NONLINEAR SURFACE SUPERCONDUCTIVITY IN THE LARGE \( \kappa \) LIMIT

Y. ALMOG

Faculty of Mathematics, Technion - Israel Institute of Technology, Haifa 32000, Israel

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The Ginzburg–Landau model for superconductivity is considered in two dimensions. We show, for smooth bounded domains, that the superconductivity order parameter decays exponentially fast away from the boundary as the Ginzburg–Landau parameter \( \kappa \) tends to infinity. We prove this result for applied magnetic fields satisfying \( h_{ex} - \kappa \gg \log \kappa / \kappa \), and therefore, improve a recent result of Pan [16].

**Keywords**: Surface superconductivity; Ginzburg–Landau; large \( \kappa \) limit.

1. Introduction

Consider a planar superconducting body which is placed in sufficiently low temperature (below the critical one) under the action of an external magnetic field. Its energy is given by the Ginzburg–Landau energy functional which can be represented in the following dimensionless form [6]

\[
E = \int_{\Omega} \left( -|\Psi|^2 + \frac{|\Psi|^4}{2} + |h - h_{ex}|^2 + \left| \frac{i}{\kappa} \nabla \Psi + A\Psi \right|^2 \right) dx dy
\]  

(1.1)

in which \( \Psi \) is the (complex) superconducting order parameter, such that \( |\Psi| \) varies from \( |\Psi| = 0 \) (when the material is at a normal state) to \( |\Psi| = 1 \) (for the purely superconducting state). The magnetic vector potential is denoted by \( A \) (the magnetic field is then given by \( h = \nabla \times A \)), \( h_{ex} \) is the constant applied magnetic field, and \( \kappa \) is the Ginzburg–Landau parameter which is a material property. Superconductors for which \( \kappa < 1/\sqrt{2} \) are termed type I superconductors, and those for which \( \kappa > 1/\sqrt{2} \) are termed type II. The superconductor lies in a smooth domain \( \Omega \) (\( \partial \Omega \) is at least \( C^{2, \alpha} \)) and its Gibbs free energy is given by \( E \). Note that \( E \) is invariant to the gauge transformation

\[
\Psi \rightarrow e^{i\kappa \eta} \psi; \quad A \rightarrow A + \nabla \eta.
\]  

(1.2)

It is known both from experiments [15] and rigorous analysis [10] that for a sufficiently strong magnetic field the normal state \( (\psi \equiv 0, h = h_{ex}) \) would prevail.
If the field is then decreased, there is a critical field, depending on the sample’s
gometry, where the material would enter the superconducting state. For samples
with boundaries, this field is known as the onset field and has been termed $H_{C3}$.

The simplest case in which the bifurcation from the normal state to the super-
conducting one was calculated is the case of a half-plane [18]. The analysis in this
case is one-dimensional: the linearized Ginzburg–Landau equations were solved on
$\mathbb{R}_+$. Even in this simple case the onset field is substantially larger than the bifurca-
tion field on $\mathbb{R}$ [9]. The situation is not different in two dimensions: it was proved in
[14] and [7] that the bifurcating mode in $\mathbb{R}^2_+$ is one-dimensional and that the value
of $H_{C3}$ is exactly the same as in the one-dimensional case. Similarly, the bifurcation
from the normal state in $\mathbb{R}^2$ takes place when the applied magnetic field is identical
with the bifurcation field for $\mathbb{R}$, which has been termed $H_{C2}$.

In addition to the difference in the values of the applied field, it was found by
Saint-James and de Gennes [18] that superconductivity is concentrated at the onset
near the boundary for a half-plane, i.e. $\psi$ decays exponentially fast away from the
boundary. This phenomenon, which appears only in the presence of boundaries have
been termed, therefore, surface superconductivity. It was later proved for general
two-dimensional domains with smooth boundaries [14, 7], that as the domain’s scale
tends to infinity the onset field tends to de Gennes’ value, and that if the boundaries
include wedges the onset field will be larger than de Gennes’ value [4, 13, 19, 12].

Surface superconductivity reflects another difference between the problems in
$\mathbb{R}^2_+$ and $\mathbb{R}^2$, where the bifurcation takes place in the form of periodic solutions
[1, 5, 2] known as Abrikosov’s lattices. The transition, as the applied magnetic
field decreases, from surface superconductivity to the experimentally-observed [8]
Abrikosov’s lattices is not yet well understood. Rubinstein [17] conjectured that
superconductivity remains limited to a neighborhood of the boundary until about
$H_{C3}$ when a new solution which is similar in bulk to Abrikosov lattice appears.

Two recent contributions [16, 3] study the behavior of the global minimizer of
the energy functional (1.1) for external fields satisfying $\kappa = H_{C3} < h_{ex} < H_{C2}$. In
[16] the limit $\kappa \to \infty$ is considered: it is demonstrated that $\psi$ decays, in $L^2$ sense,
exponentially fast away form the boundary. The results are valid whenever $h_{ex} - \kappa \gg
1$ as $\kappa \to \infty$, and are stated for the global minimizer of (1.1). In addition the energy
of the global minimizer is shown to be evenly distributed along the boundary. In
[3] the large domain limit is considered: it is demonstrated for the global minimizer
that both $\psi$ and $h$ tend, in $C^\alpha$ sense, to the normal state, exponentially fast away
from the boundary. The results are valid whenever $h_{ex} - \kappa \sim O(1)$ as the domain’s
size tends to infinity.

