PART II: THE WONDERFUL COMPACTIFICATION

Contents

1. Introduction

Let K be any algebraic group, let

$$\tau : K \longrightarrow K$$

be an involution of K, and let $H = K^\tau$ be its fixed point set. The homogeneous space

$$K/H$$

is called a *symmetric space.*

Any algebraic group G is naturally a symmetric space under the action of $K = G \times G$ by left- and right-multiplication, by the involution

$$\begin{aligned} \tau : G \times G &\longrightarrow G \times G \\ (g, h) &\longmapsto (h^{-1}, g^{-1}). \end{aligned}$$

The fixed point set is

$$H = G_\Delta = \{(g, g^{-1}) \in G \times G\}$$

and there is an isomorphism

$$G \cong (G \times G)/G_\Delta.$$

In the 1980s, DeConcini and Procesi [DP] showed that any semisimple symmetric space $\mathring{X}$ has a *wonderful compactification* X—a variety satisfying the following properties:

(1) X is smooth and complete
(2) $\mathring{X} \subset X$ is an open dense subset, and the boundary

$$X \backslash \mathring{X} = X_1 \cup \ldots \cup X_l$$

is a union of smooth prime divisors with normal crossings.
(3) The closures of the G-orbits on X are the partial intersections

$$\bigcap_{i \in I} X_i, \qquad \text{for } I \subset \{1, \ldots, l\}.$$

In a more general framework, studying equivariant compactifications of homogeneous spaces, Luna and Vust [LV] showed that any homogeneous space $\mathring{X}$ that has a wonderful compactification is necessarily *spherical*—a Borel subgroup acts on $\mathring{X}$ with an open dense orbit. For example, any reductive algebraic group G is a spherical homogeneous space under the two-sided action of $G \times G$, and the open orbit of the Borel subgroup $B \times B \subset G \times G$ is the open dense Bruhat cell.

There are two distinguished classes of equivariant compactifications of spherical homogeneous spaces. The first is the class of *toroidal* compactifications—these are generalizations of toric varieties, and their boundary structure is described combinatorially by fans. Every compactification X of $\mathring{X}$ is dominated by a toroidal compactification X', in the sense that there is a proper birational G-equivariant morphism

$$X' \longrightarrow X$$

that restricts to the identity along the open locus $\mathring{X}$.

The second class is the class of *simple* compactifications, which are compactifications on which G acts with a unique closed orbit. Brion and Pauer gave in [BP] a necessary and sufficient criterion for a spherical variety $\mathring{X}$ to have simple compactifications. When such compactifications exist, there is a unique one that is also toroidal. This compactification X has the universal property that for any toroidal compactification X', and any simple compactification X'', there are unique morphisms

$$X' \longrightarrow X \longrightarrow X''$$

that restrict to the identity along $\mathring{X}$. If X is smooth, it is the *wonderful compactification* of $\mathring{X}$ and it has the properties described by DeConcini and Procesi.

We will construct the wonderful compactification of a semisimple algebraic group of adjoint type G, following mostly the well-known survey of Evens and Jones [EJ]. Then we will describe two other realizations of the wonderful compactification, one as a variety of Lagrangian subalgebras of $\mathfrak{g} \times \mathfrak{g}$, and one as a GIT quotient of the Vinberg monoid.

2. Construction of the compactification

From now on, let G be a semisimple connected complex algebraic group of adjoint type—that is, with trivial center. Let $\widetilde{G}$ be its simply-connected cover, and choose a maximal torus and a Borel subgroup

$$\widetilde{T} \subset \widetilde{B} \subset \widetilde{G}$$

corresponding to

$$T \subset B \subset G.$$

Let $U \subset \widetilde{B}$ be the unipotent radical. Because the morphism $\widetilde{G} \longrightarrow G$ is a central quotient, it is an isomorphism on unipotent subgroups, and we can identify U with its image in B.

Let $\mathcal{X}^*(\widetilde{T})$ be the character lattice of the torus $\widetilde{T}$, Φ the set of nonzero roots, Φ^+ the set of positive roots relative to $\widetilde{B}$, and

$$\Delta = \{\alpha_1, \dots, \alpha_l\}$$

the set of simple roots, where $l = \dim \widetilde{T}$ is the rank of G. Let $W = N_{\widetilde{G}}(\widetilde{T})/\widetilde{T}$ be the corresponding Weyl group.

There is a standard ordering on $\mathcal{X}^*(\widetilde{T})$ given by

$$\lambda \geq \mu \qquad \Leftrightarrow \qquad \lambda - \mu = \sum_{i=1}^{l} n_i \alpha_i, \quad n_i \in \mathbb{Z}_{\geq 0}.$$

Definition 2.1. A weight $\lambda \in \mathcal{X}^*(\widetilde{T})$ is *dominant* if $\langle \lambda, \check{\alpha} \rangle \geq 0$ for every positive coroot $\check{\alpha} \in \check{\Phi}^+$. It is *regular* if $\langle \lambda, \check{\alpha} \rangle > 0$ for every positive coroot $\check{\alpha} \in \check{\Phi}^+$.

The dominant weights form a cone—the dominant Weyl chamber—and the regular dominant weights are exactly the ones that fall in the interior of this cone. This is dual to the notion of a regular semisimple element in the Lie algebra of G. The following lemma, whose proof is left as an exercise, will be useful.

Lemma 2.2. *Let λ be a dominant weight and let V an irreducible representation of $\widetilde{G}$ of highest weight λ. Let v_λ be a highest weight vector of V. Then the following are equivalent:*

(1) *λ is regular.*
(2) *The stabilizer of the highest weight space $\mathbb{C}v_\lambda$ in $\widetilde{G}$ is $\widetilde{B}$.*
(3) *The stabilizer of λ in the Weyl group W is trivial.*

From now on let V be an irreducible $\widetilde{G}$-representation of regular highest weight λ. In the diagram

$$\begin{array}{ccc} \widetilde{G} & \longrightarrow & \mathrm{End}\, V \backslash \{0\} \\ \downarrow & & \downarrow \\ G & \overset{\psi}{\dashrightarrow} & \mathbb{P}(\mathrm{End}\, V), \end{array} \tag{2.1}$$

the top arrow is the representation map, the left arrow is a quotient by the center, and the right arrow is a quotient by scalars. All these maps are $\widetilde{G} \times \widetilde{G}$-equivariant, and the representation map descends to the $G \times G$-equivariant morphism

$$\psi : G \longrightarrow \mathbb{P}(\text{End } V).$$

The map ψ is an injection—this is guaranteed by adjointness if G is simple, and also by the regularity of λ if it is not.

Definition 2.3. The *wonderful compactification* of G is $X = \overline{\psi(G)} \subset \mathbb{P}(\text{End } V)$.

Example 2.4. Let $G = PGL_2$ with $\widetilde{G} = SL_2$. Then all nonzero weights are regular, and we can take $V = \mathbb{C}^2$ to be the standard representation. In this case

$$\psi : G \hookrightarrow \mathbb{P}(M_{2\times 2})$$

is the embedding with image

$$\psi(G) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad - bc \neq 0 \right\},$$

and the closure of this image is

$$X = \mathbb{P}(M_{2\times 2}) \cong \mathbb{P}^3.$$

The boundary of X is

$$\partial X = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid ad - bc = 0 \right\} \cong \mathbb{P}^1 \times \mathbb{P}^1,$$

and it is a single smooth prime divisor.

Remark 2.5. Example 2.4 does not generalize. For $n \geq 3$, the standard representation of SL_n is not regular, because it is a fundamental representation and it generates one of the edges of the dominant Weyl chamber. In general, the wonderful compactification of PGL_n is not simply the projective space $\mathbb{P}^{n^2-1}$.

3. The big cell

Choose a basis of weight vectors of descending weight $v_0, \ldots, v_n$ for V, such that v_i is in the weight space V_{λ_i} of weight λ_i, and with the properties

- $v_0 \in V_\lambda$
- $i = 1, \ldots, l \quad \Rightarrow \quad v_i \in V_{\lambda - \alpha_i}$
- $\lambda_i > \lambda_j \quad \Rightarrow \quad i < j$

Let $\widetilde{B}^-$ be the opposite Borel to $\widetilde{B}$, let B^- be its image in G, and let U^- be their common unipotent radical. Then

$$U^- \cdot v_i \subset v_i + \sum_{j>i} V_{\lambda_j},$$

and so U^- stabilizes the affine space

$$\mathbb{P}_0(V) = \left\{ \left[\sum a_i v_i \right] \mid a_0 \neq 0 \right\} \cong \mathbb{C}^l.$$

Let $v_0^*, \dots, v_n^*$ be a dual basis for the dual space V^*, so that each v_i^* has weight $-\lambda_i$. Then U stabilizes the affine space

$$\mathbb{P}_0(V^*) = \left\{ \left[\sum a_i v_i^* \right] \mid a_0 \neq 0 \right\} \cong \mathbb{C}^l.$$

The following lemma is clear from Lemma 2.2, and from the fact that the unipotent groups U and U^- act on the affine spaces $\mathbb{P}_0(V^*)$ and $\mathbb{P}_0(V)$ with closed orbits.

Lemma 3.1. *The action maps*

$$U \longrightarrow U \cdot [v_0^*] \subset \mathbb{P}_0(V^*)$$

and

$$U^- \longrightarrow U^- \cdot [v_0] \subset \mathbb{P}_0(V)$$

are isomorphisms, and their images are closed.

We use the usual $G \times G$-equivariant identification

$$\begin{aligned} V \otimes V^* &\longrightarrow \operatorname{End} V \\ (v \otimes f) &\longmapsto (w \mapsto f(w)v). \end{aligned}$$

Then the set $\{v_i \otimes v_j^*\}$ is a basis for End V. The affine space

$$\mathbb{P}_0 = \left\{ \left[\sum a_{ij} v_i \otimes v_j^* \right] \mid a_{00} \neq 0 \right\} \subset \mathbb{P}(\operatorname{End} V)$$

is $U^-T \times U$-stable, by the observations before Lemma 3.1. Define

$$X_0 = X \cap \mathbb{P}_0.$$

This intersection is called the *big cell* of the wonderful compactification.

