PART I: GEOMETRY OF SEMISIMPLE LIE ALGEBRAS

Contents

1. Regular elements in semisimple Lie algebras

Let G be a connected semisimple algebraic group over $\mathbb{C}$, and let $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra.

Definition 1.1. A semisimple element $s \in \mathfrak{g}$ is *regular* if its centralizer

$$Z_{\mathfrak{g}}(s) = \{x \in \mathfrak{g} \mid [x, s] = 0\}$$

is a Cartan subalgebra.

Recall that a Cartan subalgebra is a nilpotent subalgebra which is self-normalizing. Cartans are maximal abelian subalgebras (though not all maximal abelian subalgebras are Cartans!), and they consist entirely of semisimple elements. All Cartans are conjugate under the adjoint action of G, and every semisimple element is contained in a Cartan.

Remark 1.2. Equivalently, a semisimple element $s \in \mathfrak{g}$ is regular if and only if the centralizer $Z_G(s) = \{g \in G \mid \mathrm{Ad}_g(s) = s\}$ is a maximal torus.

We will denote by $\mathfrak{g}^{rs}$ the subset of regular semisimple elements of $\mathfrak{g}$. Fix a Cartan $\mathfrak{h}$, and let Φ be the corresponding root system. This produces a corresponding root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right),$$

where

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid h \cdot x = \alpha(h)x \quad \forall h \in \mathfrak{h}\}.$$

Lemma 1.3. *An element $s \in \mathfrak{h}$ is regular iff $\alpha(s) \neq 0$ for all $\alpha \in \Phi$.*

Proof. The centralizer $Z_{\mathfrak{g}}(s)$ just the sum of the root spaces on which the operator ad_s has eigenvalue 0:

$$Z_{\mathfrak{g}}(s) = \mathfrak{h} \oplus \left(\bigoplus_{\alpha(s)=0} \mathfrak{g}_\alpha \right).$$

The statement follows. $\square$

Example 1.4. A diagonalizable element of $\mathfrak{sl}_n$ is regular if and only if its eigenvalues are distinct.

Remark 1.5. Since the semisimple elements are dense in $\mathfrak{g}$, and since every semisimple element is contained in a Cartan, Lemma 1.3 implies that $\mathfrak{g}^{rs}$ is a dense subset of $\mathfrak{g}$.

Let $l = \dim \mathfrak{h}$ be the rank of $\mathfrak{g}$.

Proposition 1.6. *Let $x \in \mathfrak{g}$. Then $Z_{\mathfrak{g}}(x)$ contains an l-dimensional abelian subalgebra.*

Proof. Let $\{x'_n\}$ be a sequence of regular semisimple elements that converges to x. Consider the Grassmannian $\mathrm{Gr} = \mathrm{Gr}(l, \mathfrak{g})$ of l-dimensional subspaces of $\mathfrak{g}$. Since Gr is projective, there is a subsequence $\{x_n\}$ such that the sequence $\{Z_{\mathfrak{g}}(x_n)\} \subset \mathrm{Gr}$ converges—call its limit $\mathfrak{u}$. We will prove that $\mathfrak{u}$ is an abelian subalgebra contained in $Z_{\mathfrak{g}}(x)$.

Let $w^1, \ldots, w^l$ be a basis for $\mathfrak{u}$, and choose for each i a sequence $\{w_n^i \in Z_{\mathfrak{g}}(x_n)\}$ such that $w_n^i \longrightarrow w^i$. Then

$$\begin{aligned} &[w_n^i, w_n^j] = 0 \quad \Rightarrow \quad [w^i, w^j] = 0 \quad \Rightarrow \quad \mathfrak{u} \text{ is abelian, and} \\ &[w_n^i, x_n] = 0 \quad \Rightarrow \quad [w^j, x] = 0 \quad \Rightarrow \quad \mathfrak{u} \subseteq Z_{\mathfrak{g}}(x). \end{aligned}$$

$\square$

Corollary 1.7. *For any $x \in \mathfrak{g}$, $\dim Z_{\mathfrak{g}}(x) \geq l$ and $\dim Z_G(x) \geq l$.*

This justifies the following definition.

Definition 1.8. An element $x \in \mathfrak{g}$ is *regular* if the dimension of $Z_{\mathfrak{g}}(x)$ is equal to l. In other words, x is regular if the dimension of its centralizer is minimal, and the dimension of the G-orbit $G \cdot x$ is maximal.

Example 1.9. There exist non-semisimple regular elements. When $\mathfrak{g} = \mathfrak{sl}_3$, the maximal nilpotent Jordan block

$$\begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix} \quad \text{is regular with centralizer} \quad \left\{ \begin{pmatrix} 0 & a & b \\ & 0 & a \\ & & 0 \end{pmatrix} \right\}.$$

We will give a more general criterion for regularity. Recall the Jordan decomposition, which says that for every $x \in \mathfrak{g}$ there exists a unique decomposition $x = x_{ss} + x_n$ into a semisimple part x_{ss} and a nilpotent part x_n such that $[x_{ss}, x_n] = 0$. Uniqueness implies in particular that

$$Z_{\mathfrak{g}}(x) = Z_{\mathfrak{g}}(x_{ss}) \cap Z_{\mathfrak{g}}(x_n).$$

Proposition 1.10. *The element $x \in \mathfrak{g}$ is regular if and only if x_n is regular in the derived subalgebra of $Z_{\mathfrak{g}}(x_{ss})$.*

Proof. The semisimple Lie algebra $\mathfrak{l}' = [Z_{\mathfrak{g}}(x_{ss}), Z_{\mathfrak{g}}(x_{ss})]$ has rank $l-k$, where k is the dimension of the center of $Z_{\mathfrak{g}}(x_{ss})$. Then

$$\begin{aligned} \dim Z_{\mathfrak{l}'}(x_n) = l-k &\iff \dim Z_{Z_{\mathfrak{g}}(x_{ss})}(x_n) = l \\ &\iff \dim Z_{\mathfrak{g}}(x_{ss}) \cap Z_{\mathfrak{g}}(x_n) = l \\ &\iff \dim Z_{\mathfrak{g}}(x) = l. \end{aligned}$$ □

Exercise 1.11. Give a criterion for an element of $\mathfrak{sl}_n$ to be regular.

2. The flag variety and the Bruhat decomposition

Most of the next four sections will follow the exposition in [CG]. Fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$, with corresponding Borel subgroup B. Let $\mathcal{B}$ denote the set of all Borel subalgebras—this is naturally a closed subvariety of the Grassmannian

$$Gr(\dim \mathfrak{b}, \mathfrak{g}),$$

and so it is a projective variety.

Borel subgroups are self-normalizing, so the stabilizer of $\mathfrak{b}$ under the G-action on $\mathcal{B}$ is $N_G(B) = B$. The isomorphism

$$\begin{aligned} G/B &\xrightarrow{\sim} \mathcal{B} \\ gB &\longmapsto g \cdot \mathfrak{b} \end{aligned}$$

gives a natural bijection

$$B\backslash G/B \xrightarrow{(1)} \{B\text{-orbits on } \mathcal{B}\}.$$

The product $G \times G$ acts on $\mathcal{B} \times \mathcal{B}$. Let G_Δ denote the diagonal embedding of G into $G \times G$. There is a bijection

$$\begin{aligned} \{B\text{-orbits on } \mathcal{B}\} &\xrightarrow{(2)} \{G_\Delta\text{-orbits on } \mathcal{B} \times \mathcal{B}\} \\ B \cdot \mathfrak{b}' &\longmapsto G_\Delta \cdot (\mathfrak{b}, \mathfrak{b}'). \end{aligned}$$

Under map (2), the unique B-fixed point $\mathfrak{b}$ is mapped to the unique closed G_Δ-orbit of minimal dimension, $G_\Delta \cdot (\mathfrak{b}, \mathfrak{b})$.

Now fix a maximal torus $T \subset B$, and let $W_T = N_G(T)/T$ denote the corresponding Weyl group. There is a third map

$$\begin{aligned} W_T &\xrightarrow{(3)} B\backslash G/B \\ w &\longmapsto B\dot{w}B, \end{aligned}$$

where $\dot{w}$ is any coset representative of w in $N_G(T)$. (This is well-defined because any two coset representative differ by an element of T, and $T \subset B$.)

These maps concatenate:

$$W_T \xrightarrow{(3)} B\backslash G/B \xrightarrow{(1)} \{B\text{-orbits on } \mathcal{B}\} \xrightarrow{(2)} \{G_\Delta\text{-orbits on } \mathcal{B}\times\mathcal{B}\}.$$

Theorem 2.1. *(The Bruhat Decomposition) Map (3) is also a bijection.*

The proof of this theorem will be an application of the Bialynicki-Birula decomposition, which we recall here. Let X be a smooth complex projective variety equipped with an algebraic action of $\mathbb{C}^*$. It is a standard fact that, for every $x \in X$, the limit

$$\lim_{z\to 0} z \cdot x$$

exists and is a $\mathbb{C}^*$-fixed point of X. Let $W \subset X$ be the set of $\mathbb{C}^*$-fixed points, and assume for our purposes that it is discrete. Then, for every $w \in W$, one defines the *attracting set*

$$X_w = \{x \in X \mid \lim_{z\to 0} z \cdot x = w\}.$$

Note that $w \in X_w$.

Since w is fixed there is an action of $\mathbb{C}^*$ on the tangent space $T_w X$, which induces a weight space decomposition

$$T_w X = \bigoplus_{n\in\mathbb{Z}} T_w X[n], \quad \text{where} \quad T_w X[n] = \{v \in T_w X \mid z \cdot v = z^n v\}.$$

Because W is discrete, $T_w Z[0] = 0$, and we get a natural decomposition

$$T_w X = T_w^+ X \oplus T_w^- X. \tag{2.1}$$

Theorem 2.2 (Bialynicki-Birula)**.** *The decomposition*

$$X = \coprod_{w\in W} X_w$$

is a decomposition into smooth locally-closed subvarieties, and there is a natural $\mathbb{C}^*$-equivariant isomorphism

$$X_w \cong T_w X_w = T_w^+ X.$$

Remark 2.3. The Bialynicki-Birula decomposition generalizes to the case where W is not discrete, and the attracting sets are parametrized by the connected components of W.

Proof of Theorem 2.1. We will prove the following sequence of bijections:

$$\begin{array}{ccccc} W_T & \xleftrightarrow{(a)} & \{T\text{-fixed points on } \mathcal{B}\} & \xleftrightarrow{(b)} & \{\mathbb{C}^*\text{-fixed points on } \mathcal{B}\} \\ & & & & \big\updownarrow BB \\ & & \{B\text{-orbits on } \mathcal{B}\} & \xleftrightarrow[(c)]{} & \{\text{attracting sets } \mathcal{B}_w\} \end{array}$$

(a) The first bijection is clear:

$$\{T\text{-fixed points on } \mathcal{B}\} \longleftrightarrow \{\mathfrak{b}' \in \mathcal{B} \mid \mathfrak{h} \subset \mathfrak{b}\} \longleftrightarrow \{w \cdot \mathfrak{b} \mid w \in W_T\} \cong W_T.$$

(b) Choose an embedding $\mathbb{C}^* \hookrightarrow T$ such that the Lie algebra Lie $\mathbb{C}^* \subset \mathfrak{h}$ is spanned by a regular semisimple element $h \in \mathfrak{h}$. This induces a $\mathbb{C}^*$-action on $\mathfrak{g}$ and on $\mathcal{B}$, and

$$\mathbb{C}^* \text{ fixes } \mathfrak{b}' \in \mathcal{B} \iff h \in \mathfrak{b}' \iff \mathfrak{h} \subset \mathfrak{b}' \iff T \text{ fixes } \mathfrak{b}'.$$

Then Bialynicki-Birula gives us a decomposition

$$\mathcal{B} = \coprod_{w \in W_T} \mathcal{B}_w.$$

(c) We will show that every $\mathcal{B}_w$ is a single B-orbit. Fix $w \in W$ and let $U \subset B$ be the unipotent radical. The $\mathbb{C}^*$-action on $\mathfrak{g}$ induces a weight space decomposition

$$\mathfrak{g} = \bigoplus_{n \in \mathbb{Z}} \mathfrak{g}_n = \left(\bigoplus_{n>0} \mathfrak{g}_n \right) \oplus \mathfrak{h} \oplus \left(\bigoplus_{n<0} \mathfrak{g}_n \right) = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-,$$

where

$$\mathfrak{g}_n = \{x \in \mathfrak{g} \mid h \cdot x = nx\}.$$

We can choose the embedding $\mathbb{C}^* \hookrightarrow G$ such that $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$, and in this case $\mathfrak{n}^+ = \text{Lie } U$.