In the present contribution we focus on the limit $\kappa \to \infty$. We prove that for
any critical point of (1.1) $(\psi, A)$ tends to the normal state exponentially fast away
from the boundary as long as $h_{ex} - \kappa \gg \log \kappa / \kappa$, which extends the validity of the
results in [16]. Furthermore, we show that the magnetic field tends to a constant
not only away from the boundaries but also near the boundary for this limit case.
The Euler–Lagrange equations associated with the energy functional defined in (1.1), or the steady state Ginzburg–Landau equations, are given by

\[
\left( \frac{i}{\kappa} \nabla + A \right)^2 \psi = \psi (1 - |\psi|^2), \quad (1.3a)
\]

\[
- \nabla \times \nabla \times A = \frac{i}{2\kappa} (\psi^* \nabla \psi - \psi \nabla \psi^*) + |\psi|^2 A, \quad (1.3b)
\]

and the natural boundary conditions by

\[
\left( \frac{i}{\kappa} \nabla + A \right) \psi \cdot \hat{n} = 0; \quad h = h_{ex}. \quad (1.4a,b)
\]

We consider two-dimensional settings where we can write \( h = (0, 0, h(x, y)) \) and \( h_{ex} = (0, 0, h_{ex}) \). In the next section we consider the global minimizer of (1.1) in smooth bounded domains as \( \kappa \to \infty \). We show that for sufficiently large \( \kappa \), the global minimizer of (1.1) which must solve (1.3) together with (1.4), tends exponentially fast away from the boundaries to a normal state as long as \( h_{ex} - \kappa \gg \log \kappa / \kappa \). Furthermore, we show that

\[
\| h - h_{ex} \|_{L^\infty[\Omega]} \leq C \left( \frac{\log \kappa}{\kappa} + \frac{1}{\kappa (h_{ex} - \kappa)} \right). \quad (1.5)
\]

To prove the above results we use a differential inequality which was proved in [3]. Let

\[
u = h - \kappa + \frac{1}{2\kappa} \rho^2. \quad (1.6)
\]

Then

\[
\nabla^2 \nu - \rho^2 \nu = \kappa |\vec{J}|^2 + \left( \kappa - \frac{1}{2\kappa} \right) \rho^4. \quad (1.7)
\]

The precise definition of \( \vec{J} \) will not concern us. We shall be interested only in its property

\[
|\vec{J}|^2 \rho^2 = |\nabla \nu|^2, \quad (1.8)
\]

which is proved in [3]. Finally, in Sec. 3 we briefly discuss a few key points which are not mentioned in Sec. 2.

**2. Exponential Rate of Decay**

We prove here the following theorem:

**Theorem 2.1.** Let \( \lambda = \sqrt{\kappa (h_{ex}(\kappa) - \kappa)} \), and let \( (\psi, A) = (\psi(\lambda, \kappa), A(\lambda, \kappa)) \) denote a solution of (1.3) and (1.4). Then, \( \exists \lambda_0 > 0, \kappa_0 > 0, \beta > 0, \) and \( \tilde{h}_\lambda \) such that for every \( \kappa > \kappa_0 \) and \( \lambda > \lambda_0 (\log \kappa)^{1/2} \) we have

\[
|D^\alpha \psi| \leq C_\alpha \kappa^\alpha e^{-\beta \lambda d(x, \partial \Omega)} \quad \text{for all } \alpha \geq 0 \text{ and } x \in \Omega \quad (2.1a)
\]

\[
|h - \tilde{h}_\lambda| \leq C e^{-\beta \lambda d(x, \partial \Omega)} \quad (2.1b)
\]
\[ |D^\alpha(h - \tilde{h}_\lambda)| \leq C_\alpha \kappa^{\alpha-1} e^{-\beta \lambda d(x, \partial \Omega)} \quad \text{for all } \alpha \geq 1 \text{ and } x \in \Omega \] (2.1c)

\[ |\tilde{h}_\lambda - h_{ex}| \leq C \frac{\log \kappa}{\kappa}. \] (2.1d)

To prove the theorem we first need a number of auxiliary results. The first of them includes the following well-known estimates:

**Lemma 2.2.** Let \( h_{ex} \geq \kappa \). Then, any solution of (1.3) and (1.4) must satisfy

\[ \|\rho\|_{L^\infty(\Omega)} < 1 \] (2.2a)

\[ \|h - h_{ex}\|_{C^1(\bar{\Omega})} \leq C \] (2.2b)

\[ \left\| \left( \frac{i}{\kappa} \nabla + A \right) \psi \right\|_{L^\infty(\bar{\Omega})} \leq C. \] (2.2c)

**Proof.** The proof of (2.2a) is well known and follows immediately from (1.3a) and the real part of the boundary condition (1.4a). The proof of (2.2b) and (2.2c) can be found in [11].

**Lemma 2.3.** Let \( h_{ex} \geq \kappa \). Then, any solution of (1.3) and (1.4) satisfies, for sufficiently large \( \kappa \)

\[ \int_\Omega \rho^4 \leq \frac{C}{\kappa} \] (2.3a)

\[ \int_\Omega |h - h_{ex}|^2 \leq C \frac{\log^2 \kappa}{\kappa^2} \] (2.3b)

where \( C \) is independent of \( \kappa \).

**Proof.** We first prove (2.3a). To this end, we integrate (1.7) over \( \Omega \). In view of (2.2b) we have,

\[ \kappa \int_\Omega \left| \nabla \frac{u}{\rho} \right|^2 + \int_\Omega \rho^2 u + \left( \kappa - \frac{1}{2\kappa} \right) \int_\Omega \rho^4 \leq \int_\partial \Omega \frac{\rho u}{\rho} \leq C. \] (2.4)

Hence, applying (2.2b) once again, we have, since \( h_{ex} \geq \kappa \)

\[ \kappa \int_\Omega \rho^4 \leq C + \left| \int_\Omega \rho^2 (h - h_{ex}) \right| \leq C \left( 1 + \left[ \int_\Omega \rho^4 \right]^{1/2} \right) \] (2.5)

from which (2.3a) is readily verified.