Proposition 3.2. *The intersection of the big cell with the open dense locus $\psi(G)$ is the image of the open Bruhat cell of G:*

$$X_0 \cap \psi(G) = \psi(U^-TU).$$

Proof. One containment is clear: $\psi(e) \in X_0$, X_0 is $U^-T \times U$-stable, and ψ is $G \times G$-equivariant, so it follows that

$$\psi(U^-TU) \subseteq X_0.$$

For the other, choose a representative $\dot{w} \in N_{\widetilde{G}}(\widetilde{T})$ for each $w \in W$. Then by the Bruhat decomposition,

$$G = \coprod_{w \in W} U^-T\dot{w}U.$$

If $w \neq 1$, then $\dot{w}v_0$ is a weight vector of weight $w\lambda$, and $w\lambda \neq \lambda$ by Lemma 2.2. It follows that

$$\begin{aligned} \psi(\dot{w}) &= \dot{w}\psi(e) \\ &= \dot{w} \left[\sum v_i \otimes v_i^* \right] \\ &= \left[\sum (\dot{w}v_i) \otimes v_i^* \right] \notin \mathbb{P}_0, \end{aligned}$$

and therefore

$$\psi(U^- T\dot{w}U) \cap X_0 = \emptyset.$$

So the only Bruhat cell whose image intersects X_0 is the open cell $U^- TU$. □

Remark 3.3. Since $U^- TU$ is dense in G, its image $\psi(U^- TU)$ is dense in X_0, and

$$X_0 = \overline{\psi(U^- TU)} \subset \mathbb{P}_0.$$

Proposition 3.4. *Let Z be the closure of $\psi(T)$ in $\mathbb{P}_0$. Then*

$$Z \cong \mathbb{C}^l.$$

Proof. Let $t \in T$ and choose a preimage $\tilde{t} \in \widetilde{T}$. Then

$$\begin{aligned} \psi(t) &= t\left[\sum v_i \otimes v_i^*\right] \\ &= \left[\sum (\tilde{t} v_i) \otimes v_i^*\right] \\ &= \left[\sum \lambda_i(\tilde{t}) v_i \otimes v_i^*\right] \\ &= \left[v_0 \otimes v_0^* + \sum \frac{\lambda_i(\tilde{t})}{\lambda(\tilde{t})} v_i \otimes v_i^*\right]. \end{aligned}$$

Since $\lambda_i \leq \lambda$,

$$\lambda - \lambda_i = \sum n_{ij}\alpha_j, \quad n_{ij} \in \mathbb{Z}_{\geq 0}.$$

Then the image of t becomes

$$\begin{aligned} \psi(t) &= \left[v_0 \otimes v_0^* + \sum \frac{1}{\prod \alpha_j(t)^{n_{ij}}} v_i \otimes v_i^*\right] \\ &= \left[v_0 \otimes v_0^* + \sum_{i=1}^{l} \frac{1}{\alpha_i(t)} v_i \otimes v_i^* + \sum_{i>l} \frac{1}{\prod \alpha_j(t)^{n_{ij}}} v_i \otimes v_i^*\right] \end{aligned}$$

Define a map

$$\begin{aligned} F : \mathbb{C}^l &\longrightarrow \overline{\psi(T)} \\ (z_1, \ldots, z_l) &\longmapsto \left[v_0 \otimes v_0^* + \sum_{i=1}^{l} z_i v_i \otimes v_i^* + \sum_{i>l} \left(\Pi z_j^{n_{ij}}\right) v_i \otimes v_i^*\right] \end{aligned}$$

It is clear that F is an isomorphism. □

Consider the action map

$$\begin{aligned} \chi : U^- \times U \times Z &\longrightarrow X_0 \\ (u, v, z) &\longmapsto uzv^{-1}. \end{aligned}$$

Theorem 3.5. *The morphism χ is an isomorphism, and therefore $X_0 \cong \mathbb{C}^{\dim G}$ is smooth.*

Lemma 3.6. *There is a $U^- \times U$-equivariant morphism $\beta : X_0 \longrightarrow U^- \times U$ such that*

$$\beta(\chi(u, v, z)) = (u, v).$$

Proof. The morphism

$$\begin{aligned} \beta_1 : \mathbb{P}_0 &\longrightarrow \mathbb{P}_0(V) \\ [A] &\longmapsto [Av_o] \end{aligned}$$

is well-defined. Moreover, for any $(u, v, t) \in U^- \times U \times T$,

$$\beta_1(\psi(u, v, t)) = utv[v_0] = u[v_0],$$

so the image $\beta_1(\psi(U^-TU))$ is the closed set $U^-[v_0] \cong U^-$ by Lemma 3.1. Extending to the closure X_0, β_1 gives a surjection

$$\beta_1 : X_0 \longrightarrow U^-.$$

Dually, define

$$\begin{aligned} \beta_2 : \mathbb{P}_0 &\longrightarrow \mathbb{P}_0(V^*) \\ [A] &\longmapsto [v_0^* \circ A^{-1}] \end{aligned}$$

Once again this induces

$$\beta_2 : X_0 \longrightarrow U.$$

Define

$$\begin{aligned} \beta : X_0 &\longrightarrow U^- \times U \\ x &\longmapsto (\beta_1(x), \beta_2(x)). \end{aligned}$$

□

Lemma 3.7. *Let A be an algebraic group acting on a variety Y. Suppose that there is an A-equivariant morphism*

$$\beta : Y \longrightarrow A,$$

where A is viewed as a left A-module. Then $Y \cong A \times f^{-1}(e)$.

Proof. Consider the maps

$$\begin{aligned} f : A \times \beta^{-1}(e) &\longrightarrow Y \\ (a, y) &\longmapsto a \cdot y \end{aligned}$$

and

$$\begin{aligned} g : Y &\longrightarrow A \times \beta^{-1}(e) \\ y &\longmapsto (\beta(y), \beta(y)^{-1}y). \end{aligned}$$

They are inverses of one another. □

Proof of Theorem 3.5. In view of the morphism β from Lemma 3.6, Lemma 3.7 implies that

$$X_0 \cong U^- \times U \times \beta^{-1}(e, e).$$

It is clear from the construction of β that $\psi(T) \subseteq \beta^{-1}(e, e)$, and since the fiber $\beta^{-1}(e, e)$ is closed, $Z \subseteq \beta^{-1}(e, e)$.

But X_0 is irreducible of dimension $\dim G$, so the fiber $\beta^{-1}(e,e)$ is irreducible of dimension $\dim T$, and the inclusion $Z \subseteq \beta^{-1}(e,e)$ is actually an equality. ☐

4. Smoothness of the compactification

We will show that X is smooth by showing that it is a union of copies of the big cell X_0. To this end, we will need the following lemmas.

Lemma 4.1. *Let A be a semisimple group acting on an irreducible representation V with highest weight vector v_0. Then $A \cdot [v_0]$ is the unique closed orbit of the action of A on $\mathbb{P}(V)$.*

Proof. An orbit $A \cdot [v]$ is closed if and only if it is projective, which is the case if and only if the stabilizer of $[v]$ is parabolic. Up to conjugation we may assume this parabolic is a standard parabolic, and then $[v]$ is stabilized by the Borel consisting of the positive roots, so it is a highest weight vector. Since V is irreducible it has a unique highest weight space, so $[v] = [v_0]$. ☐

Lemma 4.2. *Suppose A is an algebraic group acting on an irreducible variety Y with a unique closed orbit Z. If $U \subset Y$ is an open subset that intersects Z, then*

$$Y = \bigcup_{a \in A} aU.$$

Proof. The set

$$AU = \bigcup_{a \in A} aU$$

is open, so its complement

$$W = Y \backslash AU$$

is closed and A-stable. Then W contains a closed A-orbit, which by uniqueness must be the closed orbit Z. But then $Z \subset W$, so $Z \cap U = \emptyset$—a contradiction. ☐

Proposition 4.3. *Suppose that $W \subset X$ is a closed $G \times G$-stable subvariety. Then*

$$W = \bigcup_{a \in G \times G} a(W \cap X_0).$$

Proof. The tensor product $V \otimes V^*$ is an irreducible representation of $G \times G$, so by Lemma 4.1 the action of $G \times G$ on $\mathbb{P}(V \otimes V^*)$ has the unique closed orbit

$$(G \times G)[v_0 \otimes v_0^*].$$

If W is closed and $G \times G$-stable, it contains a closed orbit, so by uniqueness

$$(G \times G)[v_0 \otimes v_0^*] \subset W.$$

But since $W \cap X_0$ is open in W, and since $[v_0 \otimes v_0^*] \in X_0$, it follows by Lemma 4.2 that

$$W = \bigcup_{a \in G \times G} a(W \cap X_0).$$ ☐

The following theorem is immediate:

Theorem 4.4. *For any $G \times G$-orbit $\mathcal{O}$, the closure $\overline{\mathcal{O}}$ has the property*

$$\overline{\mathcal{O}} = \bigcup_{a \in G \times G} a(\overline{\mathcal{O}} \cap X_0).$$

In particular, $X = \bigcup_{a \in G \times G} aX_0$, and so X is smooth.

5. The $G \times G$-orbits on the compactification

First we describe the T-orbits on the closure $Z \cong \mathbb{C}^l$ of the torus T from Proposition 3.4. For each $I \subset \{1, \dots, l\}$, define

$$Z_I = \{(z_1, \dots, z_l) \mid z_i = 0 \text{ if } i \in I\} \cong \mathbb{C}^{l-|I|}$$

and

$$Z_I^\circ = \{(z_1, \dots, z_l) \in Z_I \mid z_i \neq 0 \text{ if } i \notin I\}.$$

Then it is clear that the Z_I° are exactly the T-orbits on Z, and each such orbit has a distinguished basepoint

$$z_I = (z_1, \dots, z_l), \qquad z_i = 1 \text{ if } i \notin I.$$

Each Z_I is the closure of Z_I° in Z, and the boundary

$$Z \backslash \psi(T) = \bigcup_{i=1}^{l} Z_i$$

is the union of the coordinate hyperplanes in $\mathbb{C}^l$—in particular, it is a divisor with normal crossings.