Because $\mathcal{B}$ is a homogeneous space, the tangent space of $\mathcal{B}$ at the point wB is canonically isomorphic to the quotient of $\mathfrak{g}$ by the Lie algebra of $\text{Stab}_G(wB) = w\mathfrak{b}$. The differential of the action map of G is the quotient

$$\mathfrak{g} \longrightarrow \mathfrak{g}/w\mathfrak{b} \cong T_{wB}\mathcal{B},$$

and because $\mathfrak{h} \subset w\mathfrak{b}$ it factors through $\mathfrak{g}/\mathfrak{h}$:

$$\begin{tikzcd} \mathfrak{g} \arrow[r] \arrow[d] & \mathfrak{g}/w\mathfrak{b} \cong T_{wB}\mathcal{B} \\ \mathfrak{g}/\mathfrak{h} \arrow[ur] & \end{tikzcd}$$

The surjection $\mathfrak{g}/\mathfrak{h} \longrightarrow T_{wb}\mathcal{B}$ is compatible with the decompositions

$$\mathfrak{g}/\mathfrak{h} \cong \mathfrak{n}^+ \oplus \mathfrak{n}^- \qquad \text{and} \qquad T_{wb}\mathcal{B} = T^+_{wb}\mathcal{B} \oplus T^-_{wb}\mathcal{B},$$

(the second of which comes from (2.1)), and by Theorem 2.2 this produces a surjection

$$\mathfrak{n}^+ \longrightarrow T_{wb}\mathcal{B}_w.$$

This is just the differential of the action map $U \longrightarrow \mathcal{B}_w$, and its surjectivity implies that UwB is an open dense subset of $\mathcal{B}_w$.

But U is a unipotent group acting on the affine (by Theorem 2.2) space $\mathcal{B}_w$, so all its orbits are closed by Lemma 2.4. It follows that

$$\mathcal{B}_w = UwB = BwB. \qquad \square$$

Lemma 2.4. *Suppose U is a unipotent group acting on an affine space X. Then any orbit of U is closed.*

Proof. Let $\mathcal{O}$ be a U-orbit and $\overline{\mathcal{O}}$ its closure. If $\overline{\mathcal{O}} \neq \mathcal{O}$, the boundary $C = \overline{\mathcal{O}} \backslash \mathcal{O}$ is a nonempty, closed, U-stable subvariety of $\overline{\mathcal{O}}$. Let $I \subset \mathbb{C}[\overline{\mathcal{O}}]$ be its (nonempty) defining ideal.

Because C is U-stable, the group U acts on I, and because U is unipotent it has a fixed point—a nonzero function $g \in I^G$. Because g is G-invariant, it is constant on the orbit closure $\overline{\mathcal{O}}$, and because $g \in I$, $g|_C = 0$. So, g must be identically 0 on $\overline{\mathcal{O}}$—a contradiction. □

3. The Grothendieck-Springer resolution

Proposition 3.1. *Let $\mathfrak{b}$ and $\mathfrak{b}'$ be any two Borel subalgebras. There is a canonical isomorphism*

$$\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \xrightarrow{\sim} \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}'].$$

Proof. Since all Borels are conjugate, there is some $g \in G$ with $\mathfrak{b}' = g\mathfrak{b}$. This gives an isomorphism

$$\widetilde{\varphi}_g : \mathfrak{b} \xrightarrow{\sim} \mathfrak{b}'$$

that descends to an isomorphism

$$\varphi_g : \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \xrightarrow{\sim} \mathfrak{b}'/[\mathfrak{b}', \mathfrak{b}'].$$

Suppose g' is some other group element such that $\mathfrak{b}' = g'\mathfrak{b}$. Because Borels are self-normalizing, $g' = gb$ for some $b \in B$. But then $\varphi_g = \varphi_{g'}$, because the action of the element b on $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ is trivial. □

Definition 3.2. The quotients $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}]$ are canonically identified with an l-dimensional vector space $\mathfrak{H}$ called the *universal Cartan.*

Remark 3.3. We emphasize that $\mathfrak{H}$ is not a subalgebra of $\mathfrak{g}$.

Let T be a maximal torus with Lie algebra $\mathfrak{h}$ as before, and fix a Borel subgroup B such that Lie $B = \mathfrak{b}$ contains $\mathfrak{h}$. The pair (T, B) is equipped with the data of a root system Φ_T (depending on T), and a set of simple roots Δ_{TB} (which depends also on the choice of B.) The corresponding Weyl group W_T is the Coxeter group generated by the simple reflections $\{s_\alpha \mid \alpha \in \Delta_{TB}\}$ under the usual braid relations.

The composition of the morphisms

$$\mathfrak{h} \hookrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \equiv \mathfrak{H}$$

gives an isomorphism $\mathfrak{h} \xrightarrow{\sim} \mathfrak{H}$ (which depends on the choice of $\mathfrak{b}$.) This induces a root system Φ on $\mathfrak{H}$, together with a set of simple roots Δ, a corresponding Weyl group $\mathbb{W}$, called the *universal Weyl group*, and an isomorphism

$$W_T \xrightarrow{\sim} \mathbb{W}.$$

Definition 3.4. The *Grothendieck-Springer resolution* is the incidence variety

$$\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B} \mid x \in \mathfrak{b}\}.$$

We first consider the projection $\pi : \tilde{\mathfrak{g}} \longrightarrow \mathcal{B}$. For every $\mathfrak{b} \in \mathcal{B}$,

$$\pi^{-1}(\mathfrak{b}) \cong \mathfrak{b}.$$

Fix a Borel subgroup B with Lie algebra $\mathfrak{b}$. The isomorphism

$$\begin{aligned} G \times_B \mathfrak{b} &\xrightarrow{\sim} \tilde{\mathfrak{g}} \\ (g, x) &\longmapsto (gx, g\mathfrak{b}) \end{aligned}$$

makes $\widetilde{\mathfrak{g}}$ into a G-equivariant vector bundle on $\mathcal{B}$, and π is simply the bundle map.

Now we consider the projection $\mu : \tilde{\mathfrak{g}} \longrightarrow \mathfrak{g}$. The fiber above a point $x \in \mathfrak{g}$ is the set of Borel subalgebras containing x:

$$\mu^{-1}(x) = \{\mathfrak{b} \in \mathcal{B} \mid x \in \mathfrak{b}\}$$

—in other words, it is the set of zeros of the vector field induced by x on $\mathcal{B}$.

The fiber above $0 \in \mathfrak{g}$ is the entire flag variety $\mathcal{B}$. The fiber above a regular semisimple element $s \in \mathfrak{g}^{rs}$ is finite, of the same cardinality as the Weyl group, because the Borel subalgebras containing s are freely permuted by the Weyl group corresponding to the maximal torus $Z_G(s)$. Let

$$\tilde{\mathfrak{g}}^{rs} = \mu^{-1}(\mathfrak{g}^{rs})$$

be the regular semisimple locus of $\widetilde{\mathfrak{g}}$.

Proposition 3.5. *For every $s \in \mathfrak{g}^{rs}$, there is a canonical action of the universal Weyl group $\mathbb{W}$ on the fiber $\mu^{-1}(s)$, which makes $\tilde{\mathfrak{g}}^{rs}$ into a principal $\mathbb{W}$-bundle of $\mathfrak{g}^{rs}$.*

Proof. Let $s \in \mathfrak{g}^{rs}$ and let the maximal torus $T = Z_G(s)$ be its centralizer. There is a natural action of the Weyl group W_T on the fiber $\mu^{-1}(s)$. Any $\mathfrak{b} \in \mu^{-1}(s)$ induces an isomorphism $W_T \cong \mathbb{W}$, and one defines the action of every $w \in \mathbb{W}$ on $\mathfrak{b}$ accordingly. □

We also record here the following observation, which will be useful later:

Proposition 3.6. *The map μ is proper.*

Proof. The map μ is just the restriction of the first projection $\alpha : \mathfrak{g} \times \mathcal{B} \longrightarrow \mathfrak{g}$ to the subvariety $\tilde{\mathfrak{g}} \subset \mathfrak{g} \times \mathcal{B}$, and α is proper because $\mathcal{B}$ is projective. □

There is also a natural map

$$\begin{aligned} \nu : \ \tilde{\mathfrak{g}} &\longrightarrow \mathfrak{H} \\ (x, \mathfrak{b}) &\longmapsto x + [\mathfrak{b}, \mathfrak{b}]. \end{aligned}$$

This will be useful in the proof of the following theorem, where we will work our way right-to-left along the diagram

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} & & \mathfrak{H}. \end{tikzcd} \tag{3.1}$$

Fix a Cartan $\mathfrak{h}$ with associated Weyl group W. The usual restriction $\mathbb{C}[\mathfrak{g}] \longrightarrow \mathbb{C}[\mathfrak{h}]$ descends to a homomorphism of algebras

$$\varphi : \mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]^W.$$

Theorem 3.7 (Chevalley restriction)**.** *The restricton map φ is an isomorphism.*

Proof. Injectivity is clear—if $P \in \mathbb{C}[\mathfrak{g}]^G$ is such that $P_{|\mathfrak{h}} = 0$, then by G-invariance $P_{|\mathfrak{g}^{ss}} = 0$, and because semisimple elements are dense in $\mathfrak{g}$ this means that $P = 0$.

To prove surjectivity, let $P \in \mathbb{C}[\mathfrak{h}]^W$ and let $\mathfrak{b}$ be a Borel subalgebra containing $\mathfrak{h}$. This choice of $\mathfrak{b}$ induces an isomorphism

$$\mathfrak{h} \hookrightarrow \mathfrak{b} \longrightarrow \mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \cong \mathfrak{H},$$

and this identification produces a corresponding $P_{\mathfrak{H}} \in \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$. Because $P_{\mathfrak{H}}$ is $\mathbb{W}$-invariant, it is independent of the choice of $\mathfrak{b}$.

Pulling $P_{\mathfrak{H}}$ back through the morphism ν, we obtain a polynomial $\widetilde{P} = P_{\mathfrak{H}} \circ \nu \in \mathcal{O}(\widetilde{\mathfrak{g}})$. This polynomial G-invariant:

$$\widetilde{P}(g \cdot x, g \cdot \mathfrak{b}) = P_{\mathfrak{H}}(gx + [g\mathfrak{b}, g\mathfrak{b}]) = P_{\mathfrak{H}}(x + [\mathfrak{b}, \mathfrak{b}]) = \widetilde{P}(x, \mathfrak{b})$$

because of the canonical isomorphism $\mathfrak{b}/[\mathfrak{b}, \mathfrak{b}] \cong g\mathfrak{b}/[g\mathfrak{b}, g\mathfrak{b}]$ of Proposition 3.1.

By Proposition 3.5, $\mathfrak{g}^{rs}$ is the G-equivariant quotient of $\tilde{\mathfrak{g}}^{rs}$ by the free action of the finite group $\mathbb{W}$. This gives an identification

$$\mathbb{C}(\mathfrak{g}^{rs}) \cong \mathbb{C}(\tilde{\mathfrak{g}}^{rs})^{\mathbb{W}},$$

and $\widetilde{P}$ descends to a G-invariant regular function R on $\mathfrak{g}^{rs}$.

We will show that R extends to all of $\mathfrak{g}$. Let $D \subset \mathfrak{g}$ be any relatively compact set. Because μ is proper, $\mu^{-1}(D \cap \mathfrak{g}^{rs}) \subset \tilde{\mathfrak{g}}^{rs}$ is also relatively compact. Then $\widetilde{P}_{|\mu^{-1}(D \cap \mathfrak{g}^{rs})}$ is bounded because $\widetilde{P}$ is a regular function, so $R_{|D \cap \mathfrak{g}^{rs}}$ is also bounded. Since R is bounded on every relatively compact set, it has no poles, and $R \in \mathbb{C}[\mathfrak{g}]^G$.