To prove (2.3b) we integrate (1.3a) multiplied by \( \rho^2 \psi \) and integrate over \( \Omega \). We obtain,

\[ \int_\Omega \rho^2 \left| \left( \frac{i}{\kappa} \nabla + A \right) \psi \right|^2 + \frac{1}{\kappa^2} \int_\Omega \rho^2 |\nabla \rho|^2 = \int_\Omega \rho^4 (1 - \rho^2). \]
By (1.3b) we have
\[ \int_{\Omega} |\nabla h|^2 \leq \int_{\Omega} \rho^2 \left( \frac{i}{\kappa} \nabla + A \right) \Psi^2 \leq \int_{\Omega} \rho^4. \]

We now apply Poincaré inequality and (1.4b) to obtain
\[ \int_{\Omega} |h - h_{\text{ex}}|^2 \leq C \int_{\Omega} \rho^4 \leq \frac{C}{\kappa}. \tag{2.6} \]

In a manner similar to [7, 3] we now define a local coordinate system near \( \partial \Omega \). Let \( \eta \) denote the distance from the boundary, \( s \) the arclength along the boundary, with some point \( x_0 \in \partial \Omega \) corresponding to \( s = 0 \), and \( \kappa_1(s) \) the curvature of \( \partial \Omega \), which must be uniformly bounded in \([-L/2, L/2]\). This local coordinate system is well defined in the rectangle
\[ S = \left\{ (s, \eta) \mid -\frac{L}{2} < s < \frac{L}{2}, 0 < \eta < \eta_0 \right\} \tag{2.7} \]
where \( L \) denotes the arclength of \( \partial \Omega \), and \( \eta_0 \) satisfies
\[ \inf_{s \in [-L/2, L/2]} 1 - \kappa_1(s) \eta_0 > 0. \]

Denote by \( \Omega'_{\alpha} \) the domain enclosed in \( \eta = \alpha \), i.e.,
\[ \Omega'_{\alpha} = \{ x \in \Omega \mid d(x, \partial \Omega) > \alpha \}. \]

Integrating (1.7) on \( \Omega'_{\alpha} \) yields
\[ \kappa \int_{\Omega'_{\alpha}} \left| \frac{\nabla u}{\rho} \right|^2 + \int_{\partial \Omega'_{\alpha}} \rho^2 u \leq \int_{\partial \Omega'_{\alpha}} \frac{\partial u}{\partial n} \leq C \left[ \int_{\partial \Omega'_{\alpha}} \left| \frac{\nabla u}{\rho} \right|^2 \right]^{1/2}. \tag{2.8} \]

However, by (2.6) we have
\[ -\int_{\Omega'_{\alpha}} \rho^2 u \leq -\int_{\Omega'_{\alpha}} \rho^2 (h - h_{\text{ex}}) \leq \frac{C}{\kappa}. \tag{2.9} \]

Furthermore, for every \( 0 < \delta < \eta_0 \), there exists \( \delta/2 < \alpha < \delta \) such that
\[ \int_{\partial \Omega'_{\alpha}} \left| \frac{\nabla u}{\rho} \right|^2 \leq \frac{C}{\delta} \int_{\Omega'_{\alpha}} \left| \frac{\nabla u}{\rho} \right|^2. \]

Combining the above with (2.8) and (2.9) yields
\[ \kappa \int_{\Omega'_{\delta}} \left| \frac{\nabla u}{\rho} \right|^2 \leq \frac{C}{\delta^{1/2}} \int_{\Omega'_{\delta/2}} \left[ \int_{\Omega'_{\delta/2}} \left| \frac{\nabla u}{\rho} \right|^2 \right]^{1/2} + \frac{C}{\kappa} \leq \frac{2C}{\delta^{1/2}} \int_{\Omega'_{\delta/2}} \left[ \int_{\Omega'_{\delta/2}} \left| \frac{\nabla u}{\rho} \right|^2 \right]^{1/2}, \]
which can be applied recursively to obtain
\[ \int_{\Omega'_{\delta}} |\nabla u|^2 \leq \int_{\Omega'_{\delta}} \left| \frac{\nabla u}{\rho} \right|^2 \leq \frac{C}{\kappa^2 \delta}. \]
Moreover, it is easy to show in view of (2.4), that
\[ \int_{\Omega_{\delta}} |\nabla u|^2 \leq \frac{C}{\kappa^2(\delta + 1/\kappa)}. \] (2.10)

We can now use Schwarz Inequality, and the local coordinate system defined in (2.7), to obtain
\[ |u(s, 0) - u(s, \delta)|^2 \leq \left[ \int_0^\delta |\frac{\partial u}{\partial \eta}|\,d\eta \right]^2 \leq \int_0^\delta |\frac{\partial u}{\partial \eta}|^2 \left( \eta + \frac{1}{\kappa} \right) \,d\eta \int_0^\delta \frac{d\eta}{(\eta + 1/\kappa)}. \]

Integrating the above with respect to \( s \) we obtain
\[ \int_{-L/2}^{L/2} |u(s, 0) - u(s, \delta)|^2 \,ds \leq C \log(1 + \kappa \delta) \int_{-L/2}^{L/2} \frac{d\eta}{\eta_1} \int_0^\delta |\nabla u|^2 \left( \eta + \frac{1}{\kappa} \right) \,d\eta. \]

By changing the order of integration it is easy to show that
\[ \int_{-L/2}^{L/2} \int_0^\delta |\nabla u|^2 \,d\eta = \int_0^\delta \frac{d\eta}{\eta_1} \int_{-L/2}^{L/2} \int_{\eta_1}^{\eta_2} |\nabla u|^2 \,d\eta. \]

However, in view of (2.10), we have
\[ \int_{-L/2}^{L/2} \left[ \int_\Omega \frac{d\eta}{\eta_1} \right] |\nabla u|^2 \,d\eta \leq C \int_\Omega |\nabla u|^2 \leq \frac{C}{\kappa^2(\eta_1 + 1/\kappa)}. \]

Furthermore, since
\[ \int_{-L/2}^{L/2} \left[ \int_0^\delta |\nabla u|^2 \,d\eta \right] \leq \frac{C}{\kappa}, \]

we must have
\[ \int_{-L/2}^{L/2} |u(s, 0) - u(s, \delta)|^2 \,ds \leq C \frac{\log^2(1 + \kappa \delta)}{\kappa^2}. \] (2.11)