Now we describe the $U^- T \times U$-orbits on X_0. Using the isomorphism

$$\chi : U^- \times U \times Z \longrightarrow X_0$$

of Theorem 3.5, define

$$\Sigma_I = \chi(U^- \times U \times Z_I) \cong \mathbb{C}^{\dim G - |I|}$$

and

$$\Sigma_I^\circ = \chi(U^- \times U \times Z_I^\circ).$$

Then each Σ_I° is a $U^- T \times U$-orbit on X_0, Σ_I is its closure, and the boundary

$$X_0 \backslash \psi(U^- T U) = \bigcup_{i=1}^{l} \Sigma_i$$

is a divisor with normal crossings.

The closure of every $G \times G$-orbit on X is a union of translations of its intersection with X_0—so, there are at most 2^l such orbit closures.

Lemma 5.1. *Suppose that W is a projective variety and $U \subset W$ is an open affine subset. Then the boundary $W \backslash U$ is a union of irreducible components of codimension 1.*

Proposition 5.2. *Let S_i be the closure of Σ_i in X. Then*

$$X\backslash\psi(G) = \bigcup_{i=1}^{l} S_i.$$

Proof. Since X is projective and $\psi(G)$ is an affine open subset,

$$X\backslash\psi(G) = \bigcup S_\alpha$$

is a union of irreducible components of codimension 1 by Lemma 5.1.

This union is $G\times G$-stable, and because $G\times G$ is connected every component S_α is $G\times G$-stable. By Proposition 4.3,

$$S_\alpha = \bigcup_{a\in G\times G} a(S_\alpha \cap X_0).$$

Then the intersection

$$S_\alpha \cap X_0$$

is a $U^-T\times U$-stable irreducible hypersurface in X_0, so it is equal to Σ_i for some i. It follows that $S_\alpha = \overline{\Sigma_i} = S_i$. □

Define the intersection

$$S_I = \bigcap_{i\in I} S_i.$$

Then $S_J \subseteq S_I$ if and only if $J\supseteq I$, and $S_I\cap X_0 = \Sigma_I$, so by Proposition 4.3

$$S_I = \bigcup_{a\in G\times G} a\Sigma_I.$$

In particular, S_I is smooth.

Define

$$S_I^\circ = S_I\backslash \bigcup_{I\subsetneq J} S_J.$$

Then

$$S_I^\circ = \bigcup_{a\in G\times G} a\Sigma_I^\circ = (G\times G)\Sigma_I^\circ = (G\times G)(U^-T\times U)z_I = (G\times G)z_I,$$

and S_I° is a single $G\times G$-orbit. We collect all these results into a single theorem:

Theorem 5.3. *There are exactly 2^l $G\times G$-orbits in X, given by*

$$S_I^\circ = (G\times G)z_I, \qquad I\subseteq\{1,\ldots,l\}.$$

Their closures S_I are smooth, and the boundary

$$X\backslash\psi(G)$$

of X is a divisor with normal crossings.

6. The structure of the orbits and their closures

Let $\mathfrak{g} = \text{Lie } G$, and consider the root space decomposition

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Phi} \mathfrak{g}_\alpha.$$

Fix a subset $I \subseteq \{1, \ldots, l\}$. Let

$$\Delta_I = \{\alpha_i \mid i \notin I\},$$

and let Φ_I be the set of roots spanned by Δ_I. This produces the standard Levi subalgebra

$$\mathfrak{l}_I = \mathfrak{h} + \sum_{\alpha \in \Phi_I} \mathfrak{g}_\alpha,$$

and the parabolic subalgebras $\mathfrak{p}_I^\pm = \mathfrak{l}_I + \mathfrak{b}^\pm$ with nilpotent radicals $\mathfrak{u}_I^\pm$. Denote the corresponding subgroups of G by L_I, $P_I^\pm$, $U_I^\pm$.

Let V_I be the irreducible representation of L_I generated by applying L_I to the highest weight vector v_0—that is,

$$V_I = \mathcal{U}\mathfrak{l}_I \cdot v_0$$

where $\mathcal{U}\mathfrak{l}_I$ is the universal enveloping algebra of $\mathfrak{l}_I$. For any $x \in \mathfrak{u}_I$, $x \cdot v_0 = 0$. Because $\mathfrak{u}_I$ is normal in $\mathfrak{p}_I$, it follows that V_I is $\mathfrak{p}_I$-stable.

Lemma 6.1. *The stabilizer of V_I in G is exactly the parabolic subgroup P_I.*

Proof. Let Q be the stabilizer of V_I in G. It is already clear that $P_I \subset Q$, so Q is a standard parabolic subgroup and in particular it is connected. It is enough to show that

$$\mathfrak{p}_I = \text{Lie } Q.$$

Let $t \in T$ be such that $\alpha_i(t) = 1$ whenever $\alpha_i \in \Delta_I$, and $\alpha_i(t) \neq 1$ otherwise. Then $t \in Z(L_I)$, and we pick a preimage $\tilde{t} \in \widetilde{T}$. Since V_I is an irreducible representation of L_I, $\tilde{t}$ acts on it by the scalar $\lambda(\tilde{t})$.

Let $\alpha_i \notin \Delta_I$ be a root whose corresponding root space is not contained in $\mathfrak{p}_I$, and let $x \in \mathfrak{g}_{-\alpha_i}$ be nonzero. Then $\tilde{t}$ acts on xV_I by

$$\frac{\lambda(\tilde{t})}{\alpha_i(\tilde{t})},$$

and this scalar is distinct from $\lambda(t)$ because $\alpha_i(t) \neq 1$. It follows that

$$V_I \cap xV_I = 0.$$

Because λ is regular, $xV_I \neq 0$, so $x \notin \text{Lie } Q$. So the only root spaces contained in Lie Q are the ones also contained in $\mathfrak{p}_I$. □

Now let

$$J = \left\{ j \in \{0, \ldots, n\} \mid \lambda - \lambda_j = \sum_{\alpha_i \in \Phi_I} n_i \alpha_i,\, n_i \in \mathbb{Z}_{\geq 0} \right\}.$$

The set $\{v_j \mid j \in J\}$, which consists of weight vectors whose weights can be obtained from λ by subtracting the simple roots in Δ_I, is a basis for V_I.

Proposition 6.2. *Let* $\mathrm{pr}_{V_I} \in \mathrm{End}\, V$ *denote the projection onto* V_I*. Then*

$$z_I = [\mathrm{pr}_{V_I}].$$

Proof. Recall that $z_I = (z_1, \ldots, z_l) \in \mathbb{C}^l$ is the point whose coordinates are

$$z_i = \begin{cases} 0, \; i \in I \\ 1, \; i \notin I \end{cases}$$

From the isomorphism of Proposition 3.4, it is identified with the following point in X_0:

$$z_I = \left[v_0 \otimes v_0^* + \sum_{i \notin I} z_i v_i \otimes v_i^* + \sum_{i > l} \left(\prod z_j^{n_{ij}} \right) v_i \otimes v_i^* \right] = \left[\sum_{j \in J} v_j \otimes v_j^* \right]. \qquad \square$$

Proposition 6.3. *The stabilizer of* z_I *in* $G \times G$ *is*

$$\left\{ (ux, vy) \in U_I L_I \times U_I^- L_I \mid xy^{-1} \in Z(L_I) \right\}.$$

Proof. Suppose $(r, s) \in G \times G$ stabilizes $z_I = [\mathrm{pr}_{V_I}]$. Then

$$[r \mathrm{pr}_{V_I} s^{-1}] = [\mathrm{pr}_{V_I}],$$

so in particular r stabilizes the image V_I of pr_{V_I}. By Lemma 6.1, this means $r \in P_I$. Moreover, if $r = ux \in U_I L_I$, then

$$[r \mathrm{pr}_{V_I}] = [x \mathrm{pr}_{V_I}]$$

since the action of U_I on V_I is trivial.

Dually, $s \in P_I^-$, by applying the preceding discussion to V_I^* under the isomorphism

$$\mathrm{End}\, V \cong V \otimes V^* \xrightarrow{\sim} V^* \otimes V \cong \mathrm{End}\, V^*.$$

Moreover, if $s = vy \in U_I^- L_I$, then

$$[\mathrm{pr}_{V_I} s^{-1}] = [\mathrm{pr}_{V_I} y^{-1}].$$

Then

$$[\mathrm{pr}_{V_I}] = [r \mathrm{pr}_{V_I} s^{-1}] = [x \mathrm{pr}_{V_I} y^{-1}],$$

so xy^{-1} acts trivially on $\mathbb{P}(V_I)$, and so acts by a scalar on the irreducible representation V_I of L_I. It follows that

$$xy^{-1} \in Z(L_I). \qquad \square$$

Remark 6.4. Because the stabilizer of z_I is contained in $P_I \times P_I^-$, there is a surjection

$$S_I^\circ = (G \times G)/\mathrm{Stab}_{G \times G}(z_I) \longrightarrow G/P_I \times G/P_I^-.$$

The fiber of this surjection is

$$P_I \times P_I^- / \mathrm{Stab}_{G \times G}(z_I) \cong L_I \times L_I / \{(x, y) \mid xy^{-1} \in Z(L_I)\} \cong L_I / Z(L_I),$$

which is a semisimple group of adjoint type and smaller rank than G.

In particular, this gives an isomorphism

$$S_{\{1,\dots,l\}} \cong G/B \times G/B^-$$

between the unique closed $G \times G$-orbit on X and the product of two copies of the flag variety of G.

The natural embedding

$$\text{End } V_I \hookrightarrow \text{End } V$$

induces a closed embedding of projective varieties

$$\mathbb{P}(\text{End } V_I) \hookrightarrow \mathbb{P}(\text{End } V),$$

and $z_K \in \mathbb{P}(\text{End } V_I)$ if and only if $I \subseteq K$.

Define the map

$$\begin{aligned} L_I &\longrightarrow \mathbb{P}(\text{End } V_I) \subset \mathbb{P}(\text{End } V) \\ g &\longmapsto \left[g \sum_{j \in J} v_j \otimes v_j^* \right] = [g z_I]. \end{aligned}$$

This descends to an injection

$$G_I = L_I / Z(L_I) \xrightarrow{\psi_I} \mathbb{P}(\text{End } V_I).$$

Since V_I is a regular representation of L_I, it is a regular representation of the simply-connected cover of the semisimple adjoint group G_I, and we can apply the entire previous discussion to the compactification

$$X_I = \overline{\psi_I(G_I)}$$

—the wonderful compactification of G_I.