Last, we check that R restricts to P. Let $x \in \mathfrak{h}^r$ be a regular element and let $\mathfrak{b}$ be any Borel containing x. Then

$$R(x) = \widetilde{P}(x, \mathfrak{b}) = P_{\mathfrak{H}}(x + [\mathfrak{b}, \mathfrak{b}]) = P(x).$$

Since R and P agree on the regular locus $\mathfrak{h}^r$, they agree on all of $\mathfrak{h}$. □

Remark 3.8. A Borel subalgebra $\mathfrak{b}$ containing $\mathfrak{h}$ induces an isomorphism

$$\mathfrak{h} \xrightarrow{\sim} \mathfrak{H}.$$

Because all such Borel subalgebras are permuted by the Weyl group W, the induced isomorphism of invariant coordinate rings

$$\mathbb{C}[\mathfrak{H}]^{\mathbb{W}} \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}]^W$$

is independent of the choice of $\mathfrak{b}$. So, the Chevalley restriction theorem actually gives a well-defined canonical isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}.$$

Reversing this isomorphism produces an injection

$$\mathbb{C}[\mathfrak{H}]^{\mathbb{W}} \hookrightarrow \mathbb{C}[\mathfrak{g}]. \tag{3.2}$$

Because $\mathbb{W}$ is finite, the polynomials in $\mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$ separate W-orbits on $\mathfrak{H}$ (see for example the proof of Proposition 7.3), and therefore

$$\operatorname{Specm} \mathbb{C}[\mathfrak{H}]^{\mathbb{W}} \cong \mathfrak{H}/\mathbb{W}.$$

Because the algebra of polynomial invariants is a free polynomial algebra, the quotient $\mathfrak{H}/\mathbb{W}$ is a vector space of dimension $\dim \mathfrak{H} = l$. So the injection (3.2) induces a morphism of algebraic varieties

$$\rho : \mathfrak{g} \longrightarrow \mathfrak{H}/\mathbb{W}.$$

Diagram (3.1) is now extended to

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} \arrow[dr, "\rho"'] & & \mathfrak{H} \arrow[dl, "\pi"] \\ & \mathfrak{H}/\mathbb{W} & \end{tikzcd} \tag{3.3}$$

Proposition 3.9. *Diagram* (3.3) *commutes.*

Proof. Because the polynomials in $\mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$ separate points on $\mathfrak{H}/\mathbb{W}$, it is sufficient to show that for any $P \in \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$,

$$f \circ \rho \circ \mu(x, \mathfrak{b}) = f \circ \pi \circ \nu(x, \mathfrak{b}) \quad \text{for all } (x, \mathfrak{b}) \in \tilde{\mathfrak{g}}.$$

This is equivalent to commutativity of the diagram

$$\begin{tikzcd} & \mathcal{O}(\tilde{\mathfrak{g}}) & \\ \mathbb{C}[\mathfrak{g}] \arrow[ur, "\mu^*"] & & \mathbb{C}[\mathfrak{H}] \arrow[ul, "\nu^*"'] \\ & \mathbb{C}[\mathfrak{H}]^{\mathbb{W}} \arrow[ul, "\rho^*"] \arrow[ur, "\pi^*"'] & \end{tikzcd}$$

But then, using the notation defined in the proof of Theorem 3.7,

$$\mu^*(\rho^*(P)) = \mu^*(R) = \widetilde{P} = \nu^*(\pi^*(P)).$$

□

Example 3.10. Let $\mathfrak{g} = \mathfrak{sl}_n$, and let $\mathfrak{h}$ denote the subalgebra of diagonal matrices. The Weyl group is $W = S_n$, and the algebra $\mathbb{C}[\mathfrak{h}]^W$ is generated by the elementary symmetric polynomials. Under the Chevalley restriction, they pull back to the coefficients of the characteristic polynomial. That is, for any $x \in \mathfrak{sl}_n$,

$$\operatorname{char}_x(t) = t^n + 0 \cdot t^{n-1} + p_1(x) t^{n-2} + \ldots + p_{n-1}(x)$$

and the algebra of polynomial invariants on $\mathfrak{sl}_n$ is a polynomial algebra generated by $p_1, \dots, p_{n-1}$:

$$\mathbb{C}[\mathfrak{sl}_n]^{SL_n} = \mathbb{C}[p_1, \dots, p_{n-1}].$$

The map ρ is then given by

$$\begin{aligned} \rho : \mathfrak{g} &\longrightarrow \mathbb{C}^l \\ x &\longmapsto (p_1(x), \dots, p_{n-1}(x)) \end{aligned}$$

So the image of matrix x under ρ depends only on the eigenvalues of x, with multiplicity.

Proposition 3.11. *Let $\mathfrak{b}$ be a Borel subalgebra, $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ its nilradical, and $x \in \mathfrak{b}$. Then for any $P \in \mathbb{C}[\mathfrak{g}]^G$, the restriction $P_{|x+\mathfrak{n}}$ is constant.*

Proof. Let $y = x + n$ for some $n \in \mathfrak{n}$. Then $(x, \mathfrak{b}), (y, \mathfrak{b}) \in \tilde{\mathfrak{g}}$, and

$$\nu(x, \mathfrak{b}) = \nu(y, \mathfrak{b}).$$

Let $P \in \mathbb{C}[\mathfrak{g}]^G$ and let $P_{\mathfrak{H}} \in \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$ be its image under the Chevalley restriction, so that $P \circ \mu = P_{\mathfrak{H}} \circ \nu$. Then

$$P_{\mathfrak{H}}(\nu(x, \mathfrak{b})) = P_{\mathfrak{H}}(\nu(y, \mathfrak{b})) \Rightarrow P(\mu(x, \mathfrak{b})) = P(\mu(y, \mathfrak{b})) \Rightarrow P(x) = P(y). \qquad \square$$

4. The nilpotent cone

Definition 4.1. The set of nilpotent element

$$\mathcal{N} = \{x \in \mathfrak{g} \mid x \text{ is nilpotent}\}$$

is closed, G-stable, and stable under scaling by $\mathbb{C}^*$, and it is called the *nilpotent cone.*

Let

$$\widetilde{\mathcal{N}} = \mu^{-1}(\mathcal{N}) = \{(x, \mathfrak{b}) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\}.$$

The projection $\pi : \widetilde{\mathcal{N}} \longrightarrow \mathcal{B}$ has fibers

$$\pi^{-1}(\mathfrak{b}) = [\mathfrak{b}, \mathfrak{b}].$$

Fix a point $\mathfrak{b} \in \mathcal{B}$ and let $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ be its nilradical. Because every nilpotent element in $\mathfrak{g}$ is conjugate to an element of $\mathfrak{n}$, there is an isomorphism

$$G \times_B \mathfrak{n} \xrightarrow{\sim} \widetilde{\mathcal{N}}$$

that makes $\widetilde{\mathcal{N}}$ into a G-equivariant vector bundle on $\mathcal{B}$.

Proposition 4.2. *There is an isomorphism $\widetilde{\mathcal{N}} \cong T^*\mathcal{B}$.*

Proof. Fix $\mathfrak{b} \in \mathcal{B}$ with corresponding Borel subgroup B. Because $\mathcal{B} = G/B$ is a homogeneous space,

$$T^*\mathcal{B} \cong G \times_B \mathfrak{b}^\perp,$$

where $\mathfrak{b}^\perp = \{\varphi \in \mathfrak{g}^* \mid \varphi_{|\mathfrak{b}} = 0\}$. Under the identification $\mathfrak{g} \cong \mathfrak{g}^*$ via the Killing form, the space $\mathfrak{b}^\perp$ is identified with $[\mathfrak{b}, \mathfrak{b}]$. Then

$$T^*\mathcal{B} \cong G \times_B \mathfrak{n} \cong \widetilde{\mathcal{N}}. \qquad \square$$

Let $\mathbb{C}[\mathfrak{g}]^G_+$ be the ideal of $\mathbb{C}[\mathfrak{g}]^G$ consisting of polynomials with zero constant term. This is the ideal generated by the polynomials that generate $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{H}]^{\mathbb{W}}$ as a free algebra.

Proposition 4.3. *An element $x \in \mathfrak{g}$ is nilpotent if and only if $P(x) = 0$ for every $P \in \mathbb{C}[\mathfrak{g}]^G_+$.*

Proof. The proposition amounts to showing that

$$\mathcal{N} = \rho^{-1}(0).$$

Let $x \in \mathfrak{g}$ and $\mathfrak{b}$ a Borel subalgebra containing x. Then $\rho(x) = \pi(\nu(x, \mathfrak{b}))$ because of the commutativity of diagram (3.3). The element x is nilpotent if and only if $x \in [\mathfrak{b}, \mathfrak{b}]$, if and only if $\pi(x + [\mathfrak{b}, \mathfrak{b}]) = \pi(0) = 0$, if and only if $\rho(x) = 0$. $\square$

Corollary 4.4. *The nilpotent cone $\mathcal{N}$ is an irreducible variety of dimension $2 \dim \mathfrak{n}$.*

Proof. The cotangent bundle $T^*\mathcal{B}$ is smooth and connected, so it is irreducible. Because $\mathcal{N}$ is its image under the morphism μ, $\mathcal{N}$ is also irreducible.

The nilpotent cone is the vanishing locus of the l algebraically independent polynomials that generate $\mathbb{C}[\mathfrak{g}]^G$, so

$$\dim \mathcal{N} = \dim \mathfrak{g} - l = 2 \dim \mathfrak{n}. \qquad \square$$

Proposition 4.5. *The number of G-orbits in $\mathcal{N}$ is finite.*

The proof will require the following lemma (cf. [Dix] 8.1.2 and 8.1.3.):

Lemma 4.6. *Let $\mathfrak{g}$ be a Lie algebra and suppose that $\mathfrak{a}$ is a Lie subalgebra that has an $\mathfrak{a}$-stable complement W—that is, a subspace $W \subset \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{a} \oplus W$ and $[\mathfrak{a}, W] \subset W$. Let G and A be connected algebraic groups so that $G = \mathrm{Lie}\, \mathfrak{g}$ and $A = \mathrm{Lie}\, \mathfrak{a}$. Then for any G-orbit Ω on $\mathfrak{g}$, any irreducible component of $\Omega \cap \mathfrak{a}$ is a single A-orbit.*

Proof. Let $Z \subset \Omega \cap \mathfrak{a}$ be such a component, and let $x \in Z$. Identifying $\mathfrak{g}$ with its tangent space at any point,

$$T_x\Omega = [\mathfrak{g}, x] = [\mathfrak{a}, x] + [W, x].$$

Because $x \in \mathfrak{a}$, $[\mathfrak{a}, x] \subseteq \mathfrak{a}$, and because W is $\mathfrak{a}$-stable, $[W, x] \subseteq W$. Then

$$T_xZ = T_x\Omega \cap T_x\mathfrak{a} = [\mathfrak{a}, x] = T_x(A \cdot x).$$

So the orbit $A \cdot x$ is open dense in Z. Because this is true for every x, and because intersecting orbits are equal, $A \cdot x = Z$. $\square$

Proof of Proposition 4.5. The statement is clearly true for $\mathfrak{gl}_n$, since there are finitely many configurations of nilpotent Jordan blocks. We can embed $\mathfrak{g} \hookrightarrow \mathfrak{gl}_n$, and this embedding preserves nilpotency.

Because $\mathfrak{g}$ is semisimple the image of this embedding has a $\mathfrak{g}$-stable complement. By Lemma 4.6, so that the intersection of each of the finitely many nilpotent GL_n-orbits on $\mathfrak{gl}_n$ decomposes into finitely many G-orbits on $\mathfrak{g}$. □

Corollary 4.7. *The set $\mathcal{N}^{reg}$ of regular nilpotents is an open dense G-orbit in $\mathcal{N}$.*

Proof. Because $\mathcal{N}$ is irreducible with finitely many orbits, it contains a unique open dense orbit $\mathcal{O}$. If $x \in \mathcal{O}$, then

$$2 \dim \mathfrak{n} = \dim \mathcal{O} = \dim G - \dim Z_G(x),$$

and therefore $\dim Z_G(x) = l$, so x is regular. □

Fix now a Borel subgroup B containing a maximal torus T, let U be its unipotent radical, $\mathfrak{b} = \text{Lie } B$, and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$. There is a natural action

$$T \curvearrowright \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$$

which gives a basis of weight vectors $\bar{e}_1, \dots, \bar{e}_l$ which are the images of the simple root vectors $e_1, \dots, e_l$ under the projection

$$\begin{aligned} \mathfrak{n} &\longrightarrow \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}] \\ x &\longmapsto \bar{x} \end{aligned}$$

Define the set $\mathfrak{n}^r = \{x \in \mathfrak{n} \mid \bar{x} = \sum_{i=1}^{l} a_i \bar{e}_i, a_i \in \mathbb{C}^*\}$.