Utilizing (2.11) we obtain
\[ \int_{\partial \Omega_\delta} |u - (h_{ex} - \kappa)|^2 \leq 2 \int_{-L/2}^{L/2} |u(s, 0) - u(s, \delta)|^2 \,ds + 2 \int_{\partial \Omega_\delta} |u - (h_{ex} - \kappa)|^2 \]
\[ \leq C \frac{\log^2(1 + \kappa \delta)}{\kappa^2}. \] (2.12)

However,
\[ \int_{\Omega_\delta} |u - (h_{ex} - \kappa)|^2 \leq C \int_{\partial \Omega_\delta} |u - (h_{ex} - \kappa)|^2 + C \int_{\Omega_\delta} |\nabla u|^2, \]

and hence, with the aid of (2.12), we obtain
\[ \int_{\Omega_\delta} |u - (h_{ex} - \kappa)|^2 \leq C \frac{\log^2(1 + \kappa \delta)}{\kappa^2} + \frac{C}{\kappa^2(\delta + 1/\kappa)}. \]
We now write,
\[
\int_{\Omega} |h - h_{ex}|^2 \leq 2 \int_{\Omega} |u - (h_{ex} - \kappa)|^2 + C \int_0^\delta \eta \int_{\partial \Omega_\eta} |u - (h_{ex} - \kappa)|^2 + \frac{1}{2\kappa^2} \int_{\Omega} \rho^4.
\]
Choosing \( \delta \sim O(1) \) proves (2.3b).

**Lemma 2.4.** Let \( h_{ex} > \kappa \) and \( \lambda = \sqrt{\kappa(h_{ex} - \kappa)} \). Let \( \{x\}_{\lambda \geq \lambda_0} \) denote a family of points in \( \Omega \). Let \( s_\lambda = d(x_\lambda, \partial \Omega) \) Then,
\[
F(x_\lambda, s) \leq \frac{C_e}{(\lambda s_\lambda)^{2-\epsilon}} \forall s \leq \frac{1}{2} s_\lambda \forall \epsilon > 0
\]
where \( F(x, r) \) is given by
\[
F(x, r) = \int_{B(x, r)} \lambda^2 |w^+|^2 + \rho^2 (w^+)^2 + |\nabla w^+|^2
\]
and
\[
w = \frac{u}{h_{ex} - \kappa}.
\]

**Proof.** By (1.7) \( w \) satisfies
\[
\nabla^2 w - \rho^2 w = \lambda^2 |\hat{J}|^2 + \left( \kappa^2 - \frac{1}{2} \right) \frac{1}{2\lambda^2} \rho^4.
\]
Integrating over \( B(x, r) \) the product of (2.14) by \( w^+ \) we obtain
\[
\int_{\partial B(x, r)} w^+ \frac{\partial w^+}{\partial r} \geq F(x, r).
\]
Multiplying (2.15) by \( 1/r \) and integrating between \( s \) and \( s_\lambda \) yields, in view of (2.2),
\[
\int_s^{s_\lambda} \frac{F(x, r)}{r} dr \leq \frac{1}{2} \int_0^{2\pi} \left[ (w^+(s, \theta))^2 - (w^+(s, \theta))^2 \right] d\theta \leq C.
\]
In the following we use \( C \) to denote a constant which is independent of both \( \lambda \) and \( x_\lambda \). As \( F \) is monotonically increasing in \( r \),
\[
\exists \frac{1}{2} \beta_0 < \beta_0 < 1 : F(x_\lambda, \beta_0 s_\lambda) < C.
\]
It is easy to show that \( 1/2 < \beta < \beta_0 \) exists such that
\[
\int_{\partial B_\beta} \lambda^2 w^+ |\hat{J}|^2 + \rho^2 (w^+)^2 \leq \frac{C}{s_\lambda} \int_{B_{\beta_0}} \lambda^2 w^+ |\hat{J}|^2 + \rho^2 (w^+)^2
\]
where \( B_\beta \overset{\text{def}}{=} B(x, \beta s) \). Let \( \xi_1, \xi_2 \in \partial B_\beta \). Then,
\[
|w^+|^{5/2}(\xi_1) - (w^+)^{5/2}(\xi_2)| \leq C \int_{\partial B_\beta} (w^+)^{3/2} |\nabla w|.
\]
By (1.8) $|\nabla w| = \rho |\tilde{J}|$. Hence,

\[
|(w^+)^{5/2}(\xi_1) - (w^+)^{5/2}(\xi_2)| \leq C \left[ \int_{\partial B_\beta} w^+ |\tilde{J}|^2 \right]^{1/2} \left[ \int_{\partial B_\beta} \rho^2 (w^+) \right]^{1/2} \leq \frac{C}{\lambda} \int_{\partial B_\beta} \lambda^2 w^+ |\tilde{J}|^2 + \rho^2 (w^+)^2 \leq \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda).
\]

(2.19)

Let $0 < s < \beta s_\lambda$, and let $(r, \theta)$ denote a polar coordinate system centered around $x$. Then,

\[
\int_0^{2\pi} \int_s^{\beta s_\lambda} (w^+)^{5/2} \frac{\partial w}{\partial r} dr d\theta \leq C \left[ \int_A w^+ |\tilde{J}|^2 \right]^{1/2} \left[ \int_A \rho^2 (w^+) \frac{1}{r^2} \right]^{1/2}
\]

where $A \overset{\text{def}}{=} B_\beta \setminus B(x, s)$. Hence,

\[
\int_0^{2\pi} \int_s^{\beta s_\lambda} (w^+)^{5/2} |\beta s_\lambda| \leq \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda).
\]

Utilizing (2.19) together with the inequality

\[
|x^5 - y^5| \geq |x^4 - y^4|^{5/4}
\]