The quotient

$$P_I \times P_I^- \longrightarrow P_I / U_I Z(L_I) \times P_I^- / U_I^- Z(L_I) \cong G_I \times G_I$$

induces an action of $P_I \times P_I^-$ on X_I.

Theorem 6.5. *The map*

$$\varphi : G \times G \times_{P_I \times P_I^-} X_I \longrightarrow S_I$$

is an isomorphism of $G \times G$-varieties. In particular, S_I fibers over the partial flag variety $G/P_I \times G/P_I^-$ with fiber X_I.

Proof. It is enough to show that φ is bijective, because the target S_I is smooth.

Recall that

$$S_I = \bigcup_{I \subseteq K} S_K^\circ$$

and $I \subseteq K$ if and only if $z_K \in X_I$. In this case

$$\varphi(G \times G \times \{z_K\}) = S_K^\circ.$$

So every $G \times G$-orbit is contained in the image of φ, and φ is surjective.

Similarly, it is enough to show that φ is injective on orbits. Suppose that

$$\varphi(g, h, z_K) = \varphi(e, e, z_K).$$

Then $(g, h) \in \mathrm{Stab}_{G\times G}(z_K) \subseteq P_K \times P_K^- \subseteq P_I \times P_I^-$. It follows that

$$(g, h, z_K) \sim (e, e, z_K)$$

in the fiber product $G \times G \times_{P_I \times P_I^-} X_I$. $\square$

7. Independence of regular dominant weight

Suppose that λ and μ are two regular dominant weights of $\widetilde{G}$, with corresponding irreducible representations V and W. They produce two compactifications:

$$X^1 \subset \mathbb{P}(\mathrm{End}\ V) \qquad \text{and} \qquad X^2 \subset \mathbb{P}(\mathrm{End}\ W).$$

Let $v_0, \dots v_n$ be the usual basis of V chosen in (3), and let $w_0, \dots w_n$ be the analogous basis of W. Choose identity basepoints

$$x_1 = \left[\sum v_i \otimes v_i^*\right] \in X^1 \qquad \text{and} \qquad x_2 = \left[\sum w_i \otimes w_i^*\right] \in X^2$$

and define

$$X^\Delta = (G \times G)(x_1, x_2) \in X^1 \times X^2.$$

There are natural projections

$$\begin{array}{ccccc} & & X^\Delta & & \\ & \swarrow^{p_1} & & \searrow^{p_2} & \\ X^1 & & & & X^2. \end{array}$$

Theorem 7.1. *The projections p_1 and p_2 are both isomorphisms, and they induce an isomorphism*

$$p_2 \circ p_1^{-1} : X^1 \xrightarrow{\sim} X^2.$$

We will apply superscripts to the notation of the previous sections, so that X_0^i will be the big cell of X^i, X^i the closure of the torus in the big cell, etc. Define

$$Z^\Delta = \overline{T(x_1, x_2)} \subset X_0^1 \times X_0^2.$$

Lemma 7.2. *There is an isomorphism $Z^\Delta \cong \mathbb{C}^l$ and the projections $p_i : Z^\Delta \longrightarrow Z^i$ are isomorphisms.*

Proof. The proof is exactly as in Proposition 3.4:

$$t(x_1, x_2) = \left(\left[v_0 \otimes v_0^* + \sum_{i=1}^{l} \frac{1}{\alpha_i(t)} v_i \otimes v_i^* + \sum_{i>l} \clubsuit v_i \otimes v_i^* \right], \right.$$
$$\left. \left[w_0 \otimes w_0^* + \sum_{i=1}^{l} \frac{1}{\alpha_i(t)} w_i \otimes w_i^* + \sum_{i>l} \spadesuit w_i \otimes w_i^* \right] \right),$$

where ♣ and ♠ are polynomials in

$$\frac{1}{\alpha_1(t)}, \ldots, \frac{1}{\alpha_l(t)}.$$

As before, there is an isomorphism $\mathbb{C}^l \longrightarrow Z^\Delta$ given by

$$(z_1, \ldots, z_l) \longmapsto \left(\left[v_0 \otimes v_0^* + \sum_{i=1}^{l} z_i v_i \otimes v_i^* + \sum_{i>l} \clubsuit v_i \otimes v_i^* \right], \right.$$
$$\left. \left[w_0 \otimes w_0^* + \sum_{i=1}^{l} z_i w_i \otimes w_i^* + \sum_{i>l} \spadesuit w_i \otimes w_i^* \right] \right). \qquad \square$$

Let $X_0^\Delta = p_i^{-1}(X_0^i)$ and define the action map

$$\chi^\Delta : U^- \times U \times Z^\Delta \longrightarrow X^\Delta.$$

Lemma 7.3. *The morphism χ^Δ is an isomorphism onto X_0^Δ.*

Proof. Consider the commutative diagram

$$\begin{array}{ccc} U^- \times U \times Z^\Delta & \xrightarrow{\chi^\Delta} & X^\Delta \\ \downarrow{\scriptstyle \mathrm{Id}\times\mathrm{Id}\times p_i} & & \downarrow{\scriptstyle p_i} \\ U^- \times U \times Z^i & \xrightarrow{\chi^i} & X^i. \end{array}$$

It is clear that χ^Δ is injective because the composition $\chi^i \circ (\mathrm{Id} \times \mathrm{Id} \times p_i)$ is injective.

Let Y be the image of χ^Δ, and consider the composition

$$\sigma = \chi^\Delta \circ (\mathrm{Id} \times \mathrm{Id} \times p_i)^{-1} \circ \chi^{i-1} : X_0^i \longrightarrow Y.$$

The diagram is commutative, so

$$p_i \circ \sigma = \mathrm{Id}_{X_i^0}$$

and σ is a section of p_i on X_0^i. The composition $\sigma \circ p_i$ is defined only on X_0^Δ, because σ is defined on X_0^i. Because σ is a section, it follows that the restriction of

$$\sigma \circ p_i : X_0^\Delta \longrightarrow X^\Delta$$

to the image Y of χ^Δ is also the identity.

But X_0^Δ and Y are irreducible of the same dimension as X^Δ, so $X_0^\Delta \cap Y$ is a dense subset of X_0^Δ. Then the map $\sigma \circ p_i$ is the identity on a dense subset of X_0^Δ, so it is the identity on all of X_0^Δ.

This shows that p_i gives an isomorphism $X_0^\Delta \longrightarrow X_0^i$, so

$$\chi^\Delta : U^- \times U \times Z^\Delta \longrightarrow X_0^\Delta$$

is surjective. $\square$

Lemma 7.4. *The restriction of p_i to the set*

$$U = \bigcup_{a \in G \times G} a X_0^\Delta$$

is injective.

Proof. For any subset $I \subseteq \{1, \dots, l\}$, let

$$z_I^\Delta = (z_I^1, z_I^2) \in X^1 \times X^2.$$

By Lemma 7.3, as in Section 5, X_0^Δ is $U^-T \times U$-stable and the $U^-T \times U$-orbits on X_0^Δ are exactly indexed by the basepoints z_I^Δ. It is enough to check the statement of the lemma on the intersections of $G \times G$-orbits with X_0^Δ.

Using the $G \times G$-equivariance of p_i, it is enough to suppose that

$$p_i((g,h)z_I^\Delta) = p_i(z_I^\Delta).$$

Then

$$(g,h)z_I^i = z_I^i,$$

so that $(g,h) \in \mathrm{Stab}_{G\times G}(z_I^i)$. But

$$\mathrm{Stab}_{G\times G}(z_I^\Delta) = \mathrm{Stab}_{G\times G}(z_I^1) \cap \mathrm{Stab}_{G\times G}(z_I^2) = \mathrm{Stab}_{G\times G}(z_I^i),$$

so this means that $(g,h) \in \mathrm{Stab}_{G\times G}(z_I^\Delta)$ and $(g,h)z_I^\Delta = z_I^\Delta$. □

Proof of Theorem 7.1. The projection p_i restricts to an isomorphism

$$p_i : X_0^\Delta \xrightarrow{\sim} X_0^i,$$

and by $G \times G$-equivariance it gives a surjection

$$p_i : U \longrightarrow \bigcup_{a \in G\times G} aX_0^i = X^i$$

which is injective by Lemma 7.4. Because X^i is smooth, p_i is an isomorphism.

Then $U \subseteq X^\Delta$ is a projective subvariety of the same dimension as X^Δ, so they are equal. It follows that p_i is an isomorphism between X^Δ and X^i. □

8. Compactifications in more general spaces

The results in this section are outlined in [EJ] Section 3.1. Any representation E of $\widetilde{G} \times \widetilde{G}$ induces an action

$$G \times G \curvearrowright \mathbb{P}(E).$$

A point $[x] \in \mathbb{P}(E)$ whose stabilizer is the diagonal subgroup

$$G_\Delta = \{(g,g) \mid g \in G\},$$

gives an embedding

$$\begin{aligned} \psi_E : G &\hookrightarrow \mathbb{P}(E) \\ g &\longmapsto (g,e) \cdot [x] \end{aligned}$$

and a compactification

$$X(E,[x]) = \overline{\psi_E(G)} \subset \mathbb{P}(E).$$

In the previous section we showed that if V and W are regular irreducible representations of $\widetilde{G}$, then

$$X(\text{End } V, [\text{Id}_V]) \cong X(\text{End } W, [\text{Id}_W]).$$

Suppose now that V is an irreducible $\widetilde{G}$-representation of regular highest weight λ and that $W^1, \ldots, W^r$ are irreducible $\widetilde{G}$-representations of highest weight $\mu^1, \ldots, \mu^r$. Write

$$W = W^1 \oplus \ldots \oplus W^r$$

and let F be any $\widetilde{G}$-representation. As in the previous sections, write

$$X = X(\text{End } V, [\text{Id}_V]).$$

Theorem 8.1. *Suppose that $\mu^k \leq \lambda$ for every $k = 1, \ldots, r$. Then*

$$X(\text{End } V \oplus \text{End } W \oplus F, [\text{Id}_V \oplus \text{Id}_W \oplus 0]) \cong X.$$