Proposition 4.8. *The set $\mathfrak{n}^r$ is a single B-orbit consisting of regular elements.*

Proof. The image of $\mathfrak{n}^r$ in $\mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is an open dense T-orbit, so there exists a regular element $x \in \mathfrak{n}^r$. The set $x + [\mathfrak{n}, \mathfrak{n}]$ is U-stable and

$$\dim U \cdot x \geq \dim \mathfrak{n} - \dim Z_G(x) = \dim[\mathfrak{n}, \mathfrak{n}],$$

so the orbit $U \cdot x$ is open and dense in $x + [\mathfrak{n}, \mathfrak{n}]$. But because U is unipotent this orbit is also closed by Lemma 2.4, so $U \cdot x = x + [\mathfrak{n}, \mathfrak{n}]$. Then

$$B \cdot x = T \cdot (x + [\mathfrak{n}, \mathfrak{n}]) = \mathfrak{n}^r.$$ □

Corollary 4.9. *The element $\sum_{i=1}^{l} e_i$ is a regular nilpotent.*

Remark 4.10. In $\mathfrak{sl}_n$, it follows that every regular nilpotent is conjugate to the unique maximal nilpotent Jordan block.

Proposition 4.11. *Every regular nilpotent element is contained in a unique Borel subalgebra.*

Proof. By Proposition 4.4

$$\dim \mathcal{N} = 2 \dim \mathfrak{n} = \dim T^*\mathcal{B} = \dim \widetilde{\mathcal{N}},$$

so the generic fiber of $\mu : \widetilde{\mathcal{N}} \longrightarrow \mathcal{N}$ is discrete.

It is sufficient to prove the proposition for the regular nilpotent $e = \sum_{i=1}^{l} e_i$. Let

$$\alpha_1, \dots, \alpha_l$$

be the simple roots determined by T and B, let $\mathfrak{h} = \mathrm{Lie}\, T$, and let $h \in \mathfrak{h}$ be such that $\alpha_i(h) = 1$ for every index i. Then h is regular by Lemma 1.3, and $[h, e] = e$.

Consider a one-parameter subgroup $\mathbb{C}^* \subset G$ such that $\mathrm{Lie}\, \mathbb{C}^* = \mathbb{C}h$. For any $t \in \mathbb{C}^*$,

$$t \cdot e = \exp(t)e,$$

and so $\mathbb{C}^*$ stabilizes the fiber $\mu^{-1}(e)$. Since this fiber is discrete, $\mathbb{C}^*$ fixes every point, and therefore

$$h \in \mathfrak{b}' \quad \text{for all } \mathfrak{b}' \in \mu^{-1}(e).$$

Because h is regular, this means that $\mathfrak{h} \subset \mathfrak{b}'$ for every such $\mathfrak{b}'$.

But then $\mu^{-1}(e) \subseteq W \cdot \mathfrak{b}$, and it is clear that there is only one point in this orbit—$\mathfrak{b}$ itself—containing the nilpotent e. □

Remark 4.12. This makes $\mu : \widetilde{\mathcal{N}} \longrightarrow \mathcal{N}$ a resolution of singularities, because $\widetilde{\mathcal{N}} = T^*\mathcal{B}$ is smooth and μ is an isomorphism onto the open dense regular locus in $\mathcal{N}$. It is called the *Springer resolution.*

5. The Jacobson-Morozov theorem

For any $x \in \mathfrak{g}$, let $G^x := Z_G(x)$ and $\mathfrak{g}^x := Z^x_{\mathfrak{g}}$.

Theorem 5.1 (Jacobson-Morozov)**.** *Let $e \in \mathcal{N}$ be a (not necessarily regular) nilpotent element. There exist $h, f \in \mathfrak{g}$ such that h is semisimple, f is nilpotent, and*

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Corollary 5.2. *For any $e \in \mathcal{N}$, there is a rational group homomorphism*

$$SL_2 \longrightarrow G$$

whose differential sends the nilpotent $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ *to e.*

Proof. Because SL_2 is simply-connected, the homomorphism of Lie algebras

$$\mathfrak{sl}_2 \longrightarrow \mathfrak{g}$$

given by the Jacobson-Morozov theorem descends to a morphism of groups. □

The proof will require an important lemma. Recall that the Killing form is the symmetric G-invariant bilinear form

$$\begin{aligned} (\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto \mathrm{tr}(\mathrm{ad}_x \mathrm{ad}_y) \end{aligned}$$

In particular, G-invariance implies that for any $x, y, z \in \mathfrak{g}$,

$$([x, y], z) = (x, [y, z]).$$

Lemma 5.3. *Suppose e is a nilpotent element such that the centralizer $\mathfrak{g}^e$ is consists entirely of nilpotent elements. Then $(e, \mathfrak{g}^e) = 0$.*

Proof. Let $x \in \mathfrak{g}^e$. Because e and x commute, there is some integer k such that

$$(\mathrm{ad}_e \mathrm{ad}_x)^k = (\mathrm{ad}_e)^k (\mathrm{ad}_x)^k = 0,$$

so the operator $\mathrm{ad}_e \mathrm{ad}_x$ is nilpotent. Then

$$(e, x) = \mathrm{tr}(\mathrm{ad}_e \mathrm{ad}_x) = 0. \qquad \square$$

Proof of Theorem 5.1. In order to apply the lemma, first we will reduce the proof to the case that $\mathfrak{g}^e$ consists entirely of nilpotent elements. This is by induction—suppose that the statement is known for semisimple Lie algebras of dimension smaller than $\mathfrak{g}$, and suppose that $x \in \mathfrak{g}^e$ is a non-nilpotent element.

Then x has a nontrivial Jordan decomposition $x = s + n$, and because e is an eigenvector of ad_x it is also an eigenvector of ad_s and ad_n. But n is nilpotent, so all its eigenvalues are 0. Therefore

$$0 = [x, e] = [s, e] + [n, e] = [s, e],$$

and $s \in \mathfrak{g}^e$.

The element s is nonzero and semisimple, so its centralizer $\mathfrak{g}^s$ is a proper reductive Lie subalgebra of $\mathfrak{g}$, and $e \in \mathfrak{g}^s$. Because e is nilpotent,

$$e \in [\mathfrak{g}^s, \mathfrak{g}^s].$$

But $[\mathfrak{g}^s, \mathfrak{g}^s]$ is a semisimple Lie algebra of strictly smaller dimension that $\mathfrak{g}$. By the inductive hypothesis, we are done.

So it is sufficient to consider the case where $\mathfrak{g}^e$ is nilpotent. By G-invariance, the operator ad_e is skew-symmetric with respect to the Killing form:

$$([e, x], y) = -(x, [e, y]).$$

This implies that

$$\mathrm{Im}\ \mathrm{ad}_e = (\ker\ \mathrm{ad}_e)^\perp = (\mathfrak{g}^e)^\perp.$$

Since $e \in (\mathfrak{g}^e)^\perp$ by Lemma 5.3, it follows that $e \in \mathrm{Im}\ \mathrm{ad}_e$, so there is some $h \in \mathfrak{g}$ such that

$$[h, e] = 2e.$$

Consider the Jordan decomposition $h = s + n$. As before, since e is an eigenvector of ad_h, it is an eigenvector of ad_s and ad_n, and since n is nilpotent we have

$$2e = [h, e] = [s, e].$$

So we may assume that h is semisimple.

It remains to find f. The action of h on $\mathfrak{g}$ gives a decomposition

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{C}} \mathfrak{g}_k$$

into eigenspaces $\mathfrak{g}_k = \{x \in \mathfrak{g} \mid [h, x] = kx\}$. Then

$$h \in \mathfrak{g}_0 \qquad \text{and} \qquad e \in \mathfrak{g}_2,$$

and the commutation relation $[h,e]=2e$ implies that

$$e:\mathfrak{g}_k\longrightarrow\mathfrak{g}_{k+2}.$$

To find an element $f\in\mathfrak{g}_{-2}$ such that $[e,f]=h$, it is enough to show as before that $h\in\operatorname{Im}\operatorname{ad}_e$, or in other words that $h\in(\mathfrak{g}^e)^\perp$.

Since $[h,e]=2e$, $[h,\mathfrak{g}^e]\subset\mathfrak{g}^e$, and so the subspace

$$\mathbb{C}h+\mathfrak{g}^e$$

is a solvable Lie subalgebra of $\mathfrak{g}$. By the theorems of Lie and Engel, for an appropriate choice of basis in $\operatorname{End}\mathfrak{g}$, ad_h is upper triangular and any $x\in\mathbb{C}h+\mathfrak{g}^e$ is strictly upper triangular. So the product

$$\operatorname{ad}_h\operatorname{ad}_x$$

is also strictly upper triangular, and its trace is 0:

$$(h,x)=\operatorname{tr}(\operatorname{ad}_h\operatorname{ad}_x)=0\quad\text{for any } x\in\mathfrak{g}^e.$$

So there is some $f\in\mathfrak{g}$ such that $[e,f]=h$.

It remains to show that f is nilpotent. This is clear because the triple (e,h,f) induces a Lie algebra homomorphism

$$\mathfrak{sl}_2\longrightarrow\mathfrak{g},$$

and the element which maps to f under this homomorphism is a nilpotent element. $\square$

Proposition 5.4. *Let $e\in\mathcal{N}$. All $\mathfrak{sl}_2$-triples containing e are conjugate by G^e.*

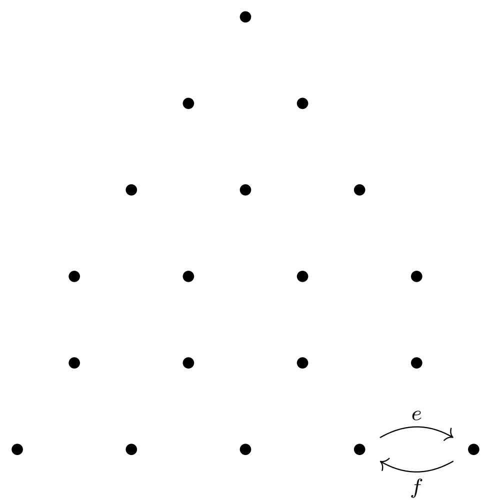
Fix a triple (e,h,f). Then any representation V of $\mathfrak{g}$ decomposes into a direct sum of irreducible representations of $\mathfrak{sl}_2$. These are indexed by non-negative integers and are usually represented diagrammatically as:

$$\begin{tikzcd}\bullet & \bullet & \bullet & \bullet \arrow[r, bend left, "e"] & \bullet \arrow[l, bend left, "f"]\end{tikzcd}$$

where each dot is an h-eigenspace, the action of e shifts the dots to the right, and the action of f shifts them to the left.

Then the representation V looks like:

(5.1)



The left-most dots in each row span the kernel of f, and their complement spans the image of e. This gives a decomposition

$$V = \ker\, f \oplus \operatorname{Im}\, e.$$

The Lie algebra $\mathfrak{g}$ also decomposes in this way, with

$$\mathfrak{g} = \mathfrak{g}^f \oplus [\mathfrak{g}, e]. \tag{5.2}$$

As before, action of h on $\mathfrak{g}$ induces a grading

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_k, \qquad [\mathfrak{g}_k, \mathfrak{g}_j] \subseteq \mathfrak{g}_{k+j}$$

by the eigenvalues of h.

Remark 5.5. In general,

$$\dim \mathfrak{g}^e = \dim \mathfrak{g}^f \geq \dim \mathfrak{g}^h.$$

Equality holds if and only if all the eigenvalues of h are even, so that every irreducible representation contains a 0-eigenspace. Equivalently, this happens if and only if every $\mathfrak{sl}_2$-irrep is of odd dimension.

This inequality also shows that if e is regular, then h and f are also regular, and the number of irreducible representations of $\mathfrak{sl}_2$ in the decomposition of $\mathfrak{g}$ is equal to $\dim \mathfrak{g}^h = l$. In this case (e, h, f) is called a *principal* $\mathfrak{sl}_2$*-triple*. Because all regular nilpotents are conjugate, and in view of Proposition 5.4, it is clear that all principal triples are conjugate.