(2.21)

and Hölder inequality we obtain

\[
\int_0^{2\pi} (w^+)^{5/2} \int_s^{\beta s_\lambda} \frac{dr d\theta}{s_\lambda} + \frac{2\pi C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda)
\]

\[
= \int_0^{2\pi} \left\{ \left( (w^+)^{5/2} (\beta s_\lambda) + \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda) \right)^{1/5} - (w^+)^{5/2} (s) \right\} d\theta
\]

\[
\geq \int_0^{2\pi} \left\{ \left( (w^+)^{5/2} (\beta s_\lambda) + \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda) \right)^{4/5} - (w^+)^{2} (s) \right\}^{5/4} d\theta
\]

\[
\geq C \left\{ \int_0^{2\pi} \left( (w^+)^{5/2} (\beta s_\lambda) + \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda) \right)^{4/5} - (w^+)^{2} (s) \right\}^{5/4} d\theta
\]

In view of (2.16)

\[
\int_0^{2\pi} (w^+)^{2} \int_s^{\beta s_\lambda} \frac{dr d\theta}{s_\lambda} \geq 0.
\]

Consequently,

\[
\left\{ \int_0^{2\pi} \left( (w^+)^{5/2} (\beta s_\lambda) + \frac{C}{\lambda s_\lambda} F(x_\lambda, \beta_0 s_\lambda) \right)^{4/5} - (w^+)^{2} (s) \right\}^{5/4} d\theta
\]

\[
\geq \left\{ \int_0^{2\pi} (w^+)^{2} \int_s^{\beta s_\lambda} \frac{dr d\theta}{s_\lambda} \right\}^{5/4}.
\]
Combining the above inequalities yields
\[
\int_0^{2\pi} (w^+)^2 |_{s=0}^\theta \, d\theta \leq C \left( \frac{F(x_\lambda, \beta_0 s_\lambda)}{\lambda s} \right)^{4/5}
\]
and by (2.16) we have
\[
\int_s^{s_\lambda} \frac{F(x_\lambda, r)}{r} \, dr \leq C \left( \frac{F(x_\lambda, \beta_0 s_\lambda)}{\lambda s} \right)^{4/5}.
\]
Thus, since \( F \) is monotone increasing
\[
\exists \frac{1}{2} < \beta_1 < \beta_0 \text{ s.t. } F(x_\lambda, \beta_1 s_\lambda) \leq C_1 \left( \frac{F(x_\lambda, \beta_0 s_\lambda)}{\lambda s_\lambda} \right)^{4/5}.
\] (2.22)
It is possible to repeat the above procedure recursively (cf. [3]) to prove the existence of a monotone decreasing sequence \( \{\beta_n\}_{n=1}^\infty \), which is strictly bounded from below by \( 1/2 \), such that
\[
F(x_\lambda, \beta_n s_\lambda) \leq C_n \left( \frac{F(x_\lambda, \beta_{n-1} s_\lambda)}{\lambda s_\lambda} \right)^{4/5} \quad \forall n \geq 1.
\] (2.23)
Utilizing the above inequality together with (2.17) proves the lemma.

Lemma 2.4 allows us to obtain uniform convergence in \( \Omega \) of \( w^+ \) to a constant, except for a boundary layer of \( O(1/\lambda) \) size (as \( \lambda \to \infty \)).

Lemma 2.5. For any family of points \( \{x_\lambda\}_{\lambda > \lambda_0} \)
\[
\exists \tilde{\omega}_\lambda : |w^+(x_\lambda) - \tilde{\omega}_\lambda| \leq C \frac{1}{\lambda^{1/2} d(x_\lambda, \partial \Omega)^{1/2}}.
\]
The lemma can be proved by applying the same arguments as in the proof of Lemma 3.4 in [3].

We now find the value of the constant \( \tilde{\omega}_\lambda \) by using the estimates in Lemma 2.3.

Lemma 2.6. Let \( h_{ex} > \kappa \). Then,
\[
|\tilde{\omega}_\lambda - 1| \leq C \left( \frac{1}{\lambda^{1/2}} + \frac{\log \kappa}{\lambda^2} \right).
\] (2.24)

Proof. Let \( x \in \Omega \) such that \( \partial B(x, r) \subset \text{int}(\Omega) \), where \( r \) is independent of \( \lambda \). By Lemma 2.3 we have
\[
\|1 - w\|_{L^2[\Omega \setminus B(x, r)]} \leq \|1 - w\|_{L^2[\Omega]} \leq \frac{2}{h_{ex} - \kappa} \|h_{ex} - h\|_{L^2[\Omega]} + \frac{1}{\kappa(h_{ex} - \kappa)} \|\rho^+\|_{L^2[\Omega]} \leq C \frac{\log \kappa}{\lambda^2}.
\]
However, by the previous lemma \( |w^+ - \tilde{\omega}_\lambda| \leq C/\lambda^{1/2} \) in \( B(x, r) \), and hence, since
\[
\|\tilde{\omega}_\lambda - 1\|_{L^2[\Omega \setminus B(x, r)]} \leq \|1 - w^+\|_{L^2[\Omega \setminus B(x, r)]} + \|\tilde{\omega}_\lambda - w^+\|_{L^2[\Omega \setminus B(x, r)]} \leq C \left( \frac{1}{\lambda^{1/2}} + \frac{\log \kappa}{\lambda^2} \right),
\]
the lemma immediately follows.
We can now obtain better estimates for the rate of decay of \(|w^+ - \overline{w}_\lambda|\) away from the boundaries as \(\lambda \to \infty\).