Remark 8.2. The G-orbit

$$G \cdot [\text{Id}_V \oplus \text{Id}_W \oplus 0]$$

lies in the image of the closed embedding

$$\mathbb{P}(\text{End } V \oplus \text{End } W) \hookrightarrow \mathbb{P}(\text{End } V \oplus \text{End } W \oplus F),$$

so there is an identification

$$X(\text{End } V \oplus \text{End } W \oplus F, [\text{Id}_V \oplus \text{Id}_W \oplus 0]) = X(\text{End } V \oplus \text{End } W, [\text{Id}_V \oplus \text{Id}_W]).$$

Denote by ψ the injection

$$\psi : G \longrightarrow \mathbb{P}(\text{End } V \oplus \text{End } W)$$
$$g \longmapsto g \cdot [\text{Id}_V \oplus \text{Id}_W]$$

and by X' the closure of its image inside $\mathbb{P}(\text{End } V \oplus \text{End } W)$. To prove the theorem it will be sufficient to show that

$$X' \cong X.$$

As before, let $v_0, \ldots, v_n$ be a basis of T-weight vectors for V satisfying the conditions (3). Let $w_0, \ldots, w_m$ be a basis of T-weight vectors for W such that w_i has weight μ_i. This gives a basis

$$\{v_i \otimes v_j^*, w_i \otimes w_j^*\}$$

for the space End $V \oplus$ End W. Let

$$\mathbb{P}'_0 = \left\{ \left[\sum a_{ij} v_i \otimes vj^* + \sum b_{ij} w_i \otimes w_j^* \right] \mid a_{00} \neq 0 \right\},$$

and let Z' be the closure of the torus inside this affine space:

$$Z' = \overline{\psi(T)} \subset \mathbb{P}'_0.$$

Proposition 8.3. *There is an isomorphism $Z' \cong \mathbb{C}^l$.*

Proof. Let $t \in T$ and $\tilde{t} \in \widetilde{T}$ be some preimage of t in the simply-connected cover $\widetilde{G}$. Then

$$\begin{aligned}\psi(t) &= t\left[\sum v_i \otimes v_i^* + \sum w_i \otimes w_i^*\right]\\ &= \left[\sum \lambda_i(\tilde{t}) v_i \otimes v_i^* + \sum \mu_i(\tilde{t}) w_i \otimes w_i^*\right]\\ &= \left[v_0 \otimes v_0^* + \sum_{i=1}^{l} \frac{1}{\alpha_i(t)} v_i \otimes v_i^* + \sum_{i>l} \frac{\lambda_i(\tilde{t})}{\lambda(\tilde{t})} v_i \otimes v_i^* + \sum \frac{\mu_i(\tilde{t})}{\lambda(\tilde{t})} w_i \otimes w_i^*\right]\end{aligned}$$

The coefficients

$$\frac{\lambda_i(\tilde{t})}{\lambda(\tilde{t})} \quad \text{and} \quad \frac{\mu_i(\tilde{t})}{\lambda(\tilde{t})}$$

are polynomial in the terms

$$\frac{1}{\alpha_1(t)}, \ldots, \frac{1}{\alpha_l(t)}$$

because each μ_i is less than some μ^k in the partial ordering of the weight lattice, and each $\mu^k \le \lambda$ by the assumption of Theorem 8.1. We can define an isomorphism $\mathbb{C}^l \longrightarrow Z'$ just as in the proof of Proposition 3.4:

$$(z_1, \ldots, z_l) \longmapsto \left[v_0 \otimes v_0^* + \sum_{i=1}^{l} z_i v_i \otimes v_i^* + \sum_{i>l} \clubsuit v_i \otimes v_i^* + \sum \spadesuit w_i \otimes w_i^*\right],$$

where $\clubsuit$ and $\spadesuit$ are polynomials in $z_1, \ldots, z_l$. □

Define

$$\widetilde{\mathbb{P}} = \{[A \oplus B] \in \mathbb{P}(\operatorname{End} V \oplus \operatorname{End} W) \mid A \neq 0\}.$$

Then there is a natural projection

$$\pi : \widetilde{\mathbb{P}} \longrightarrow \mathbb{P}(\operatorname{End} V),$$

and Proposition 8.3, together with Proposition 3.4, imply that the restriction

$$\pi_{|Z'} : Z' \longrightarrow Z$$

is an isomorphism.

Fix $I \subset \{1, \ldots, l\}$, and under the identification of Proposition 8.3 define

$$z_I' = (z_1, \ldots, z_l) \in Z', \qquad z_i = \begin{cases} 1, & \text{if } i \notin I \\ 0, & \text{if } i \in I. \end{cases}$$

(Cf. the definitions at the start of Section 5.) Then

$$\pi(z_I') = z_I,$$

and each T-orbit on Z' contains exactly one basepoint of the form z_I'.

As in Section 6, let $\Delta_I = \{\alpha_i \mid i \notin I\}$ and let $\mathfrak{l}_I$ be the corresponding Levi subalgebra of $\mathfrak{g} = \operatorname{Lie} G$. Define

$$V_I = \mathcal{U}\mathfrak{l}_I \cdot v_0$$

to be the subspace of V generated by applying $\mathfrak{l}_I$ to the highest weight vector v_0, and recall that the unipotent radical U_I of the corresponding positive parabolic acts on V_I trivially.

For each index k such that $\lambda - \mu^k$ is in the span of the simple roots Δ_I, let $w_0^k, \ldots, w_{n_k}^k$ be a basis of T-weight vectors for W^k, such that w_i^k has weight μ_i^k and which satisfies the conditions of (3). Define

$$W_I^k = \mathcal{U}\mathfrak{l}_I \cdot w_0^k$$

to be the subspace of W^k generated by applying $\mathfrak{l}_I$ to the highest weight vector w_0^k.

In Section 6, the set of indices

$$J = \left\{ j \in \{0, \ldots, n\} \mid \lambda - \lambda_j = \sum_{\alpha_i \in \Phi_I} n_i \alpha_i,\, n_i \in \mathbb{Z}_{\geq 0} \right\}$$

indexed a basis of weight vectors $\{v_j \mid j \in J\}$ for V_I. Similarly, for each W^k as above, define

$$J^k = \left\{ j \in \{0, \ldots, n\} \mid \lambda - \mu_j^k = \sum_{\alpha_i \in \Phi_I} n_i \alpha_i,\, n_i \in \mathbb{Z}_{\geq 0} \right\}.$$

Because

$$\lambda - \mu_j^k = (\lambda - \mu^k) + (\mu^k - \mu_j^k),$$

and each term is a linear combinations of roots with non-negative coefficients, an index j is in J^k if and only if both $\lambda - \mu^k$ and $\mu^k - \mu_j^k$ are in the span of Δ_I. Then the set of weight vectors $\{w_j^k \mid j \in J^k\}$ is a basis for W_I^k, and it is guaranteed to be nonempty because $0 \in J^k$.

Lemma 8.4. *Let* $\mathrm{pr}_{W_I^k} \in \mathrm{End}\, W$ *denote the projection onto* W_I^k*. Then*

$$z_I' = \left[\mathrm{pr}_{V_I} \oplus \left(\sum \mathrm{pr}_{W_I^k}\right)\right],$$

where the sum is taken over all k *such that* $\lambda - \mu^k$ *is in the span of* Δ_I*.*

Proof. From Proposition 8.3,

$$z_I' = \left[v_0 \otimes v_0^* + \sum_{i=1}^{n} \delta_i v_i \otimes v_i^* + \sum_{k=1}^{r} \left(\sum_{i=0}^{n_k} \delta_i^k w_i^k \otimes w_i^{k*} \right) \right].$$

Here $\delta_i = 1$ if $\lambda - \lambda_i$ is in the span of Δ_I, and 0 otherwise. Likewise, $\delta_i^k = 1$ if $\lambda - \mu_i^k$ is in the span of Δ_I, and 0 otherwise.

It follows immediately that

$$z_I' = \left[v_0 \otimes v_0^* + \sum_{j \in J} v_i \otimes v_i^* + \sum_{k} \left(\sum_{j \in J^k} w_i^k \otimes w_i^{k*} \right) \right],$$

where the only indices k appearing in the second sum are those for which $\lambda - \mu^k$ is in the span of Δ_I. □

Lemma 8.5. *The points* $z_I' \in X'$ *and* $z_I \in X$ *have the same stabilizer in* $G \times G$*.*

Proof. The inclusion

$$\mathrm{Stab}_{G\times G}(z_I') \subseteq \mathrm{Stab}_{G\times G}(z_I).$$

is clear, since $z_I' \in \mathbb{P}(\mathrm{End}\ V \oplus \mathrm{End}\ W)$, $z_I \in \mathbb{P}(\mathrm{End}\ V)$, and the action of $G \times G$ is block-diagonal.

Conversely, recall from Proposition 6.3 that

$$\mathrm{Stab}_{G\times G}(z_I) = \{(ux, vy) \in U_I L_I \times U_I^- L_I \mid xy^{-1} \in Z(L_I)\}.$$

For any point (ux, vy),

$$(ux, vy) \cdot \left[\mathrm{pr}_{V_I} \oplus \left(\sum \mathrm{pr}_{W_I^k}\right)\right] = \left[x\mathrm{pr}_{V_I} y^{-1} \oplus \left(\sum x\mathrm{pr}_{W_I^k} y^{-1}\right)\right],$$

because $u \in U_I$ and $v \in U_I^-$ both act trivially. (Cf. the proof of Proposition 6.3.)