Remark 5.6. It is immediate that $\mathfrak{g}^e \subseteq \oplus_{k\geq 0}\mathfrak{g}^k$. However, $\mathfrak{g}^e$ is not generally contained in the strictly positive eigenspaces if e is not regular. Define

$$\mathfrak{u} = \mathfrak{g}^e \cap \left(\bigoplus_{k>0} \mathfrak{g}_k \right).$$

This is a nilpotent ideal of $\mathfrak{g}^e$, and from diagram (5.1) it is easy to see that

$$\mathfrak{u} = \mathfrak{g}^e \cap [\mathfrak{g}, e].$$

Let $U \subset G^e$ be the unipotent normal subgroup such that Lie $U = \mathfrak{u}$.

Lemma 5.7. *The affine space $h + \mathfrak{u}$ is a single U-orbit.*

Proof. Because of the grading, $[\mathfrak{u}, \mathfrak{u}] \subseteq \mathfrak{u}$, and because $\mathfrak{u}$ is a sum of h-eigenspaces, $[h, \mathfrak{u}] = \mathfrak{u}$. Therefore $h + \mathfrak{u}$ is U-stable.

Because $\mathfrak{u}^h = 0$, we also have

$$\dim U \cdot h = \dim h + \mathfrak{u},$$

so $U \cdot h \subseteq h + \mathfrak{u}$ is open dense.

Because U is unipotent, $U \cdot h$ is closed by Lemma 2.4, so $U \cdot h = h + \mathfrak{u}$. □

Proof of Proposition 5.4. Let (e, h, f) and (e, h', f') be two $\mathfrak{sl}_2$-triples containing e. Notice that if $h = h'$, then

$$[e, f'] = [e, f] \qquad \Rightarrow \qquad f' - f \in \mathfrak{g}^e.$$

But $\mathfrak{g}^e \subseteq \oplus_{k \geq 0} \mathfrak{g}_k$ and $f' - f \in \mathfrak{g}_{-2}$, so $f' - f = 0$.

Otherwise,

$$\begin{aligned} [h', e] = [h, e] \qquad &\Rightarrow \qquad h' - h \in \mathfrak{g}^e \\ [e, f' - f] = h' - h \qquad &\Rightarrow \qquad h' - h \in [\mathfrak{g}, e]. \end{aligned}$$

So $h' - h \in \mathfrak{u}$, which means that $h' \in h + \mathfrak{u}$ and by Lemma 5.7 there is some $u \in U$ such that $h' = u \cdot h$.

Now (e, h', f') and $(e, h', u \cdot f)$ are two $\mathfrak{sl}_2$-triples containing e with the same semisimple element, so by the first part of this proof $f' = u \cdot f$. □

Let H be the simultaneous centralizer in G of the $\mathfrak{sl}_2$-triple (e, h, f). When e is regular, $\mathfrak{g}^e = \mathfrak{u}$ is nilpotent and $H = Z(G)$. More generally,

Proposition 5.8. *The unipotent subgroup U is the unipotent radical of the centralizer G^e, and H is a maximal reductive subgroup of G^e.*

Proof. Because the centralizer of a reductive subgroup is reductive, H is reductive. It is enough to show that

$$G^e = HU.$$

Because the intersection $H \cap U$ is a normal unipotent subgroup of the reductive group H, it is trivial.

Let $g \in G^e$, and consider the $\mathfrak{sl}_2$-triple

$$(g^{-1}e, g^{-1}h, g^{-1}f) = (e, g^{-1}h, g^{-1}f).$$

By Proposition 5.4 there is some $u \in U$ such that

$$(e, uh, uf) = (e, g^{-1}h, g^{-1}f).$$

But then $(e, h, f) = (e, guh, guf)$, and so $gu \in H$, and $g \in HU$. □

6. The exponents of $\mathfrak{g}$

The restriction map $\mathbb{C}[\mathfrak{g}] \longrightarrow \mathbb{C}[\mathfrak{h}]$ is a graded algebra homomorphism, and so the Chevalley isomorphism

$$\mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]^W$$

of Theorem 3.7 is an isomorphism of graded subalgebras.

The algebra of invariants $\mathbb{C}[\mathfrak{h}]^W$ is a free polynomial algebra with $l = \operatorname{rk} \mathfrak{g}$ homogeneous generators by the Chevalley-Shephard-Todd theorem [Bou], and these generators pull back to homogeneous generators of $\mathbb{C}[\mathfrak{g}]^G$:

$$\mathbb{C}[\mathfrak{g}]^G = \mathbb{C}[P_1, \ldots, P_l].$$

Let $d_i = \deg P_i$, ordered so that $d_1 \leq \ldots \leq d_l$. The integers $d_1, \ldots, d_l$ are called the *exponents* of the Lie algebra $\mathfrak{g}$, and they are independent of the choice of homogeneous generators. The following theorem, due to Kostant [Kos2], gives a way of computing these exponents explicitly.

Theorem 6.1 (Kostant)**.** *Let (e, h, f) be a principal $\mathfrak{sl}_2$-triple, and let $\mathfrak{g} = \oplus V_i$ be the corresponding decomposition into irreducible representations of $\mathfrak{sl}_2$. Let*

$$\dim V_i = 2\lambda_i + 1,$$

ordered so that $\lambda_1 \leq \ldots \leq \lambda_l$. Then $d_i = \lambda_i + 1$.

The proof on the following proposition, which will be proved at the end of Section 10:

Proposition 6.2. *Let $\mathfrak{g}$ be a semisimple Lie algebra of rank l and $d_1, \ldots, d_l$ its exponents. Then*

$$d_1 + \ldots + d_l = \frac{1}{2}(l + \dim \mathfrak{g}).$$

Proof of Theorem 6.1. (See [Dix] 8.1.1. for this proof and the next.) Consider the decomposition

$$\mathfrak{g} = \mathfrak{g}^f \oplus [\mathfrak{g}, e],$$

as in (5.2), and let $\{\zeta_1, \ldots, \zeta_l\}$ be a basis of h-eigenvectors for $\mathfrak{g}^f$ such that

$$[h, \zeta_i] = -2\lambda_i \zeta_i.$$

Define the map

$$\psi : G \times \mathbb{C}^l \longrightarrow \mathfrak{g}$$
$$(g, a_1, \ldots, a_l) \longmapsto g \cdot \left(e + \sum a_i \zeta_i\right)$$

The differential of ψ at $(1, a_1, \ldots, a_l)$ is

$$\mathrm{d}_{(1,a_1,\ldots,a_l)}\psi : \mathfrak{g} \times \mathbb{C}^l \longrightarrow \mathfrak{g} \tag{6.1}$$
$$(x, b_1, \ldots, b_l) \longmapsto \left[x, e + \sum a_i \zeta_i\right] + \sum b_i \zeta_i. \tag{6.2}$$

The image of $\mathrm{d}_{(1,a_1,\ldots,a_l)}\psi$ is $[\mathfrak{g}, e] \oplus \mathfrak{g}^f = \mathfrak{g}$, so the differential is surjective. It follows that the image $G(e + \mathfrak{g}^f)$ of ψ is open dense in $\mathfrak{g}$, and the restriction map

$$\mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[e + \mathfrak{g}^f]$$

is an injection.

For every homogeneous generator P_j of $\mathbb{C}[\mathfrak{g}]^G$, define $R_j \in \mathbb{C}[a_1, \ldots, a_n]$ by

$$R_j(a_1, \ldots, a_l) = P_j \circ \psi(1, a_1, \ldots, a_l) = P_j\left(e + \sum a_i \zeta_i\right).$$

Because the polynomials $P_1, \ldots, P_l$ are algebraically independent, so are $R_1, \ldots, R_l$.

We use the following criterion of Euler: For any $P \in \mathbb{C}[x_1, \ldots, x_n]$,

$$x_1 \frac{\partial P}{\partial x_i} + \ldots + x_n \frac{\partial P}{\partial x_n} = \deg(P) P(x_1, \ldots, x_l).$$

So for every P_j, considering u as a tangent vector in $T_u\mathfrak{g}$ and denoting by $\langle \cdot, \cdot \rangle$ the usual pairing between vector and covectors,

$$\langle u, \mathrm{d}_u P_j \rangle = d_j P_j(u).$$

On the other hand, any element

$$u = e + \sum a_i \zeta_i \in e + \mathfrak{g}^f$$

can be written as

$$\begin{aligned} u &= e + \sum a_i \zeta_i \\ &= \left(e - \sum \lambda_i a_i \zeta_i\right) + \sum (1 + \lambda_i) a_i \zeta_i \\ &= \mathrm{d}_{(1, a_1, \ldots, a_l)} \psi \left(\frac{h}{2}, (1 + \lambda_1) a_1, \ldots, (1 + \lambda_l) a_l\right). \end{aligned}$$

Then

$$\begin{aligned} \langle u, \mathrm{d}_u P_j \rangle &= \left\langle \mathrm{d}_{(1, a_1, \ldots, a_l)} \psi \left(\frac{h}{2}, (1 + \lambda_1) a_1, \ldots, (1 + \lambda_l) a_l\right), \mathrm{d}_u P_j \right\rangle \\ &= \left\langle \left(\frac{h}{2}, (1 + \lambda_1) a_1, \ldots, (1 + \lambda_l) a_l\right), \mathrm{d}_{(1, a_1, \ldots, a_l)} (P_j \circ \psi) \right\rangle \\ &= \left\langle ((1 + \lambda_1) a_1, \ldots, (1 + \lambda_l) a_l), \mathrm{d}_{(a_1, \ldots, a_l)} R_j \right\rangle \\ &= (1 + \lambda_1) a_1 \frac{\partial R_j}{\partial a_1} + \ldots + (1 + \lambda_l) a_l \frac{\partial R_j}{\partial a_l}. \end{aligned}$$

The second-to-last equality follows because P_j is G-invariant, so $P_j \cdot \psi$ is independent of the first coordinate. We get the equality of polynomials

$$(1 + \lambda_1) a_1 \frac{\partial R_j}{\partial a_1} + \ldots + (1 + \lambda_l) a_l \frac{\partial R_j}{\partial a_l} = d_j R_j(a_1, \ldots, a_l).$$

The polynomial R_j is a sum of monomials of the form $a_1^{m_{1j}} \ldots a_k^{m_{kj}}$ such that

$$\sum (1 + \lambda_i) m_{ij} = d_j.$$

Suppose towards a contradiction that there is some j_0 such that $d_{j_0} < 1 + \lambda_{j_0}$. Then for any $j \leq j_0$, $d_j < 1 + \lambda_{j_0}$, and $m_{ij} = 0$ for all $i \geq j_0$. So for all $j \leq j_0$, R_j only depends on the variables

$$x_1, \ldots, x_{j_0 - 1}.$$

But this contradicts the algebraic independence of the polynomials $R_1, \ldots, R_{j_0}$. So, we must always have $d_j \geq 1 + \lambda_j$.

But by (6.2),

$$d_1 + \ldots + d_l = \frac{1}{2}(l + \dim \mathfrak{g}) = (1 + \lambda_1) + \ldots + (1 + \lambda_l),$$

so equality must hold everywhere and $d_j = 1 + \lambda_j$. □

Proposition 6.3. *The differentials* $\mathrm{d}P_1, \ldots, \mathrm{d}P_l$ *are linearly independent at every point of the slice* $e + \mathfrak{g}^f$*.*

Proof. Because $d\psi$ is a bijective linear transformation, it is enough to show that the differentials $\mathrm{d}R_1, \ldots, \mathrm{d}R_l$ are linearly independent at every point of $\mathbb{C}^l$.

From the proof of Theorem 6.1, for every index j we have the formula

$$\sum_i d_i m_{ij} = d_j,$$

where m_{ij} is the cumulative exponent of a_i in R_j. Then

$$d_i > d_j \qquad \Rightarrow \qquad m_{ij} = 0,$$

so R_j depends only on the set $\{a_r \mid d_r \leq d_j\}$. Moreover, if $d_i = d_j$, then either $m_{ij} = 0$ or $m_{ij} = 1$ and for all $i' \neq i$ we have $m_{i'j} = 0$.

Let $C_j = \{i \mid d_i = d_j\}$. Then

$$R_j(a_1, \ldots, a_l) = \sum_{i \in C_j} \alpha_{ij} a_i^{m_{ij}} + g_j,$$

where $\alpha_{ij} \in \mathbb{C}$ are constants and g_j is a polynomial that depends only on $\{a_r \mid d_r < d_j\}$.