**Lemma 2.7.** Let \(h_{\kappa x} > \kappa\) and \(\{x_\lambda\}_{\lambda \geq \lambda_0}\) denote a family of points such that \(x_\lambda \in \Omega\). Let \(\lambda s_\lambda = \lambda d(x_\lambda, \partial \Omega) \to \infty\). Then,

\[
\forall n \in \mathbb{N} \quad \exists \frac{1}{2} < \beta_n < 1, \quad C_n > 0 : F(x_\lambda, \beta_n s_\lambda) \leq \frac{C_n}{s_\lambda^{n}} \quad (2.25a)
\]

where \(F\) is defined in (2.13)

\[
\exists \overline{w}_\lambda : |w^+(x_\lambda) - \overline{w}_\lambda| \leq \frac{C_n}{\lambda^ns_\lambda} \quad (2.25b)
\]

\[
|\overline{w}_\lambda - 1| \leq C \frac{\log \kappa}{\lambda^2} \quad (2.25c)
\]

**Proof.** The proof of (2.25a) and (2.25b) is obtained by following the same line of arguments as in the proof of Lemma 3.6 in [3]. To prove (2.25c) we use (2.25b) and apply arguments of the proof of Lemma 2.6 once again.

**Lemma 2.8.** Let \(h_{\kappa x} > \kappa\) and \(\{x_\lambda\}_{\lambda \geq \lambda_0}\) denote a family of points such that \(x_\lambda \in \Omega\). Let \(s_\lambda = d(x_\lambda, \partial \Omega) \to \infty\). Then,

\[
\forall n \in \mathbb{N} \quad \exists C_n > 0 \quad \int_{B(x_\lambda, s_\lambda/2)} \rho^2 \leq \frac{C_n}{\lambda^ns_\lambda^2} \quad (2.26)
\]

**Proof.** By (2.25a)

\[
\exists \frac{1}{2} < \beta_n < 1 : \int_{B(x_\lambda, \beta_n s_\lambda)} \rho^2 (w^+)^2 \leq \frac{C_n}{\lambda^ns_\lambda^2} \quad (2.27a)
\]

Writing

\[
\|\rho\|_{L^2[B(x_\lambda, \beta_n s_\lambda)]} \leq \|\rho w^+\|_{L^2[B(x_\lambda, \beta_n s_\lambda)]} + \|\rho (w^+ - \overline{w}_\lambda)\|_{L^2[B(x_\lambda, \beta_n s_\lambda)]} + \|\rho (1 - \overline{w}_\lambda)\|_{L^2[B(x_\lambda, \beta_n s_\lambda)]} \quad (2.27b)
\]

we obtain, in view of (2.24) and (2.25b),

\[
\|\rho\|_{L^2[B(x_\lambda, \beta_n s_\lambda)]} \leq \frac{C_n}{(\lambda s_\lambda)^{n/2}} \quad (2.27c)
\]

**Proof of Theorem 2.1.** We prove the theorem by invoking blow up arguments. We first prove that \(\exists \lambda_0\) and \(\beta > 0\) such that

\[
\|\psi\|_{L^2[B(x, \delta)]} \leq C \delta e^{-\beta \lambda d(x, \partial \Omega)} \quad (2.28)
\]

\[
\forall \lambda > \lambda_0 \log^{1/2} \kappa, \quad 0 < \delta < \frac{1}{\lambda}, \quad \forall x \in \Omega : d(x, \partial \Omega) \geq \frac{1}{\lambda} \quad (2.28)
\]
Let
\[ \Omega(\lambda, k, s) = \left\{ x \in \Omega \mid d(x, \partial\Omega) \geq k^2 \frac{\alpha}{\lambda} \right\}. \]

We prove (2.28) by showing that
\[ \sup_{x \in \Omega(\lambda, k, s)} \| \psi(\kappa, \lambda) \|_{L^2[B(x, \delta)]} \leq \frac{1}{2} \sup_{x \in \Omega(\lambda, k+1, s)} \| \psi(\kappa, \lambda) \|_{L^2[B(x, \delta)]} \]
\[ \forall s > s_0 \kappa > \lambda_0 \log^{1/2} \kappa, \ k \in \mathbb{N}, \ 0 < \delta < \frac{1}{\lambda}. \] (2.29)

Suppose, for a contradiction, that (2.29) does not hold. Then, sequences
\[ \lambda_j \uparrow \infty, \ \kappa_j \uparrow \infty, \ s_j \uparrow \infty, \ k_j \in \mathbb{N}, \ 0 < \delta_j < 1/\lambda_j, \] and
\[ \sup_{x \in \Omega(\lambda_j, k_j+1, s_j)} \| \psi(\kappa_j, \lambda_j) \|_{L^2[B(x, \delta_j)]} \geq \frac{1}{2} \sup_{x \in \Omega(\lambda_j, k_j, s_j)} \| \psi(\kappa_j, \lambda_j) \|_{L^2[B(x, \delta_j)]} \]
\[ \defeq \frac{1}{2} m_j. \] (2.30)

Let
\[ \psi_j \defeq \psi(\kappa_j, \lambda_j). \]

By (2.29) there exists \( x_j \in \Omega(\lambda_j, k_j+1, s_j) \) such that \( \| \psi_j \|_{L^2[B(x_j, \delta_j)]} \geq \frac{1}{2} \). Furthermore, since \( B(x_j, \delta_j) \in \Omega(\lambda_j, k_j, s_j) \) we have
\[ \frac{1}{2} \leq \| \psi_j \|_{L^2[B(x_j, \delta_j)]} \leq 1. \] (2.31)

Define
\[ f_j = \frac{\psi_j}{\lambda_j} \left( x_j + \frac{x}{\lambda_j} \right) e^{iA_j(\kappa_j, \lambda_j)} f_j, \]
where \( A_j = A(\kappa_j, \lambda_j) \). In view of (2.31) we have
\[ \frac{1}{2} \leq \| f_j \|_{L^2[B(x_j, \delta_j)]} \leq 1. \] (2.32)

Let \( w_\lambda \) be the same as in (2.25c) and let
\[ h_\lambda = w_\lambda (h_{ex} - \kappa) + \kappa. \]

Clearly,
\[ \| h - h_\lambda \|_{L^\infty[\Omega(\lambda, k, s)]} \leq \frac{C_n \lambda^2}{\kappa s^2}. \] (2.33)