Since V_I is an irreducible representation of L_I, the central element xy^{-1} acts on it by the scalar

$$\lambda(xy^{-1}).$$

Likewise, xy^{-1} acts on each W_I^k by

$$\mu^k(xy^{-1}).$$

Because xy^{-1} is central in L_I, for any α_i in the set Δ_I of simple roots that generate $\mathfrak{l}_I$,

$$\alpha_i(xy^{-1}) = 1.$$

But $\lambda - \mu^k$ is in the span of Δ_I, so it follows that

$$\frac{\lambda(xy^{-1})}{\mu^k(xy^{-1})} = 1.$$

Retracing our steps,

$$\begin{aligned}(ux, vy) \cdot \left[\mathrm{pr}_{V_I} \oplus \left(\sum \mathrm{pr}_{W_I^k}\right)\right] &= \left[x\mathrm{pr}_{V_I} y^{-1} \oplus \left(\sum x\mathrm{pr}_{W_I^k} y^{-1}\right)\right] \\ &= \left[\lambda(xy^{-1})\mathrm{pr}_{V_I} \oplus \left(\sum \mu^k(xy^{-1})\mathrm{pr}_{W_I^k}\right)\right] \\ &= \left[\mathrm{pr}_{V_I} \oplus \left(\sum \mathrm{pr}_{W_I^k}\right)\right],\end{aligned}$$

so (ux, vy) stabilizes z_I'. □

Consider the open subset $X_0' = X' \cap \mathbb{P}_0'$ of X'. It is $U^-T \times U$-stable, and we define the action map

$$\chi' : U^- \times U \times Z' \longrightarrow X_0'.$$

Proposition 8.6. *The morphism χ' is an isomorphism, and therefore $X_0' \cong \mathbb{C}^{\dim G}$.*

Proof. Applying the construction of Lemma 3.6, there is a $U^- \times U$-equivariant morphism $\beta' : X_0' \longrightarrow U^- \times U$ such that

$$\beta'(\chi'(u, v, z')) = (u, v).$$

Then by Lemma 3.7, there is an isomorphism

$$X_0' \cong U^- \times U \times \beta'^{-1}(e, e),$$

and $Z' \subseteq \beta'^{-1}(e, e)$.

But X'_0 is irreducible of dimension $n = \dim G$, so the fiber $\beta'^{-1}(e,e)$ is irreducible of dimension $\dim T$, and the inclusion $Z' \subseteq \beta'^{-1}(e,e)$ is an equality. □

Now define

$$Y = \bigcup_{a \in G\times G} aX'_0$$

to be the union of all $G \times G$-translates of the open affine cell X'_0. The open subvariety Y of X' is contained in $\widetilde{\mathbb{P}}$.

Proposition 8.7. *The restriction of*

$$\pi : \widetilde{\mathbb{P}} \longrightarrow \mathbb{P}(\text{End } V)$$

to Y is injective.

Proof. It is sufficient to check that π is injective on $G \times G$-orbits. Every T-orbit on Z' contains a basepoint of the form z'_I. Then every $U^-T \times U$-orbit on X'_0 contains some point z'_I, and therefore every $G \times G$-orbit on Y contains such a point.

Suppose without loss of generality that

$$\pi(gz'_I h^{-1}) = \pi(z'_I).$$

Because π is $G \times G$-equivariant,

$$gz_I h^{-1} = z_I,$$

so $(g,h) \in \text{Stab}_{G\times G}(z_I)$. By Lemma 8.5, this is the same as the stabilizer of z'_I, so

$$gz'_I h^{-1} = z'_I.$$ □

Proof of Theorem 8.1. The restriction

$$\pi_{|Z'} : Z' \longrightarrow Z$$

is an isomorphism by Proposition 8.3. By Proposition 8.6 and 3.5, it follows that

$$\pi_{|X'_0} : X'_0 \longrightarrow X_0$$

is also an isomorphism. So the restriction

$$\pi_{|Y} : Y \longrightarrow \bigcup_{a \in G\times G} aX_0 = X$$

is surjective, and by Proposition 8.7 it is also injective. Since the wonderful compactification X is smooth, $\pi_{|Y}$ is an isomorphism of algebraic varieties.

This means that $Y \subseteq X'$ is a projective (and therefore complete, and therefore closed) algebraic subvariety of X' of the same dimension, so they are equal. Then π gives an isomorphism

$$X' \cong X.$$ □

Remark 8.8. The compactification X' is contained in the closed subvariety

$$\mathbb{P}\left(\text{End } V \oplus \left(\oplus \text{ End } W^k\right) \oplus F\right) \subset \mathbb{P}\left(\text{End } V \oplus \text{End } W \oplus F\right).$$

Therefore we could replace X' in the discussion above by

$$X\left(\mathrm{End}\ V \oplus \left(\oplus\ \mathrm{End}\ W^k\right) \oplus F, \left[\mathrm{Id}_V \oplus \left(\sum c_k \mathrm{Id}_{W^k}\right) \oplus 0\right]\right),$$

for some scalars $c_k \in \mathbb{C}$.

9. The Lie algebra realization of the compactification

This section outlines another realization of the wonderful compactification, using the results of Section 8, and following the construction in [EJ] Section 3.2. Let n be the dimension of G, and consider the action of $G \times G$ on the Grassmannian

$$\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}).$$

The stabilizer in $G \times G$ of the diagonal subalgebra

$$\mathfrak{g}_\Delta = \{(x, x) \mid x \in \mathfrak{g}\} \subset \mathfrak{g} \oplus \mathfrak{g}$$

is the diagonal subgroup

$$G_\Delta = \{(g, g) \mid g \in G\} \subset G \times G.$$

The orbit of this diagonal subalgebra in the Grassmannian is

$$(G \times G) \cdot \mathfrak{g}_\Delta \cong (G \times G)/G_\Delta \cong G,$$

and we consider its closure

$$\overline{G} = \overline{(G \times G) \cdot \mathfrak{g}_\Delta} \subset \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}).$$

Theorem 9.1. *The compactification $\overline{G}$ is isomorphic to the wonderful compactification X.*

Consider the Plücker embedding

$$\mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g}) \hookrightarrow \mathbb{P}\left(\wedge^n(\mathfrak{g} \oplus \mathfrak{g})\right),$$

which takes a subspace spanned by a basis $u_1, \ldots, u_n$ to the line $[u_1 \wedge \ldots \wedge u_n]$. Let $[\mathfrak{g}_\Delta]$ be the image of $\mathfrak{g}_\Delta$. Because this is a closed embedding,

$$\overline{G} = \overline{(G \times G) \cdot \mathfrak{g}_\Delta} \cong \overline{(G \times G) \cdot [\mathfrak{g}_\Delta]} \subset \mathbb{P}\left(\wedge^n(\mathfrak{g} \oplus \mathfrak{g})\right).$$

For a nonzero vector $v_\Delta \in [\mathfrak{g}_\Delta]$, define the subspace

$$E = \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) \cdot v_\Delta \subset \wedge^n(\mathfrak{g} \oplus \mathfrak{g}).$$

It is clear that E does not depend on the choice of v_Δ inside $[\mathfrak{g}_\Delta]$, and the compactification $\overline{G}$ is contained in the projectivization of E:

$$\overline{G} \subset \mathbb{P}(E).$$

We will show that E is of the form

$$\mathrm{End}\ V \oplus \left(\oplus\ \mathrm{End}\ W^k\right) \oplus F$$

for some irreducible representation V of G of highest weight λ and some irreducible representations W^k of highest weights μ^k with $\mu^k \leq \lambda$ in the partial order on the weight lattice. We will show that under this identification

$$[\mathfrak{g}_\Delta] = \left[\mathrm{Id}_V \oplus \left(\sum c_k \mathrm{Id}_{W^k}\right) \oplus 0\right],$$

so that Theorem 9.1 will follow from Theorem 8.1 and Remark 8.8.

Let $h_1, \dots, h_l$ be a basis for the Cartan $\mathfrak{h} = \mathrm{Lie}\, T$, and for each $\alpha \in \Phi$ let $e_\alpha \in \mathfrak{g}$ be a root vector of weight α. There is a basis of $T \times T$-weight vectors for $\mathfrak{g} \oplus \mathfrak{g}$:

- $\{(h_i, \pm h_i) \mid i = 1, \dots, l\}$ of weight $(0, 0)$,
- $\{(e_\alpha, 0) \mid \alpha \in \Phi\}$ of weight $(\alpha, 0)$,
- $\{(0, e_\alpha) \mid \alpha \in \Phi\}$ of weight $(0, \alpha)$.

This gives a basis of $T \times T$-weight vectors of $\wedge^n(\mathfrak{g} \oplus \mathfrak{g})$ indexed by triples (A, B, S), where

- $A, B \subset \Phi$ are such that $|A| + |B| \leq n$,
- $S \subset \{(h_i, \pm h_i) \mid i = 1, \dots, l\}$ is such that $|A| + |B| + |S| = n$.

The weight vector corresponding to such a triple is

$$v_{ABS} = \left(\bigwedge_{\alpha \in A} (e_\alpha, 0)\right) \wedge \left(\bigwedge_{s \in S} s\right) \wedge \left(\bigwedge_{\beta \in B} (0, e_\alpha)\right),$$

and it has weight

$$\left(\sum_{\alpha \in A} \alpha, \sum_{\beta \in B} \beta\right).$$

Remark 9.2. Let $B^+ \subset G$ be a positive choice of Borel subgroup containing the maximal torus T, and let B^- be the opposite Borel. Denote by $\Phi^+ \subset \Phi$ the positive roots. Then the $T \times T$-weight vector

$$v_0 = \left(\bigwedge_{\alpha \in \Phi^+} (e_\alpha, 0)\right) \wedge \left(\bigwedge_{i=1^l} (h_i, h_i)\right) \wedge \left(\bigwedge_{\beta \in -\Phi^+} (0, e_\alpha)\right),$$

has weight $(\lambda, -\lambda)$, where

$$\lambda = \sum_{\alpha \in \Phi^+} \alpha$$

is the sum of the positive roots.

Any other $T \times T$-weight vector v_{ABS} has weight (μ, μ') with $\mu \leq \lambda$ and $\mu' \geq -\lambda$, so v_0 is a highest weight vector with respect to the Borel subgroup

$$B \times B^- \subset G \times G.$$

Proposition 9.3. *The vector v_0 is in the subspace $E = \mathcal{U}(\mathfrak{g} \oplus \mathfrak{g}) \cdot v_\Delta$ of $\wedge^n(\mathfrak{g} \oplus \mathfrak{g})$.*

Let $h \in \mathfrak{h}$ be a regular element such that

$$\alpha_i(h) = 1, \quad \forall i = 1, \dots, l.$$

This element induces an injection

$$\gamma : \mathbb{C}^* \longrightarrow T$$

such that Lie $\mathbb{C}^* = \mathbb{C}h$. The proof of Proposition 9.3 will follow from the following Lemma.