Then the Jacobian matrix

$$\left(\frac{\partial R_j}{\partial a_i} \right)$$

is block-upper triangular, and its diagonal blocks are the matrices (α_{ij}), $i, j \in C_k$, listed over all equivalence sets C_k without multiplicity.

For each C_k, the matrix (α_{ij}), $i, j \in C_k$, has exactly one nonzero entry in each row and in each column, because the polynomials $R_1, \ldots, R_l$ are algebraically independent and so no subset $R_1, \ldots, R_k$ can depend on less than k distinct variables. Therefore,

$$\det \left(\frac{\partial R_j}{\partial a_i} \right) = \det (\alpha_{ij}) \neq 0,$$

and the polynomials $R_1, \ldots, R_l$ have linearly independent differentials at every point. □

Corollary 6.4. *The morphism*

$$\begin{aligned} \rho : e + \mathfrak{g}^f &\longrightarrow \mathbb{C}^l \\ x &\longmapsto (P_1(x), \ldots, P_l(x)) \end{aligned}$$

is an isomorphism.

Proof. The map

$$\psi : e + \mathfrak{g}^f \longrightarrow \mathbb{C}^l$$
$$e + \sum a_i z_i \longmapsto (a_1, \ldots, a_l)$$

from the proof of Theorem 6.1 is an isomorphism whose pullback takes each polynomial P_j to $R_j = \alpha_j a_j + g_j$. (Without loss of generality we can rearrange the indices so that the leading term of R_j is a_j, and then g_j is a polynomial in the variables $a_1, \ldots, a_{j-1}$.)

Its composition with ρ,

$$\rho \circ \psi^{-1} : \mathbb{C}^l \longrightarrow \mathbb{C}^l$$
$$(a_1, \ldots, a_l) \longmapsto (\alpha_1 a_1, \alpha_2 a_2 + g_2(a_1), \ldots, \alpha_l a_l + g_l(a_1, \ldots, a_{l-1})),$$

is an isomorphism.

□

7. Regular elements and the principal slice

Now let $\mathfrak{b}^-$ be the unique Borel containing the regular nilpotent f.

Proposition 7.1. *Every element of $f + \mathfrak{b}^-$ is regular.*

Proof. (See [Kos2] Lemma 10.) The action of the regular semisimple element h gives $\mathfrak{g}$ a grading by even eigenvalues (cf. Remark 5.5):

$$\mathfrak{g} = \bigoplus_{k \in 2\mathbb{Z}} \mathfrak{g}_k.$$

The principal nilpotent e lives in degree 2, and maps

$$\mathrm{ad}_e : \mathfrak{g}_k \longrightarrow \mathfrak{g}_{k+2}.$$

Consider the ascending filtration

$$\mathfrak{g}_{\leq j} = \bigoplus_{k \leq j} \mathfrak{g}_k,$$

and let $x = e + v \in e + \mathfrak{b}^-$. The element

$$v \in \mathfrak{b}^- = \bigoplus_{k \leq 0} \mathfrak{g}_k$$

preserves this filtration, and there is an induced filtration on $\mathfrak{g}^x$:

$$\mathfrak{g}^x_{\leq j} = \mathfrak{g}^x \cap \mathfrak{g}_{\leq j}.$$

We claim that the image of the injection

$$\mathfrak{g}^x_{\leq j} / \mathfrak{g}^x_{\leq (j-2)} \hookrightarrow \mathfrak{g}_j$$

lies in the centralizer $\mathfrak{g}^e$.

Let $y \in \mathfrak{g}^x_{\leq j}$—then $y = y_j + y'$, where $y_j \in \mathfrak{g}_j$ and $y' \in \mathfrak{g}_{\leq (j-2)}$. We have

$$0 = [y, x] = [y_j, e] + [y', e] + [y, v],$$

where the first term is in $\mathfrak{g}_{j+2}$ and the other terms are in $\mathfrak{g}_{\leq j}$. This implies that $[y_j, e] = 0$, proving the claim.

Therefore,

$$\dim \mathfrak{g}^x = \sum_j \dim \left(\mathfrak{g}^x_{\leq j} / \mathfrak{g}^x_{\leq (j-2)} \right) \leq \sum_j \dim \left(\mathfrak{g}^e \cap \mathfrak{g}_j \right) = \dim \mathfrak{g}^e = l. \qquad \square$$

Theorem 7.2. *The composition*

$$e + \mathfrak{g}^f \hookrightarrow \mathfrak{g}^{reg} \longrightarrow \mathfrak{g}^{reg}/G$$

is a bijection.

Proof. Because the polynomials $P_1, \dots, P_l$ are G-invariant, the morphism ρ of Corollary 6.4 descends to a map

$$\overline{\rho} : \mathfrak{g}^{reg}/G \longrightarrow \mathbb{C}^l.$$

This gives the diagram

$$\begin{tikzcd} e + \mathfrak{g}^f \arrow[r] \arrow[dr, "\rho"'] & \mathfrak{g}^{reg}/G \arrow[d, "\overline{\rho}"] \\ & \mathbb{C}^l \end{tikzcd}$$

where the top arrow is the composition we are interested in. We know that ρ is an isomorphism from Corollary 6.4, and we will prove in the next proposition that $\overline{\rho}$ is an injection. The theorem then follows. $\square$

Proposition 7.3. *The map $\overline{\rho} : \mathfrak{g}^{reg}/G \longrightarrow \mathbb{C}^l$ is injective.*

Proof. It is enough to show that invariant polynomials on $\mathfrak{g}$ separate regular G-orbits. Suppose $x, y \in \mathfrak{g}^{reg}$ have the same semisimple part in their Jordan decompositions:

$$x = s + n, \quad y = s + n'.$$

Then n, n' are regular nilpotent elements in $\mathfrak{g}^s$ by Proposition 1.10, so they lie in the same G^s-orbit, and x and y lie in the same G-orbit.

In particular, $x = s + n$ is conjugate to $x = s + cn$ for any $c \in \mathbb{C}^*$. So for any $P \in \mathbb{C}[\mathfrak{g}]^G$,

$$P(x) = P(s + cn) \quad \forall c \in \mathbb{C}^*,$$

and by continuity

$$P(x) = P(s).$$

Because of this, it is equivalent to show that invariant polynomials separate semisimple G-orbits, or equivalently that $\mathbb{C}[\mathfrak{h}]^W$ separates W-orbits.

Let $s, t \in \mathfrak{h}$ such that $Ws \cap Wt = \emptyset$, and let $R \in \mathbb{C}[\mathfrak{h}]$ be a polynomials such that $R(ws) = 1$ and $R(wt) = 0$ for all $w \in W$. Consider the invariant averaged polynomial

$$P = \frac{1}{\# W} \sum w \cdot R \in \mathbb{C}[\mathfrak{h}]^W$$

—$P(s) = 1$ and $P(t) = 0$, so P separates the orbits of s and t. $\square$

Let B^- be the Borel whose Lie algebra is $\mathfrak{b}^-$, let N^- be its unipotent radical, and let $\mathfrak{n}^- = \operatorname{Lie} N^-$.

Proposition 7.4. *The action map*

$$\alpha : N^- \times (e + \mathfrak{g}^f) \longrightarrow e + \mathfrak{b}^-$$

is an isomorphism.

This result is due to Kostant [Kos1], but the proof will use a different approach, see [Gin] Theorem 7.5 and the subsequent Remarks.

Definition 7.5. Let X be an affine variety with an algebraic action of $\mathbb{C}^*$. This action is called *contracting* if there exists a point $x_0 \in X$ such that for all $x \in X$,

$$\lim_{t \to 0} t \cdot x = x_0, \qquad t \in \mathbb{C}^*.$$

Lemma 7.6 ([Gin], (7.7))**.** *Let X_1 and X_2 be smooth irreducible affine varieties with contracting $\mathbb{C}^*$-actions, to $x_1 \in X_1$ and $x_2 \in X_2$ respectively. Suppose $\alpha : X_1 \longrightarrow X_2$ is a $\mathbb{C}^*$-equivariant morphism such that*

$$d_{x_1}\alpha : T_{x_1}X_1 \longrightarrow T_{x_2}X_2$$

is an isomorphism. Then α is an isomorphism.

Proof of Proposition 7.4. From diagram (5.1), it is clear that

$$\mathfrak{b}^- = [\mathfrak{n}^-, e] \oplus \mathfrak{g}^f.$$

In view of (6.1), the differential of α at $(1, e)$ is

$$\begin{aligned} \mathrm{d}_{(1,e)}\alpha : \mathfrak{n}^- \times \mathfrak{g}^f &\longrightarrow \mathfrak{b}^- \\ (n, x) &\longmapsto [n, e] + x, \end{aligned}$$

so the image of $\mathrm{d}_{(1,e)}\alpha$ is $[\mathfrak{n}^-, e] \oplus \mathfrak{g}^f = \mathfrak{b}^-$. Since the two tangent spaces have the same dimension and since $\mathrm{d}_{(1,e)}\alpha$ is surjective, it is an isomorphism.

We will define compatible $\mathbb{C}^*$-actions so that we can apply the previous lemma. The group homomorphism

$$SL_2 \longrightarrow G$$

given by the $\mathfrak{sl}_2$-triple (e, h, f) induces a homomorphism

$$\gamma : \mathbb{C}^* \longrightarrow G$$

such that the Lie algebra of the image of γ is $\mathbb{C}h$.

Define an action of $\mathbb{C}^*$ on $e + \mathfrak{b}^-$ by

$$t \cdot (e + x) = t^2\gamma(t^{-1})(e + x) = e + t^2\gamma(t^{-1}) \cdot x,$$

where the second equality follows because $e \in \mathfrak{g}_2$ under the grading by h-eigenvalues. Then

$$\lim_{t \to 0} t \cdot (e + x) = e,$$

because $\mathfrak{b}^-$ consists of the non-positive h-eigenspaces.

Similarly, define the action of $\mathbb{C}^*$ on $N^- \times (e + \mathfrak{g}^f)$ by

$$t \cdot (g, e + x) = (\gamma(t^{-1}) n \gamma(t), t^2 \gamma(t^{-1})(e + x)) = (\gamma(t^{-1}) n \gamma(t), e + t^2 \gamma(t^{-1}) \cdot x).$$

This action is also contracting, because N^- is generated by the root groups corresponding to negative h-eigenspaces:

$$\lim_{t \to 0} t \cdot (n, e + x) = (1, e).$$

The action map α is $\mathbb{C}^*$-equivariant with respect to these actions, so from Lemma 7.6 it follows that α is an isomorphism. $\square$

8. The first Kostant theorem

Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and identify it with the universal Cartan $\mathfrak{H}$. The Grothendieck-Springer diagram becomes

$$\begin{tikzcd} & \tilde{\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\nu"] & \\ \mathfrak{g} \arrow[dr, "\rho"'] & & \mathfrak{h} \arrow[dl, "\pi"] \\ & \mathfrak{h}/W & \end{tikzcd} \tag{8.1}$$

and for every $\bar{h} \in \mathfrak{h}/W$ we are interested in the fiber $V_h = \rho^{-1}(\bar{h})$.

When $\bar{h} = 0$, the fiber V_0 is exactly the nilpotent cone $\mathcal{N}$. We have already shown that $\mathcal{N}$ is a G-stable variety of dimension $\dim \mathfrak{g} - \mathrm{rk}\ \mathfrak{g}$ (Corollary 4.4), that it consists of finitely many G-orbits (Proposition 4.5), and that the regular locus $\mathcal{N}^{reg}$ is its unique open dense G-orbit (Corollary 4.7). Moreover, for any $c \in \mathbb{C}^*$, the element $n \in \mathcal{N}$ is G-conjugate to cn (see the proof of Proposition 7.3). Therefore, the point $0 \in \mathcal{N}$ is the unique closed G-orbit of minimal dimension that lies in the closure of every other nilpotent orbit.

We will generalize these results to arbitrary fibers of ρ.

