The maximal current carried by a normal/superconducting interface in the absence of magnetic field

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Abstract: Modeling a normal/superconducting interface, we consider a semi-infinite wire whose edge is adjacent to a normal mater, assuming asymptotic convergence, away from the boundary, to the purely superconducting state. We obtain that the maximal current which can be carried by the interface diminishes in the small normal conductivity limit.

1 Introduction

We consider the time dependent Ginzburg-Landau model in the absence of magnetic field, presented here in the dimensionless form [2, 7, 11]

\[
\frac{\partial \psi}{\partial t} + i \phi \psi = \Delta \psi + \psi(1 - |\psi|^2) \quad \text{in } \Omega \times \mathbb{R}_+ \tag{1a}
\]

\[
\sigma \Delta \phi = \nabla \cdot [\Im (\bar{\psi} \nabla \psi)] \quad \text{in } \Omega \times \mathbb{R}_+ \tag{1b}
\]

\[
\psi = 0 \quad \text{on } \partial \Omega_c \times \mathbb{R}_+ \tag{1c}
\]

\[-\sigma \frac{\partial \phi}{\partial v} = J \quad \text{on } \partial \Omega_c \times \mathbb{R}_+ \tag{1d}
\]

\[
\frac{\partial \psi}{\partial v} = 0 \quad \text{on } \partial \Omega_i \times \mathbb{R}_+ \tag{1e}
\]

\[
\frac{\partial \phi}{\partial v} = 0 \quad \text{on } \partial \Omega_i \times \mathbb{R}_+ \tag{1f}
\]

\[
\psi(x, 0) = \psi_0 \quad \text{in } \Omega \tag{1g}
\]

In (1) \(\psi\) denotes the superconductivity order parameter, which implies that \(|\psi|^2\) is proportional to the number density of pairs of superconducting electrons (Cooper pairs). Superconductors with \(|\psi| = 1\) are called purely superconducting, whereas those for which \(\psi = 0\) are said to be at the normal state. The scalar electric potential is denoted by \(\phi\), while the

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constant \( \sigma \) represents the normal conductivity of the superconducting material. In the presence of magnetic field the normal current is given by \(-\sigma(A_t + \nabla \phi)\), where \( A \) is the magnetic vector potential, but since in our case \( A = 0 \) the normal current is given by \(-\sigma \nabla \phi\). The domain \( \Omega \subset \mathbb{R}^n (n \in \{1, 2, 3\}) \), where the sample resides has interface \( \partial \Omega_c \) with a conducting metal which is at normal state. The remaining boundary \( \partial \Omega_i \) is adjacent to an insulator. The function \( J : \partial \Omega_c \to \mathbb{R} \) represents the normal current entering the sample.

The above model, with various boundary conditions, has been studied by both physicists [7, 8, 9, 5] and mathematicians [3, 12, 10, 4]. We mention in particular [2] which addresses precisely the same one-dimensional simplification we consider in the sequel.

Assuming a one-dimensional wire lying in \( \mathbb{R}_+ \), a stationary solution of (1) must satisfy

\[
\begin{align*}
-\psi'' + i\phi \psi - \psi(1 - |\psi|^2) &= 0 & \text{in } \mathbb{R}_+ & \quad (2a) \\
-\sigma \phi'' + \Im[\phi' \bar{\phi}'] &= 0 & \text{in } \mathbb{R}_+ & \quad (2b) \\
\psi(0) &= 0 & \quad (2c) \\
|\psi| &\to \rho_\infty & \text{as } x \to \infty & \quad (2d) \\
\phi &\to 0 & \text{as } x \to \infty & \quad (2e)
\end{align*}
\]