It is easy to show that
\[ \left( \frac{i\lambda_j}{\kappa_j} \nabla + \frac{\hat{h}_j}{\lambda_j} B_j \right)^2 = f_j \left( 1 - m_j f_j \right)^2 \quad x \in B(0, s_j) \] (2.34a)
wherein

\[ B_j(x) = [A_j(x) + x] - A_j(x)] \frac{1}{h_j}, \tag{2.34b} \]

and \( \tilde{h}_j = \tilde{h}_j \). We now define a cut-off function

\[ \eta_r = \begin{cases} 1 & \text{in } B(0, r) \\ 0 & \text{in } \mathbb{R}^2/B(0, 2r) \end{cases} \]

\[
|\nabla \eta_r| \leq \frac{C}{r} \quad \text{in } \mathbb{R}^2.
\]

Multiplying (2.34a) by \( \eta_r^2 \), and integrating over \( B(0, 2r) \) we obtain, for all \( r \leq \frac{\varepsilon^2}{2} \) (cf. [14]), that

\[
\int_{B(0,2r)} \left| \left( \frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} B_j \right)(\eta_r f_j) \right|^2 = \int_{B(0,2r)} \eta_r^2 f_j^2 \left( 1 - m_{f_j}^2 \right) + \frac{\lambda^2}{\kappa_j^2} |\nabla \eta_r|^2 |f_j|^2.
\]  

(2.35)

Let \( \hat{A} : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) denote any vector field satisfying \( \nabla \times \hat{A} = i_z \) and \( (\hat{A} - B) \cdot \hat{n}|_{\partial B(0, s)} = 0 \). Then,

\[
\int_{B(0,2r)} \left| \left( \frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} \hat{A} \right)(\eta_r f_j) \right|^2
\]

\[
= \int_{B(0,2r)} \left| \left( \frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} \hat{A} \right)(\eta_r f_j) \right|^2
\]

\[
+ \int_{B(0,2r)} \frac{\tilde{h}_j}{\kappa_j} (B_j - \hat{A}) \eta_r^2 \left[ i (f_j \nabla f_j - f_j \nabla \tilde{f}_j) + 2 |f_j|^2 \frac{\tilde{h}_j}{\lambda_j} B_j \right]
\]

\[
- \int_{B(0,2r)} \frac{\tilde{h}_j}{\lambda_j} \left| B_j - \hat{A} \right|^2 \eta_r^2 |f_j|^2.
\]

Clearly,

\[
\eta_r^2 \left[ \frac{i\lambda_j}{\kappa_j} (f_j \nabla f_j - f_j \nabla \tilde{f}_j) + 2 |f_j|^2 \frac{\tilde{h}_j}{\lambda_j} B_j \right] = 2 \Re \left\{ \eta_r \tilde{f}_j \left( \frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} B_j \right)(\eta_r f_j) \right\},
\]

and hence,

\[
\int_{B(0,2r)} \left| \left( \frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} B_j \right)(\eta_r f_j) \right|^2 \geq \int_{B(0,2r)} \left| \left( \frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} \hat{A} \right)(\eta_r f_j) \right|^2
\]

\[
- 2 \frac{\tilde{h}_j}{\lambda_j} M_j \left[ \int_{B(0,2r)} \eta_r^2 |f_j|^2 \right]^{1/2} \left[ \int_{B(0,2r)} \left| \left( \frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} B_j \right)(\eta_r f_j) \right|^2 \right]^{1/2}
\]

\[
- \left( \frac{\tilde{h}_j}{\lambda_j} \right)^2 M_j^2 \int_{B(0,2r)} \eta_r^2 |f_j|^2
\]  

(2.36a)
where

\[ M_j = \sup_{x \in B(x,s_j)} |B_j - \hat{A}|. \]  

(2.36b)

In [14, 7] it was shown that

\[ \int_{\mathbb{R}^2} \left| \left( \frac{i\lambda_j}{\kappa_j} \nabla + \frac{\tilde{h}_j}{\lambda_j} \hat{A} \right) (\eta_f f_j) \right|^2 \geq \frac{\tilde{h}_j}{\kappa_j} \int_{\mathbb{R}^2} \eta_r^2 |f_j|^2. \]  

(2.37)

Combining the above with (2.35) and (2.36a) we obtain

\[ \left( \frac{\tilde{h}_j}{\kappa_j} - 1 \right) \int_{B(0,2r)} \eta_r^2 |f_j|^2 \leq \frac{\lambda_j^2}{\kappa_j^2} \int_{B(0,2r)} |\nabla \eta_r|^2 f_j^2 + M_j \frac{\tilde{h}_j}{\lambda_j} \int_{B(0,2r)} 2 \eta_r^2 f_j^2 \]

\[ + \frac{\lambda_j^2}{\kappa_j^2} \int_{B(0,2r)} |\nabla \eta_r|^2 f_j^2 + \left( \frac{\tilde{h}_j}{\lambda_j} \right)^2 M_j^2 \int_{B(0,2r)} \eta_r^2 |f_j|^2. \]  

(2.38)

By (2.33) we have

\[ M_j \leq \frac{\lambda_j^2 C_n}{\kappa_j^2 s_j} \quad \forall n \in \mathbb{N}. \]

By (2.25c) we have, for sufficiently large \( j \) (recall that \( \lambda_j \gg \log \kappa_j \)),

\[ |\tilde{h}_j - h_{ex}(\kappa_j)| \leq \frac{1}{2} |h_{ex}(\kappa_j) - \kappa_j| = \frac{1}{2} \lambda_j^2 \frac{1}{\kappa_j} \]

and hence by (2.38) we have

\[ \int_{B(0,r)} |f_j|^2 \leq C \int_{B(0,2r)} |\nabla \eta_r|^2 f_j^2. \]  

(2.39)

Choosing \( r = \lambda_j \delta_j \) we obtain, by applying (2.39) \( n \) consecutive times

\[ \int_{B(0,\lambda_j \delta_j)} |f_j|^2 \leq \frac{C^n}{(\lambda_j \delta_j)^{2n(n+1)}} \int_{B(0,2^n \lambda_j \delta_j)} |f_j|^2. \]  