Theorem 8.1. *The morphism ρ is surjective, and for every $\bar{h} \in \mathfrak{h}/W$,*

(1) *The dimension of V_h is $\dim \mathfrak{g} - rk\ \mathfrak{g}$.*
(2) *The fiber V_h is irreducible, G-stable, and consists of finitely many G-orbits.*
(3) *The regular locus V_h^{reg} is the unique open dense G-orbit.*
(4) *The semisimple locus V_h^{ss} is the unique closed G-orbit of minimal dimension.*

Proof. The surjectivity of ρ follows from diagram (8.1). Fix a semisimple $h \in \mathfrak{h}$. Then $\mathfrak{g}^h$ is a reductive Lie algebra of the same rank as $\mathfrak{g}$, and it decomposes as

$$\mathfrak{g}^h = Z(\mathfrak{g}^h) \oplus [\mathfrak{g}^h, \mathfrak{g}^h].$$

The element h is contained in the center $Z(\mathfrak{g}^h)$, and the cone of nilpotent elements $\mathcal{N}^h \subset \mathfrak{g}^h$ is contained in the semisimple derived subalgebra $[\mathfrak{g}^h, \mathfrak{g}^h]$.

Suppose $n \in \mathcal{N}^h$—then $h + n$ is a Jordan decomposition and by the argument in the proof of Proposition 7.3 we have

$$P(h) = P(h+n) \qquad \text{for any } P \in \mathbb{C}[\mathfrak{g}]^G,$$

so $h + n \in V_h$. It follows by G-invariance that $G(h + \mathcal{N}^h) \subset V_h$.

Claim. *There is an isomorphism*

$$\begin{aligned} G \times_{G^h} (h + \mathcal{N}^h) &\longrightarrow V_h \\ (g, x) &\longmapsto g \cdot x. \end{aligned}$$

Proof of claim. Take any element $x \in V_h$ and write its Jordan decomposition $x = s + v$. Then s is conjugate to some $h' \in \mathfrak{h}$, and for any $P \in \mathbb{C}[\mathfrak{g}]^G$,

$$P(h') = P(s) = P(s+v) = P(h).$$

It follows that h and h' are G-conjugate, so x is in fact conjugate to an element of $h + \mathcal{N}^h$. This proves surjectivity.

Now suppose $g \cdot (h + n) = g' \cdot (h + n')$. By uniqueness of the Jordan decomposition, $gh = g'h$ and $gn = g'n'$. Then $g^{-1}g' = a \in G^h$, and

$$(g', h + n') = (ga, h + n') \sim (g, h + n),$$

proving injectivity. $\square$

Now we can prove parts (1)–(4).

(1) If h is regular, $\mathfrak{g}^h = \mathfrak{h}$, and $\mathcal{N}^h = 0$. Then $V_h = G \cdot h$, and

$$\dim V_h = \dim G - \dim G^h = \dim \mathfrak{g} - \text{rk } \mathfrak{g},$$

so the generic fibers have the desired dimension. Since fiber dimension can only jump up, for arbitrary h we have

$$\begin{aligned} \dim \mathfrak{g} - \text{rk } \mathfrak{g} &\le \dim V_h \\ &= \dim G \times_{G^h} (h + \mathcal{N}^h) \\ &= \dim G - (\dim G^h - \dim \mathcal{N}^h) \\ &= \dim \mathfrak{g} - \text{rk } \mathfrak{g}^h \\ &= \dim \mathfrak{g} - \text{rk } \mathfrak{g}, \end{aligned}$$

where the first equality follows from the Claim, and second-to-last quality follows from Corollary 4.4. Equality must hold throughout, proving (1).

(2) The isomorphism proved in the Claim shows that V_h is irreducible. The G-orbits on $G \times_{G^h} (h + \mathcal{N}^h)$ are in bijection with the G^h-orbits on $h + \mathcal{N}^h$. The latter set is finite by Proposition 4.5.

(3) By (2), V_h contains a unique open dense G-orbit, which must be of dimension

$$\dim \mathfrak{g} - \text{rk } \mathfrak{g}.$$

But this is exactly the orbit of a regular element.

(4) Since h is the unique semisimple element in $h+\mathcal{N}^h$, its orbit $G\cdot h$ is the semisimple locus V_h^{ss} of V_h. Every G^h-orbit in $\mathcal{N}^h$ contains 0 in its closure, so every G-orbit on $G\times_{G^h}(h+\mathcal{N}^h)=V_h$ contains $G\cdot h$ in its closure. It follows that $G\cdot h$ is the unique closed orbit of minimal dimension in V_h. □

Corollary 8.2. *The fiber V_h is a single G-orbit if and only if h is a regular semisimple element.*

Example 8.3. Concretely, let $\mathfrak{g}=\mathfrak{sl}_4$ and let

$$h=\begin{bmatrix}2 & & & \\ & 2 & & \\ & & -2 & \\ & & & -2\end{bmatrix}.$$

Theorem 8.1 says that the fiber V_h contains an open dense orbit of elements conjugate to

$$\begin{bmatrix}2 & 1 & & \\ & 2 & & \\ & & -2 & 1\\ & & & -2\end{bmatrix},$$

a unique closed semisimple orbit of elements conjugate to

$$\begin{bmatrix}2 & & & \\ & 2 & & \\ & & -2 & \\ & & & -2\end{bmatrix},$$

and two intermediate orbits of elements conjugate to

$$\begin{bmatrix}2 & 1 & & \\ & 2 & & \\ & & -2 & \\ & & & -2\end{bmatrix} \quad\text{and}\quad \begin{bmatrix}2 & & & \\ & 2 & & \\ & & -2 & 1\\ & & & -2\end{bmatrix}.$$

9. The second Kostant theorem

View the point $\bar{h}\in\mathfrak{h}/W=\operatorname{Spec}\mathbb{C}[\mathfrak{g}]^G$ as a maximal ideal in $\mathfrak{m}_h\subset\mathbb{C}[\mathfrak{g}]^G$.

Theorem 9.1. (1) *The ring of regular functions on V_h is*

$$\mathbb{C}[V_h]\cong\mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}]\mathfrak{m}_h.$$

(2) *The variety V_h is normal.*

Proof. Part (1) follows from Lemma 9.2 below, after noting that V_h is a complete intersection, and so Cohen-Macaulay, and that the differentials $dP_1,\dots,dP_l$ are linearly independent at every point of the open dense subset

$$G(e+\mathfrak{g})^f = \mathfrak{g}^{reg}.$$

For (2), since V_h is a complete intersection, it is sufficient to check that it has no singularities in codimension 1. In other words, to check that

$$\operatorname{codim} V_h \backslash V_h^{reg} \geq 2.$$

This follows immediately from the standard fact that the G-orbits on $\mathfrak{g}$ are even-dimensional, which we prove in Lemma 9.3. ☐

Lemma 9.2 ([CG], 2.2.11)**.** *Let $I = (f_1, \dots, f_l) \subset \mathbb{C}[x_1, \dots, x_n]$ be an ideal such that the quotient*

$$\mathbb{C}[x_1, \dots, x_l]/I$$

is Cohen-Macaulay and such that the differentials $df_1, \dots, df_l$ are linearly independent at every points of an open dense subset $X^\circ \subset X$. Then $I = \sqrt{I}$.

Lemma 9.3. *Any G-orbit in $\mathfrak{g}$ has even dimension.*

Proof. Let $\xi \in \mathfrak{g}$—we define a bilinear form on

$$T_\xi(G\cdot\xi) = [\mathfrak{g}, \xi] \cong \mathfrak{g}/\mathfrak{g}^\xi$$

by the formula

$$\omega_\xi(x,y) = (\xi, [x,y]).$$

where $(\cdot,\cdot)$ is the Killing form.

Then

$$\omega_\xi(y,x) = (\xi,[y,x]) = -(\xi,[x,y]) = -\omega_\xi(x,y),$$

so ω_ξ is skew-symmetric. Moreover,

$$\begin{aligned}
\ker\, \omega_\xi &= \{x \in \mathfrak{g} \mid \omega_\xi(x,y) = 0 \forall y \in \mathfrak{g}\} \\
&= \{x \in \mathfrak{g} \mid (\xi,[x,y]) = 0 \forall y \in \mathfrak{g}\} \\
&= \{x \in \mathfrak{g} \mid ([\xi,x],y) = 0 \forall y \in \mathfrak{g}\} \\
&= \{x \in \mathfrak{g} \mid [\xi,x] = 0\} \\
&= \mathfrak{g}^\xi,
\end{aligned}$$

so ω_ξ is nondegenerate on the tangent space $T_\xi(G\cdot\xi)$. ☐

Remark 9.4. The form defined in Lemma 9.3 is usually defined on the dual $\mathfrak{g}^*$. This form is in fact symplectic, called the Kirillov-Kostant-Souriau form, and it gives every coadjoint orbit of G in $\mathfrak{g}^*$ the structure of a symplectic variety.

10. The third Kostant theorem

Theorem 10.1. (1) *The polynomial algebra $\mathbb{C}[\mathfrak{g}]$ is a free $\mathbb{C}[\mathfrak{g}]^G$-module.*

(2) *Let $\mathcal{H} \subset \mathbb{C}[\mathfrak{g}]$ denote the space of G-harmonic polynomials. The multiplication map*

$$\mathbb{C}[\mathfrak{g}]^G \otimes \mathcal{H} \longrightarrow \mathbb{C}[\mathfrak{g}]$$

is an isomorphism of graded $\mathbb{C}[\mathfrak{g}]^G$-modules.

(3) *The ring $\mathcal{O}(V_h)$ is a sum of finite-dimensional simple G-modules, and any such G-module W appears with multiplicity*

$$[\mathcal{O}(V_h) : W] = \dim W^T$$

—the dimension of the set of fixed points of the action of a maximal torus on W.

The proof will follow [CG], which takes the approach of [BL]. It will make essential use of the following theorem, cf. [Ste], Theorem 2.2 and Remark 2.3:

Theorem 10.2 (Pittie-Steinberg)**.** *The algebra $\mathbb{C}[\mathfrak{h}]$ is a free graded $\mathbb{C}[\mathfrak{h}]^W$-module. There is a natural embedding*

$$\begin{aligned} \mathbb{C}[W] &\hookrightarrow \mathbb{C}[\mathfrak{h}] \\ w &\longmapsto w^{-1}\lambda_w \end{aligned}$$

where $\lambda_w = \prod \alpha$ is the product of all positive roots $\alpha \in \mathfrak{h}^*$ such that $w\alpha$ is a negative root. The multiplication map

$$\mathbb{C}[\mathfrak{h}]^W \otimes \mathbb{C}[W] \longrightarrow \mathbb{C}[\mathfrak{h}]$$

is an isomorphism.

For any vector space V, the coordinate ring of V has a natural grading by degree:

$$\mathbb{C}[V] = \oplus_i \mathbb{C}^i[V].$$

Proposition 10.3. *Suppose that $A \subset \mathbb{C}[V]$ is a graded subalgebra such that the restriction*

$$res : A \longrightarrow \mathbb{C}[E]$$

is injective, and such that $\mathbb{C}[E]$ is a free graded A-module. Then the multiplication map

$$\mathbb{C}[V/E] \otimes A \longrightarrow \mathbb{C}[V]$$

is injective, and $\mathbb{C}[V]$ is a free graded $\mathbb{C}[V/E] \otimes A$-module.

Proof of Theorem 10.1 (1). Apply Proposition 10.3 with $V = \mathfrak{g}$, $E = \mathfrak{h}$, and $A = \mathbb{C}[\mathfrak{g}]^G$. By the Chevalley isomorphism, the restriction

$$\mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]$$

factors through the isomorphism $\mathbb{C}[\mathfrak{g}]^G \cong \mathbb{C}[\mathfrak{h}]^W$, so it is injective. By the Pittie-Steinberg theorem, $\mathbb{C}[\mathfrak{h}]$ is a free module over its image $\mathbb{C}[\mathfrak{h}]^W$.

Then Proposition 10.3 implies that

$$\mathbb{C}[\mathfrak{g}/\mathfrak{h}] \otimes \mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{g}]$$

is an injection, and that $\mathbb{C}[\mathfrak{g}]$ is free over its image. $\square$

Now let V be a finite-dimensional vector space and let G be a reductive group acting on V. Let $\mathcal{D}$ denote the algebra of differential operators with constant coefficients—this is a commutative algebra with a natural G-action. Let $\mathcal{D}_+^G$ be the ideal of the invariant subalgebra $\mathcal{D}^G$ consisting of invariant differential operators with zero constant term.

Definition 10.4. A polynomial $P \in \mathbb{C}[V]$ is *G-harmonic* if $DP = 0$ for every $D \in \mathcal{D}_+^G$.

Let $\mathcal{H}$ be the space of G-invariant polynomials on V. The goal is to prove the following proposition, from which part (2) of Theorem 10.1 follows in view of part (1).