In (2), the current \( J \) is constant. The boundary conditions at \( x = 0 \) represent an interface with a metal at the normal state. As \( x \to \infty \) the sample assumes the fully superconducting state. The latter is given, for this simple setting (cf. [2, 13]) by

\[
\psi_s = \rho_\infty e^{i\alpha x} \quad ; \quad \phi \equiv 0,
\]

with \( \alpha = [1 - \rho_\infty^2]^{1/2} \), and

\[
J^2 = \rho_\infty^2 (1 - \rho_\infty^2).
\]

Accordingly, in (2), \( J \) and \( \rho_\infty \) must be related by (4). It can be easily verified that, as \( 0 \leq \rho_\infty \leq 1 \), that the values of \( J \) for which (4) can be satisfied are limited to \( J \in [0, J_c] \) where \( J_c = [4/27]^{1/2} \), and that for \( J = J_c \) we have \( \rho_\infty^2 = 2/3 \). This critical current is well known and has frequently been documented in the literature [6, 7, 13].

Using the polar representation \( \psi = \rho e^{ix} \) we obtain from (2a,c) that

\[
\chi' = \frac{\sigma \phi' + J}{\rho^2},
\]
whenever $\rho \neq 0$. For $(\rho, \phi)$ we then obtain the following system of equations

$$
-\rho'' + \frac{(\sigma \phi' + J)^2}{\rho^3} - \rho(1 - \rho^2) = 0 \quad \text{in} \ R_+ \tag{5a}
$$

$$
-\sigma \phi'' + \rho^2 \phi' = 0 \quad \text{in} \ R_+ \tag{5b}
$$

$$
\rho(0) = 0 \tag{5c}
$$

$$
\rho \xrightarrow{x \to \infty} \rho_\infty \tag{5d}
$$

$$
\phi'(0) = -\frac{J}{\sigma} \tag{5e}
$$

$$
\phi \xrightarrow{x \to \infty} 0 \tag{5f}
$$

The present contribution focuses on the numerical evaluation of the values of $J$ and $\sigma$ for which solutions of (5) exist. As stated above an infinite wire may admit the solution (3) for all $J \in [0, J_c]$ and positive $\sigma$. When an interface with a normal metal at $x = 0$ is added we expect that the maximal value of $J$ for which solutions of (5) exist would depend on $\sigma$. In [1], it is proven that the maximal value of $J$ for which solutions of (5) can exist decays as $\sigma$ tends to zero. However as $\sigma$ gets sufficiently large, the maximal value for $J$ asymptotically approaches $J_c = \left[\frac{1}{27}\right]^{\frac{1}{2}}$

It was proven in [2] that letting

$$
S(\sigma) = \{ J \in R_+ \mid \exists (\rho, \phi) \in C^2(R_+) \times C^2(R_+) \text{satisfying (5)} \}
$$

exists $C > 0$ such that

$$
\sup S(\sigma) \leq C \sigma^{\frac{1}{4}}.
$$

The leading order behavior as $\sigma \to 0$ has been formally obtained in [2] as well.

The rest of this contribution is arranged as follows. In the next section we present the numerical computation of $\sup S(\sigma)$. In §3 we present the formal asymptotic expansion of $\sup S(\sigma)$ obtained in [2] and compare it with the numerical results of §2. In addition we obtain in §3 the potential drop over the boundary layer (i.e. $\phi(0)$).

## 2 Critical Current

In this section we obtain the relation between the maximal current, for which a solution of (3) can exist, and $\sigma$. To this end, we need to plot the solution $(\rho, \phi)$ of (3). A typical plot of $\rho(x)$ is provided in Fig. 1 for multiple values of $J$ and $\sigma = .2$. 

3
Figure 1: Graph of $\rho(x)$ in (3) for $\sigma = .2$. The left two graphs asymptotically match (3) as $x \to \infty$. If $J > .35$ the graph becomes nonphysical.

Similarly, Fig. 2 presents a plot of $\phi(x)$ for the same values, as in Fig. 1, of $J$ and $\sigma$.

