$ respectively.
2. for $b \notin B_0$, $f^{-1}(b)$ and $\widehat{f}^{-1}(b)$ are singular SL 3-folds in X, $\widehat{X}$.

We call f, $\widehat{f}$ special Lagrangian fibrations, and $\Delta = B \setminus B_0$ the discriminant.

Harvey-Lawson cone as foam.

Let D^3 be a 3-dimensional ball. A **foam** in D^3 is a stratified subset $F^2 \subset F^1 \subset F \subset D^3$ satisfying certain conditions that are topological analogue for Plateau conditions for soap bubbles. A foam gives a regular cell-complex structure on D^3, whose dual complex is a "tetrahedronation" of D^3 in the similar way a planar trivalent graphs give triangulation.

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Recall, a HL 3-cone is a cone over a two torus with a conic singularity at the origin. The Harvey-Lawson cone is therefore a singular Lagrangian filling of the Legendrian surface associated to the tetrahedron.

"Mathematics should be learnt very much like languages. Fluency will come slowly, over a period of time." T. W. Frankel