Lemma 9.4.

$$\lim_{z\to\infty} \gamma(z) \cdot [\mathfrak{g}_\Delta] = [v_0].$$

Proof. First we decompose $[\mathfrak{g}_\Delta]$ into a projectivized sum of $T \times T$-weight vectors. Write

$$[\mathfrak{g}_\Delta] = \left[\left(\bigwedge_{\delta\in\Phi} (e_\delta, e_\delta) \right) \wedge \left(\bigwedge_{i=1}^{l} (h_i, h_i) \right) \right].$$

Then

$$v_\Delta = \left(\bigwedge_{\delta\in\Phi} ((e_\delta, 0) + (0, e_\delta)) \right) \wedge \left(\bigwedge_{i=1}^{l} (h_i, h_i) \right) = \sum v_{ABS}, \tag{9.1}$$

where the sum is taken over all triples (A, B, S) such that

- $A \subset \Phi$,
- $B = \Phi \backslash A$,
- $S = \{(h_i, h_i) \mid i = 1, \ldots, l\}$.

For any $\alpha = \sum_{i=1}^{l} n_i \alpha_i \in \Phi$,

$$\gamma(z) \cdot e_\alpha = z^{\alpha(h)} e_\alpha = z^{\mathrm{ht}(\alpha)} e_\alpha,$$

where $\mathrm{ht}(\alpha) = \sum_{i=1}^{l} n_i$ is the height of the root α. Then

$$\gamma(z) \cdot v_{ABS} = \left(\bigwedge_{\alpha\in A} z^{\mathrm{ht}(\alpha)} (e_\alpha, 0) \right) \wedge \left(\bigwedge_{s\in S} s \right) \wedge \left(\bigwedge_{\beta\in B} (0, e_\alpha) \right) = z^{n_A} v_{ABS}, \tag{9.2}$$

where

$$n_A = \sum_{\alpha\in A} \mathrm{ht}(\alpha)$$

is the sum of the heights of the roots appearing in A.

Let

$$n_0 = \sum_{\alpha\in\Phi^+} \mathrm{ht}(\alpha).$$

Then

$$n_0 \geq n_A \quad \text{for all } A \subset \Phi,$$
$$n_0 = n_A \quad \text{if and only if } A = \Phi^+.$$

We can now compute

$$\begin{aligned}\lim_{z\to\infty}\gamma(z)[\mathfrak{g}_\Delta] &= \lim_{z\to\infty}[\gamma(z)v_\Delta] \\ &= \lim_{z\to\infty}\left[\sum_{A\subset\Phi} z^{n_A} v_{ABS}\right] \\ &= \lim_{z\to\infty}\left[v_0 + \sum_{A\subset\Phi} z^{n_A-n_0} v_{ABS}\right] = [v_0].\end{aligned}$$ □

Proof of Proposition 9.3. The closed subvariety

$$\mathbb{P}(E) \subset \mathbb{P}(\wedge^n(\mathfrak{g}\oplus\mathfrak{g}))$$

is $T\times T$-stable and closed, so

$$\overline{(T\times T)[\mathfrak{g}_\Delta]} \subset \mathbb{P}(E).$$

It follows that $[v_0]\in\mathbb{P}(E)$, and $v_0\in E$. □

Proof of Theorem 9.1. By Proposition 9.3 and Remark 9.2, $v_0\in E$ is a highest weight vector of weight $(\lambda,-\lambda)$, with $\lambda=\sum_{\alpha\in\Phi}\alpha$. Then

$$\mathcal{U}(\mathfrak{g}\oplus\mathfrak{g})\cdot v_0 \cong V\otimes V^* \cong \text{End } V \subset E,$$

where V is the irreducible G-representation of regular highest weight λ.

Because G is semisimple, we can decmopose

$$E = \text{End } V \oplus (\oplus \text{ End } W^k) \oplus F,$$

where the second summand consists of all irreducible representations of $G\times G$ of the form $W\otimes W^*$, and the third summand consists of all irreducible representations $G\times G$ of the form $U\otimes W^*$ with $U\ncong W$.

Each representation End W^k has highest weight $(\mu^k,-\mu^k)$, and from Remark 9.2 it follows that $\mu^k\le\lambda$. It remains to show that

$$[\mathfrak{g}_\Delta] = \left[\text{Id}_V \oplus \left(\sum c_k \text{Id}_{W^k}\right)\oplus 0\right].$$

This will follow from the next lemma.

Lemma 9.5. *An irreducible representation of $G\times G$ has a G_Δ-stable one-dimensional subspace if and only if it is of the form End W for some irreducible representations W of G. In this case, the unique such space is*

$$\mathbb{C}\text{Id}_W.$$

Proof. Any irreducible representation of $G\times G$ is of the form

$$U\otimes W^* \cong \text{Hom}(W,U)$$

for irreducible representations U and W of G.

There is a G_Δ-stable line in $U \otimes W^*$ if and only if there is a G-equivariant homomorphism in $\mathrm{Hom}(W, U)$. By Schur's lemma, such a homomorphism exists if and only if $U \cong W$, in which case it is unique up to scaling. □

Because $[\mathfrak{g}_\Delta] \in \mathbb{P}(E)$ is G_Δ-fixed, the line

$$\mathbb{C}v_\Delta \subset E = \mathrm{End}\ V \oplus (\oplus\ \mathrm{End}\ W^k) \oplus F$$

is G_Δ-stable. Then, its projection onto each summand is G_Δ-stable.

Lemma 9.5 then implies that the projection of v_Δ onto End V is

$$a_0 \mathrm{Id}_V$$

for some $a_0 \in \mathbb{C}$, that its projection onto End W^k is

$$a_k \mathrm{Id}_{W^k}$$

for some $a_k \in \mathbb{C}$, and that its projection onto F is 0 because F has no one-dimensional G_Δ-stable subspaces.

So we have

$$v_\Delta = c_0 \mathrm{Id}_V \oplus \left(\sum c_k \mathrm{Id}_{W^k}\right) \oplus 0.$$

But recall from (9.1) that

$$v_\Delta = v_0 + \sum v_{ABS}$$

as a sum of $T \times T$-weight vectors in $\wedge^n(\mathfrak{g} \oplus \mathfrak{g})$, so the projection of v_Δ onto End V is nonzero and so $c_0 \neq 0$. It follows that

$$[\mathfrak{g}_\Delta] = \left[\mathrm{Id}_V \oplus \left(\sum c_k \mathrm{Id}_{W^k}\right) \oplus 0\right].$$ □

Theorem 9.1 gives an isomorphism

$$\varphi : X \xrightarrow{\sim} \overline{G} \subset \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$$

such that for any interior point $g \in G \subset X$ of the wonderful compactification,

$$\varphi(g) = (g, e) \cdot \mathfrak{g}_\Delta. \tag{9.3}$$

We will describe which n-dimensional subspaces of $\mathfrak{g} \oplus \mathfrak{g}$ appear in the boundary of $\overline{G}$ in the Grassmannian. Because the map φ is $G \times G$-equivariant, it is enough to find the image

$$\varphi(z_I) \in \mathrm{Gr}(n, \mathfrak{g} \oplus \mathfrak{g})$$

of each $G \times G$-orbit basepoint z_I.

Recall the notation defined at the beginning of Section 6.

Theorem 9.6. *The image of the orbit basepoint z_I under the isomorphism φ is the n-dimensional space*

$$\varphi(z_I) = \{(u + x, v + x) \mid u \in \mathfrak{u}_I, v \in \mathfrak{u}_I^-, x \in \mathfrak{l}_I\} = \mathfrak{p}_I \times_{\mathfrak{l}_I} \mathfrak{p}_I^-.$$

Remark 9.7. Suppose $I = \{1, \ldots, l\}$ and $z_{\{1,\ldots,l\}}$ is the basepoint of the unique closed $G \times G$-orbit of minimal dimension. Theorem 9.6 says that

$$\varphi(z_{\{1,\ldots,l\}}) = \mathfrak{b} \times_{\mathfrak{h}} \mathfrak{b}^-,$$

and the image of this subspace under the Plücker embedding is exactly the point

$$[v_0] \in \mathbb{P}\left(\wedge^n(\mathfrak{g} \oplus \mathfrak{g})\right)$$

defined in Remark 9.2.

The proof is similar to the discussion in Lemma 9.4.

Proof of Theorem 9.6. Let $h \in \mathfrak{h}$ be such that

$$\alpha_i(h) = \begin{cases} 0, & \alpha_i \in \Delta_I \\ 1, & \alpha_i \notin \Delta_I \end{cases}$$

This produces a one-parameter subgroup

$$\gamma : \mathbb{C}^* \longrightarrow T$$

such that Lie $\gamma(\mathbb{C}^*) = \mathbb{C}h$. Then

$$\alpha_i(\gamma(z)) = z^{\alpha_i(h)} = \begin{cases} 1, & \alpha_i \in \Delta_I \\ z, & \alpha_i \notin \Delta_I. \end{cases}$$

The in $X \subset \mathbb{P}(\mathrm{End}\ V)$, this one-parameter subgroup is

$$\gamma(z) = \left[v_0 \otimes v_0^* + \sum_{i=1}^{l} \frac{1}{\alpha_i(\gamma(z))} v_i \otimes v_i^* + \sum_{i>l} \clubsuit v_i \otimes v_i^* \right],$$

(where $\clubsuit$ is polynomial in the first l coefficients, cf. Proposition 3.4) and as z tends to infinity we obtain

$$\lim_{z\to\infty} \gamma(z) = \left[v_0 \otimes v_0^* + \sum_{\alpha_i \in \Delta_I} v_i \otimes v_i^* + \sum_{i>l} \clubsuit v_i \otimes v_i^* \right] = z_I.$$

So γ is a one-parameter subgroup that tends to the orbit basepoint z_I in the boundary of the wonderful compactification.