Figure 2: Graph of $\phi(x)$ in (5) for $\sigma = .2$. The left two graphs asymptotically match (3) as $x \to \infty$. If $J > .35$ the graph becomes nonphysical.
We use MATLAB routine BV4PC to obtain the solution of (5). To this end we must first change it to a system of first order ODEs.

\[
\begin{align*}
  f_1 &= \rho' \\
  f_2 &= \frac{(\sigma \phi + J)^2}{\rho^3} - \rho(1 - \rho^2) \\
  f_3 &= \phi' \\
  f_4 &= \frac{\rho^2 \phi}{\sigma}
\end{align*}
\]

with boundary conditions at \( x = 0 \) and \( x = b \) for some constant \( b \gg 1 \).

\[
\begin{align*}
  \rho(0) &= 0 \\
  \rho(b) &= \rho_\infty \\
  \phi(b) &= 0 \\
  \phi'(0) &= -\frac{J}{\sigma}
\end{align*}
\]

Clearly, any change in the values of \( J \) and \( \sigma \) will produce a change in \((\rho, \phi)\). To determine the maximal value of \( J \), for which a solution of (3) can exist for a given \( \sigma \), we increase \( J \) incrementally over a set of evenly spaced numbers \( 0 = J_0 < J_1 < \ldots < J_{400} = J_c = \sqrt{\frac{4}{27}} \) (clearly, \( J_k = kJ_c/400 \)). For each \( J_k \) we graphed \( \rho(x) \). The smallest value of \( J \) for which \( \rho \) does not tend asymptotically to \( \rho_\infty \), should be close to the above maximal value. We denote this critical value by \( J_c(\sigma) = \sup S(\sigma) \).

3 Asymptotic Expansion

We begin by repeating the formal asymptotic expansion, as \( \sigma \to 0 \), of \( J_c(\sigma) \) from [2]. We then compare it with the numerical solution described in the previous section.

Since by Lemma 2.1 in [2], \( \rho' \leq \sqrt{\frac{2}{3}} \). It therefore seems safe to assume that \( \rho \sim \alpha x \) in the close vicinity of \( x = 0 \) where \( \alpha = \rho'(0) \). The equation for \( \phi \) (5b) takes the form

\[
\begin{align*}
  -\sigma \phi'' + \alpha^2 x^2 \phi &= 0 \quad \text{in } \mathbb{R}_+ \\
  \phi'(0) &= -\frac{J}{\sigma} \\
  \phi &\xrightarrow{x \to \infty} 0.
\end{align*}
\]

Consider the scaled coordinate \( \xi = \alpha^{\frac{1}{2}} \sigma^{-\frac{1}{2}} x \) and the function

\[
\Phi(\xi) = \frac{\alpha^{\frac{1}{2}} \sigma^{\frac{3}{2}}}{J} \phi(x)
\]
The rescaled form of (10) is

\[ \begin{align*}
-\Phi'' + \xi^2 \Phi &= 0 \quad &\text{in } \mathbb{R}_+ \\
\Phi'(0) &= -1 \quad &\Phi \rightarrow 0 \quad &\xi \rightarrow 0
\end{align*} \] (8a)

Let

\[ H = \frac{1}{2} \left( |\rho'|^2 + \frac{(\sigma \phi' + J)^2}{\rho^2} + \rho^2 - \frac{1}{2} \rho^4 \right). \] (9)

It can be easily verified that

\[ H' = (\sigma \phi' + J) \phi. \]

Integrating the above between 0 and \( \infty \) yields, by (9) and (5c-f),

\[ \int_0^\infty (\sigma \phi' + J) \phi \, dx = \frac{J^2}{\alpha \sigma^{1/2}} \int_0^\infty (\Phi' + 1) \Phi \, d\xi = 2 \rho_\infty^2 - \frac{3}{2} \rho_\infty^4 - \alpha^2. \]

Let then

\[ A = \int_0^\infty (\Phi' + 1) \Phi \, d\xi, \]