Proposition 10.5. *Suppose that $\mathbb{C}[V]$ is a free graded $\mathbb{C}[V]^G$-module. Then the multiplication map*

$$\mathbb{C}[V]^G \otimes \mathcal{H} \longrightarrow \mathbb{C}[V]$$

is an isomorphism of graded $\mathbb{C}[V]^G$-modules.

Consider the symmetric algebras

$$\mathrm{Sym}\ V = \bigoplus \mathrm{Sym}\ ^i V \qquad \text{and} \qquad \mathrm{Sym}\ V^* = \bigoplus \mathrm{Sym}\ ^i V^*$$

with the usual gradings. There are canonical graded isomorphisms

$$\mathrm{Sym}\ V = \mathcal{D} \qquad \text{and} \qquad \mathrm{Sym}\ V^* = \mathbb{C}[V]$$

and there is a natural pairing

$$\begin{aligned} \langle \cdot, \cdot \rangle : \mathcal{D} \times \mathbb{C}[V] &\longrightarrow \mathbb{C} \\ \langle D, P \rangle &\longmapsto (DP)(0). \end{aligned}$$

If D and P are homogeneous of the same degree, then DP is a constant, and so this pairing is a perfect pairing on $\mathcal{D}^i \times \mathbb{C}^i[V]$.

For any graded G-algebra A, let I^A be the ideal generated by A_+^G in A.

Lemma 10.6. *A polynomial P is G-harmonic if and only if $P \in (I^{\mathcal{D}})^{\perp}$.*

Proof. The forward direction is clear. Suppose conversely that $P \in (I^{\mathcal{D}})^{\perp}$, and fix $D \in \mathcal{D}_+^G$. Then

$$(D'DP)(0) = \langle D'D, P \rangle = 0 \qquad \forall D' \in \mathcal{D},$$

so every derivative of DP vanishes at 0—therefore $DP = 0$. $\square$

Let $K \subset G$ be a maximal compact subgroup. Then every connected component of G intersects K, and $\mathfrak{g}$ is the complexification of Lie K, so K is Zariski-dense in G. It follows that for every G-module A,

$$A^G = A^K.$$

Fix a K-invariant positive-definite Hermitian form on V—this gives a K-equivariant skew-linear isomorphism of vector spaces

$$V \longrightarrow V^*,$$

which extends to a K-equivariant skew-linear isomorphism

$$\varphi : \text{Sym } V \longrightarrow \text{Sym } V^*.$$

Moreover, φ takes G-invariants to G-invariants:

$$\varphi((\text{Sym } V)^G) = \varphi((\text{Sym } V)^K) = (\text{Sym } V^*)^K = (\text{Sym } V^*)^G.$$

Recall that if $E = \oplus E_i$ is any graded vector space, the *Poincaré polynomial* of E is

$$P(E) = \sum (\dim E_i) t^i.$$

Lemma 10.7.

$$P(\mathcal{H}) = P(\mathbb{C}[V]/I^{\mathbb{C}[V]}).$$

Proof. This is immediate from basic properties of the Poincaré polynomial:

$$\begin{aligned} P(\mathcal{H}) &= P((I^{\mathcal{D}})^{\perp}) \\ &= P(\mathcal{D}/I^{\mathcal{D}}) \\ &= P(\mathbb{C}[V]/I^{\mathbb{C}[V]}), \end{aligned}$$

where the last equality follows from the isomorphism φ. ☐

Lemma 10.8. *There is a G-stable graded direct sum decomposition*

$$\mathbb{C}[V] = \mathcal{H} \oplus I^{\mathbb{C}[V]}.$$

Proof. Under the isomorphism φ, the pairing $\langle \cdot, \cdot \rangle$ becomes the K-invariant positive-definite Hermitian form

$$\begin{aligned} (\cdot, \cdot) : \text{Sym } V \times \text{Sym } V &\longrightarrow \mathbb{C} \\ (v_1, v_2) &\longmapsto \langle v_1, \varphi(v_2) \rangle. \end{aligned}$$

By positive-definiteness,

$$(I^{\text{Sym } V})^{\perp} \cap I^{\text{Sym } V} = 0,$$

where the orthogonal complement is with respect to the form $(\cdot, \cdot)$. Pulling this through φ,

$$(I^{\mathcal{D}})^{\perp} \cap I^{\text{Sym } V^*} = 0,$$

where the orthogonal complement is now with respect to the pairing $\langle \cdot, \cdot \rangle$. But the left-hand side is exactly $\mathcal{H} \cap I^{\mathbb{C}[V]}$.

Then the graded composition

$$\mathcal{H} \hookrightarrow \mathbb{C}[V] \longrightarrow \mathbb{C}[V]/I^{\mathbb{C}[V]}$$

is injective. Since it is injective on any graded component, and each graded component is finite-dimensional, it is enough to check that the graded components have the same dimension. But

$$\dim \mathcal{H}_i = \dim \left(\mathbb{C}[V]/I^{\mathbb{C}[V]} \right)_i$$

by the equality of Poincaré polynomials from Lemma 10.7. ☐

We are now ready to prove Proposition 10.5.

Proof of Proposition 10.5. First we will show that the multiplication map is surjective, by showing inductively that every graded component $\mathbb{C}^k[V]$ is contained in the image. When $k = 0$ this is clear, so assume now that it is true for degrees less than k and let $P \in \mathbb{C}^k[V]$.

By the direct sum decomposition in Lemma 10.8,

$$P = P_0 + \sum_i Q_i Z_i,$$

where $P_0 \in \mathcal{H}$, $Q_i \in \mathbb{C}[V]$, and $Z_i \in \mathbb{C}[V]^G_+$. Because the decomposition of Lemma 10.8 is graded, every summand must have degree k. In particular, because deg $Z_i > 0$, each Q_i has degree strictly less than k.

Applying the induction hypothesis,

$$Q_i = \sum_j P_{ij} Y_{ij}$$

where $P_{ij} \in \mathcal{H}$ and $Y_{ij} \in \mathbb{C}[V]^G$. Then

$$P = P_0 + \sum_{i,j} P_{ij} Y_{ij} Z_i$$

is in the image of the multiplication map.

Because the multiplication is surjective on every graded component, it is again enough to check that the graded components have the same dimension—in other words, that $\mathbb{C}[V]^G \otimes \mathcal{H}$ and $\mathbb{C}[V]$ have the same Poincaré polynomials.

Since $\mathbb{C}[V]$ is free over $\mathbb{C}[V]^G$ by assumption, let $E \subset \mathbb{C}[V]$ be a graded subspace such that

$$\mathbb{C}[V] = \mathbb{C}[V]^G \otimes E,$$

Then

$$\begin{aligned} \mathbb{C}[V] &= \mathbb{C}[V]^G \otimes E \\ &= (\mathbb{C}[V]^G_+ \oplus \mathbb{C}) \otimes E \\ &= (\mathbb{C}[V]^G_+ \otimes E) \oplus E \\ &= I^{\mathbb{C}[V]} \oplus E, \end{aligned}$$

and on Poincaré polynomials

$$\begin{aligned} P(\mathbb{C}[V]^G \otimes \mathcal{H}) &= P(\mathbb{C}[V]^G) P(\mathcal{H}) \\ &= P(\mathbb{C}[V]^G) P(\mathbb{C}[V]/I^{\mathbb{C}[V]}) \\ &= P(\mathbb{C}[V]^G) P(E) \\ &= P(\mathbb{C}[V]). \end{aligned}$$

□

Proof of Theorem 10.1 (3). Let $\bar{h} = (a_1, \ldots, a_l) \in \mathbb{H}/W$, consider the maximal ideal

$$\mathfrak{m}_h = (P_1 - a_1, \ldots, P_l - a_l) \subset \mathbb{C}[\mathfrak{g}]^G,$$

and let $(\mathfrak{m}_h)$ be the ideal generated by $\mathfrak{m}_h$ inside $\mathbb{C}[\mathfrak{g}]$.

Since G is semisimple there is a direct sum decomposition of G-modules

$$\mathbb{C}[\mathfrak{g}]^G = \mathfrak{m}_h \oplus \mathbb{C},$$

and therefore, using part (2) of this theorem,

$$\begin{aligned}\mathbb{C}[\mathfrak{g}] &= \mathbb{C}[\mathfrak{g}] \otimes \mathcal{H} \\ &= (\mathfrak{m}_h \otimes \mathcal{H}) \oplus \mathcal{H} \\ &= (\mathfrak{m}_h) \oplus \mathcal{H}.\end{aligned}$$

This implies that the composition

$$\mathcal{H} \longrightarrow \mathbb{C}[\mathfrak{g}] \longrightarrow \mathbb{C}[\mathfrak{g}]/(\mathfrak{m}_h) \cong \mathbb{C}[V_h]$$

is an isomorphism of G-modules, so the G-module structure of

$$\mathbb{C}[V_h] \cong \mathcal{H}$$

does not depend on h.

Without loss of generality assume then that h is regular semisimple, so that $V_h \cong G/T$ is a single closed G-orbit. Then

$$\mathbb{C}[V_h] \cong \mathbb{C}[G]^T$$

is the algebra of regular functions on G invariant under right-translation by T.

Recall the Peter-Weyl theorem:

$$\mathbb{C}[G] = \bigoplus V \otimes V^*,$$

where the sum is taken over all irreducible representations of G. For any fixed representation W,

$$\begin{aligned}[\mathbb{C}[G]^T : W] &= [\bigoplus V \otimes (V^*)^T : W] \\ &= \sum [V : W] \dim (V^*)^T \\ &= \dim (W^*)^T\end{aligned}$$

and the statement of Theorem 10.1 (3) follows. □

It remains to prove only Proposition 6.2. We will use the Pittie-Steinberg Thereom (10.2), and the formalism of Poincaré polynomials.

Remark 10.9. If $\mathbb{C}[x]$ is a graded polynomial ring in one variable, and $\deg x = d$, then

$$P(\mathbb{C}[x]) = 1 + t^d + t^{2d} + \ldots = \frac{1}{1-t^d}.$$

If $\mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}[x_1] \otimes \ldots \otimes \mathbb{C}[x_n]$ is graded with $\deg x_i = d_i$, then

$$P(\mathbb{C}[x_1, \ldots, x_n]) = \prod_{i=1}^{n} \frac{1}{1-t^{d_i}}.$$

Proof of Proposition 6.2. As before, $P_1, \ldots, P_l$ are the homogeneous polynomials generators of $\mathbb{C}[\mathfrak{h}]^W$, and $\deg P_i = m_i$. We can then compute

$$P(\mathbb{C}[\mathfrak{h}]) = \frac{1}{(1-t)^l}$$
$$P(\mathbb{C}[\mathfrak{h}]^W) = \prod_{i=1}^{n} \frac{1}{1-t^{m_i}}$$

Recall from Theorem 10.2 the embedding

$$\begin{aligned} \mathbb{C}[W] &\hookrightarrow \mathbb{C}[\mathfrak{h}] \\ w &\longmapsto w^{-1}\lambda_w, \end{aligned}$$

where $\lambda_w = \prod \alpha$ is the product over all positive roots α such that $w\alpha$ is a negative root. Since the degree of $\alpha \in \mathfrak{h}^*$ is 1, the degree of λ_w is

$$\deg \lambda_w = \#\{\alpha > 0 \mid w\alpha < 0\} = l(w),$$

which is called the *length* of w. Then

$$P(\mathbb{C}[W]) = \sum_{w \in W} t^{l(w)}.$$

Again by Theorem 10.2, there is an equality of Poincaré polynomials

$$P(\mathbb{C}[\mathfrak{h}]^W) P(\mathbb{C}[W]) = P(\mathbb{C}[\mathfrak{h}]),$$

which implies that

$$\sum_{w \in W} t^{l(w)} = \prod_{i=1}^{n} \frac{1-t^{m_i}}{1-t}.$$

The degree of the left-hand side is the length of the longest word in the Weyl group, which is the number of positive roots:

$$\frac{1}{2}(\dim \mathfrak{g} - \mathrm{rk}\ \mathfrak{g}).$$

The degree of the right-hand side is

$$\sum_{i=1}^{l} (m_i - 1) = \left(\sum_{i=1}^{l} m_i \right) - \mathrm{rk}\ \mathfrak{g}.$$

Since these two are equal, we obtain

$$\sum_{i=1}^{l} m_i = \frac{1}{2}(\dim \mathfrak{g} + \mathrm{rk}\ \mathfrak{g}). \qquad \square$$

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