(2.40)

By (2.32) we have, however,

\[ \int_{B(0,2^n \lambda_j \delta_j)} \frac{f_j}{\lambda_j} \leq C 2^{2n} \]

where \( C \) is independent of \( n \) and \( j \). Consequently,

\[ \int_{B(0,\lambda_j \delta_j)} \left| \frac{f_j}{\lambda_j} \right|^2 \leq \left( \frac{C}{\lambda_j \delta_j 2^{n-1}} \right)^n \]

which is true for all \( n \in \mathbb{N} \) such that \( 2^n \lambda_j \delta_j \leq s_j \). Substituting in the above an integer \( n_j \) satisfying

\[ \frac{s_j}{2} < 2^{n_j} \lambda_j \delta_j \leq s_j \]
we easily obtain
\[
\lim_{j \to \infty} \int_{B(0, \lambda, \delta_j)} \left| \frac{f_j}{\lambda_j} \right|^2 = 0 \tag{2.41}
\]
contradicting (2.32), and therefore proving (2.28).

In order to obtain exponential decay in \(C^\alpha\) norm we first write the equation for
\[
\phi(z) = \psi(x_0 + z/\kappa)e^{-iA(x_0)z}
\]
which is given by
\[
\nabla^2 \phi = 2\tilde{A} \cdot (i\nabla + \tilde{A}) \phi - \phi(1 - |\phi|^2 + |\tilde{A}_j|^2) \tag{2.42}
\]
where
\[
\tilde{A}(z) = A(x_0 + z/\kappa) - A(x_0).
\]
It is possible to show, using the identity (2.35) and the boundedness of \(\tilde{A}(z)\) in \(B(0, 2)\), that
\[
\int_{B(0, 2)} \left| 2\tilde{A} \cdot (i\nabla + \tilde{A}) \phi - \phi(1 - |\phi|^2 + |\tilde{A}_j|^2) \right|^2 dz \leq C \int_{B(0, 2)} \phi^2 dz.
\]
Using standard elliptic estimates we then have
\[
\|\phi\|_{L^2(B(0,1))} \leq C\|\phi\|_{L^2(B(0,2))}
\]
where \(C\) is independent of \(\kappa, \lambda,\) and \(x_0\). Choosing \(\delta = 2/\kappa\) in (2.28) we obtain
\[
\|\phi\|_{L^2(B(0,2))} \leq Ce^{-\beta\lambda d(x_0, \partial \Omega)}.
\]
Sobolev embedding then implies
\[
\|\phi\|_{L^\infty(B(0,1))} \leq Ce^{-\beta\lambda d(x_0, \partial \Omega)}, \tag{2.43}
\]
which proves (2.1a) for \(\alpha = 0\).

We now write the equation for \(\tilde{A}(z)\),
\[
\nabla \times H = \frac{1}{\kappa^2} \Re \left[ \phi(i\nabla + \tilde{A}) \right] \tag{2.44}
\]
where \(H(z) = \nabla \times \tilde{A}(z)\). By (2.2c) we then have
\[
\|\nabla H\|_{L^\infty(B(0,1))} \leq \frac{C}{\kappa^2} e^{-\beta\lambda d(x_0, \partial \Omega)},
\]
and hence,
\[
\|\nabla h\|_{L^\infty(B(x_0,1/\kappa))} \leq Ce^{-\beta\lambda d(x_0, \partial \Omega)}. \tag{2.45}
\]
We can now integrate (2.45) to obtain
\[
\exists \hat{h}_\lambda : |h(x_0) - \hat{h}_\lambda| \leq Ce^{-\beta\lambda d(x_0, \partial \Omega)}.
\]

To prove (2.1c), and (2.1a) for \(\alpha \geq 1\) we use (2.42) and (2.44) together with bootstrapping and Sobolev embedding. To prove (2.1d) we use (2.25c), which gives
\[
|h_{xx} - \hat{h}_\lambda| \leq C \frac{\log \kappa}{\kappa}.
\]
3. Conclusion
In [16] Pan obtains that in the limit \( \kappa \to \infty \)
\[
\int_{\Omega} \left\{ |\psi|^2 + \left| \frac{1}{\kappa} \nabla \psi - iA\right|^2 \right\} e^{\beta \sqrt{\kappa(h_{ex} - \kappa)d}\partial\Omega} dx \leq \frac{C}{\sqrt{\kappa(h_{ex} - \kappa)}} \tag{3.1}
\]
whenever \( h_{ex} - \kappa \gg 1 \), for some \( \beta > 0 \) which is independent of \( \kappa \). In the present contribution we extend the validity of the above result to external fields satisfying
\[ h_{ex} - \kappa \gg \log \frac{\kappa}{\kappa}. \]

We also obtain, in Theorem 2.1, convergence in \( C^\alpha \) norms in contrast to the above \( L^2 \) convergence which is proved in [16]. It should be mentioned, however, that once \( L^2 \) convergence is obtained, it is possible to prove (2.43) and (2.45) and then proceed using bootstrapping and Sobolev embedding. The main advantage of the results in this work is therefore the greater range of external fields for which exponential rate of decay is guaranteed. This is facilitated by better \textit{a priori} estimates of the magnetic field: in [16] it is first proved that \( |h - h_{ex}| \leq C \) in \( \Omega \), whereas here, (2.25) provides a much better estimate on the magnetic field.

In addition to (3.1) it is demonstrated in [16] that for \( h_{ex} - \kappa \gg 1 \) the energy of the global minimizer is evenly distributed along the boundary (for a more precise definition the reader is referred to [16]). In view of the better estimate of \( h \) in the present contribution it appears reasonable to believe that the validity of this result can be extended to external fields satisfying \( h_{ex} - \kappa \gg \log \kappa/\kappa \). However, since the analysis in [16] is heavily based on the assumption \( h_{ex} - \kappa \gg 1 \), significant modification is necessary before it can be applied to a greater range of applied magnetic fields.

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