Since \( J \) must be small as \( \sigma \rightarrow 0 \), we must have by (4) that \( \rho_\infty \sim 1 \), and hence we reach the asymptotic identity

\[ \alpha^2 + \frac{AJ^2}{\alpha \sigma^{1/2}} = \frac{1}{2}. \]

The minimum of the left-hand-side, with respect to \( \alpha \), is given by \( [AJ^2/(2\sigma^{1/2})]^{2/3} \). Consequently,

\[ J_c(\sigma) = \sqrt{\frac{2}{A}} \left( \frac{1}{6} \right)^{3/4} \sigma^{1/4}. \] (10)

We now express \( \Phi \) in terms of the parabolic cylinder function \( U(0, \xi) \). It can be easily verified that

\[ A = -\Phi^2(0) + 2 \int_0^\infty \Phi \, d\xi \] (11)

By [1, Chapter 19] we have \( \Phi = CU(0, \sqrt{2}\xi) \). To obtain \( C \) we write

\[ \Phi'(0) = C \sqrt{2} U'(0, 0) = -1, \]

And hence,

\[ \Phi(\xi) = \frac{2^{-1/4} \Gamma(1/4)}{\sqrt{2\pi}} U(0, \sqrt{2}\xi). \] (12)
We now use [1, 19.3.1, 19.3.3–4] together with [1, 19.2.5–6] to obtain

\[
U(0, \sqrt{2} x) = \cos \left( \frac{\pi}{4} \right) \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \frac{1}{4} \right)}{2^{-\frac{3}{4}}} \cdot \left( 1 + \frac{2x^4}{4!} + \frac{60x^8}{8!} + \ldots \right) \\
- \sin \left( \frac{\pi}{4} \right) \frac{1}{\sqrt{\pi}} \frac{\Gamma \left( \frac{3}{4} \right)}{2^{-\frac{3}{4}}} \cdot \left( \sqrt{2} x + \frac{6\sqrt{2}x^5}{5!} + \frac{252\sqrt{2}x^9}{9!} + \ldots \right) .
\]  

We estimate \( \phi(x) \) using (13) for \( x \leq 7 \) and [1, 19.8.1] for \( x > 7 \) to obtain from (11)

\[ A \approx 0.4336, \]

and consequently by (10) that

\[ J^c(\sigma) \sim 0.5602 \cdot \sigma^{\frac{1}{4}} . \]  

For small \( \sigma \), the asymptotic curve of \( J^c(\sigma) \) aligns with the critical \( J \) values found numerically, as can be viewed in Fig. 3

\[ J^c(\sigma) \approx 0.5602 \cdot \sigma^{\frac{1}{4}}. \]

\[ y = J^c(\sigma) \]

\[ y = \text{Numerical Method} \]

\[ y = (4/27)^{1/2} \]

Figure 3: A plot of \( J^c(\sigma) \) for \( 0 < \sigma \leq 0.5 \)

Note that the asymptotic approximation for \( J^c(\sigma) \) begins to diverge from the numerical value at about \( \sigma \approx 0.13 \). Such a divergence is expected since (14) cannot tend to \( J_c \) as \( \sigma \to \infty \).
It was established in [2] that the potential drop for $J = J^c(\sigma)$ formally satisfies, as $\sigma \to 0$, 
\[
\phi^c(0, \sigma) \sim [3A\sigma]^{-\frac{1}{2}} \cdot \Phi(0) \approx 2.3861 \cdot \sigma^{-\frac{1}{2}} .
\] (15)

In Fig. 4 we plot the numerical value of $\phi^c(\sigma)$ (the solid curve) and the asymptotic estimate given by (15).

![Figure 4: $\phi^c(\sigma)$ for $0 < \sigma \leq .5$](image)

Unlike the approximation for critical current, the asymptotic approximation for potential drop does not diverge from its numerical counterpart.

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References