Then

$$\varphi(z_I) = \lim_{z\to\infty} \varphi(\gamma(z)) = \lim_{z\to\infty} (\gamma(z), e) \cdot \mathfrak{g}_\Delta,$$

as in (9.3). To compute $(\gamma(z), e) \cdot \mathfrak{g}_\Delta$ we work in the projective space $\mathbb{P}(\wedge^n(\mathfrak{g} \oplus \mathfrak{g}))$ under the Plücker embedding.

Recall that

$$[\mathfrak{g}_\Delta] = \left[\sum v_{ABS} \right],$$

where as in (9.1) the sum is taken over all triples (A, B, S) such that

- $A \subset \Phi$,
- $B = \Phi \backslash A$,

- $S = \{(h_i, h_i) \mid i = 1, \ldots, l\}$.

For any root $\alpha \in \Phi$, write

$$\alpha = \sum_{\alpha_i \in \Delta_I} n_i \alpha_i + \sum_{\alpha_j \notin \Delta_I} n_j \alpha_j$$

and define

$$\mathrm{ht}_I(\alpha) = \sum_{\alpha_j \notin \Delta_I} n_j.$$

Then

$$\gamma(z) \cdot e_\alpha = z^{\alpha(h)} e_\alpha = z^{\mathrm{ht}_I(\alpha)} e_\alpha.$$

Applying the one-parameter subgroup γ to $T \times T$-weight vectors in $\wedge^n(\mathfrak{g} \oplus \mathfrak{g})$,

$$\gamma(z) \cdot v_{ABS} = z^{m_A} v_{ABS},$$

where

$$m_A = \sum_{\alpha \in A} \mathrm{ht}_I(\alpha).$$

(Cf. the computation in (9.2).)

Let

$$m_0 = \sum_{\alpha \in \Phi^+} \mathrm{ht}_I(\alpha).$$

Then

$$\begin{aligned} m_0 &\geq m_A \quad \text{for all } A \subset \Phi, \\ m_0 &= m_A \quad \text{if and only if } \Phi^+ \backslash \Phi_I \subseteq A \subseteq \Phi^+ \cup \Phi_I, \end{aligned}$$

—that is, $m_0 = m_A$ if and only if A differs from Φ^+ by roots in Φ_I, which do not contribute to the sum m_A.

Then we can compute

$$\begin{aligned} \lim_{z \to \infty} \gamma(z)[\mathfrak{g}_\Delta] &= \lim_{z \to \infty} \left[\gamma(z) \cdot \sum v_{ABS}\right] \\ &= \lim_{z \to \infty} \left[v_0 + \sum z^{m_A - m_0} v_{ABS}\right] = \left[\sum v_{A'B'S}\right], \end{aligned}$$

where the last sum is taken over triples (A', B', S) such that

- $\Phi^+ \backslash \Phi_I \subseteq A \subseteq \Phi^+ \cup \Phi_I$
- $B = \Phi \backslash A$
- $S = \{(h_i, h_i) \mid i = 1, \ldots, l\}$.

This sum can be written

$$\left[\sum_{(A', B', S)} \left(\bigwedge_{\alpha \in A'} (e_\alpha, 0)\right) \wedge \left(\bigwedge_{i=1}^{l} (h_i, h_i)\right) \wedge \left(\bigwedge_{\beta \in B'} (0, e_\beta)\right)\right], \tag{9.4}$$

and we notice that every root vector e_δ with $\delta \in \Phi_I$ appears as both $(e_\delta, 0)$ and as $(0, e_\delta)$ in this sum. Then we can rewrite (9.4) as

$$\left(\bigwedge_{\alpha \in \Phi^+ \setminus \Phi_I} (e_\alpha, 0) \right) \wedge \left(\bigwedge_{\delta \in \Phi_I} (e_\delta, e_\delta) \right) \wedge \left(\bigwedge_{i=1}^{l} (h_i, h_i) \right) \wedge \left(\bigwedge_{\beta \in -\Phi^+ \setminus \Phi_I} (0, e_\beta) \right)$$

The vectors in the first wedge give a basis for

$$\mathfrak{u}_I \oplus 0 \subset \mathfrak{g} \oplus \mathfrak{g},$$

the vectors in the last wedge give a basis for

$$0 \oplus \mathfrak{u}_I^- \subset \mathfrak{g} \oplus \mathfrak{g},$$

and the diagonal vectors in the two middle wedges give a basis for the diagonal subspace

$$\mathfrak{l}_{I\Delta} = \{(x, x) \mid x \in \mathfrak{l}_I\} \subset \mathfrak{g} \oplus \mathfrak{g}.$$

It follows that

$$\varphi(z_I) = \lim_{z \to \infty} \gamma(z)[\mathfrak{g}_\Delta] = [\mathfrak{p}_I \times_{\mathfrak{l}_I} \mathfrak{p}_I^-]. \qquad \square$$

10. Log-homogeneous varieties

In this section we introduce some general notions about log-homogeneous varieties, following the exposition in Sections 1.1 and 2.1 of the lecture notes [Bri1]. For now, let G be a connected complex algebraic group with Lie algebra $\mathfrak{g}$, and let X be a smooth connected G-variety. Denote by

$$\mathcal{T}_X = \mathcal{D}er(\mathcal{O}_X)$$

the tangent sheaf of X, whose sections are derivations of the ring of regular functions $\mathcal{O}_X$. This is the locally-free sheaf associated to the tangent bundle TX of X.

The action of G on the variety X gives a map

$$\begin{aligned} \mathrm{op}_X : \mathfrak{g} &\longrightarrow \Gamma(X, TX) \\ \xi &\longmapsto v_\xi, \end{aligned}$$

where v_ξ is the vector field induced by the differential of the G-action:

$$v_\xi(x) = \frac{d}{dt}_{|t=0} \left(\exp(-t\xi) x \right).$$

(The negative sign is necessary to make op_X a homomorphism of Lie algebras.) There is a corresponding morphism of sheaves

$$\underline{\mathrm{op}}_X : \mathcal{O}_X \otimes \mathfrak{g} \longrightarrow \mathcal{T}_X.$$

Definition 10.1. The variety X is *homogeneous* if the action of G on X is transitive.

Proposition 10.2. *The variety X is homogeneous if and only if the morphism $\underline{\mathrm{op}}_X$ is surjective.*

Proof. Choose a basepoint $x \in X$. If X is homogeneous, the action map

$$\begin{aligned} \varphi_x : G &\longrightarrow X \\ g &\longmapsto g \cdot x \end{aligned}$$

is surjective, and so is its differential

$$d\varphi_x : \mathfrak{g} \longrightarrow T_x X.$$

But

$$d\varphi_x = \underline{\mathrm{op}}_{X,x},$$

so it follows that $\underline{\mathrm{op}}_X$ is also surjective.

Conversely, suppose that $\underline{\mathrm{op}}_X$ is surjective, so that the induced map on stalks $d\varphi_x$ is surjective at every x. Then φ_x is a submersion, and its image $G \cdot x$ is open in X. Since X is connected and $x \in X$ was chosen arbitrarily, it follows that X is homogeneous. □

Definition 10.3. An effective reduced divisor $D \subset X$ has *normal crossings* if at each $x \in X$ there exist local coordinates $x_1, \dots, x_n$ such that

$$D = \{(x_1, \dots, x_n) \mid x_1 \cdot \ldots \cdot x_k = 0\}.$$

That is, in the completed local ring

$$\hat{\mathcal{O}}_{X,x} = \mathbb{C}[[x_1, \dots, x_n]]$$

the ideal of D is generated by $x_1 \cdot \ldots \cdot x_k$.

Definition 10.4. Suppose that $D \subset X$ is a normal crossing divisor. The *logarithmic tangent sheaf* is the subsheaf

$$\mathcal{T}_X(-\log D) \subset \mathcal{T}_X$$

whose sections are the derivations of $\mathcal{O}_X$ that preserve the ideal sheaf of D. In other words, these sections are vector fields on X that are tangent to the divisor D, called *logarithmic vector fields.*

Example 10.5. Let $X = \mathbb{C}^n$ and let

$$D = \{x_1 \cdot \ldots \cdot x_k = 0\}$$

be the union of the first k coordinate hyperplanes. At the origin, the logarithmic tangent sheaf is generated by

$$x_1\partial_1, \dots, x_k\partial_k, \partial_{k+1}, \dots, \partial_n.$$

Remark 10.6.

(1) Because D is a normal crossing divisor, the logarithmic tangent sheaf is locally-free of rank $\dim X$, and the associated vector bundle is the *logarithmic tangent bundle*

$$TX(-\log D).$$

It is not a subbundle of the tangent bundle – on the contrary, the two bundles have the same rank.

(2) The restriction of $\mathcal{T}_X(-\log D)$ to the open piece $X^\circ = X \backslash D$ is the usual tangent sheaf $\mathcal{T}_{X^\circ}$.
(3) The dual of $\mathcal{T}_X(-\log D)$ is the sheaf $\Omega^1_X(\log D)$ of logarithmic 1-forms with poles along D. (A logarithmic form is an algebraic form with simple poles whose differential also has simple poles.) In Example 10.5, this sheaf is locally generated by

$$\frac{\mathrm{d}x_1}{x_1}, \ldots, \frac{\mathrm{d}x_k}{x_k}, \mathrm{d}x_{k+1}, \ldots, \mathrm{d}x_n.$$

Its associated bundle is the *logarithmic cotangent bundle*

$$T^*X(-\log D).$$

Now let G act on X and let $D \subset X$ be a G-stable normal crossing divisor. Then the differential of the action map induces the morphism of Lie algebras

$$\mathrm{op}_{X,D} : \mathfrak{g} \longrightarrow \Gamma(X, TX(-\log D))$$

and the associated morphism of sheaves

$$\underline{\mathrm{op}}_{X,D} : \mathcal{O}_X \otimes \mathfrak{g} \longrightarrow \mathcal{T}_X(-\log D).$$

Definition 10.7. The pair (X, D) is *log-homogeneous* if the morphism $\underline{\mathrm{op}}_{X,D}$ is surjective.

Example 10.8. (1) Suppose that $X = \mathbb{C}^n$ is affine space, $D = \{x_1 \cdot \ldots \cdot x_n = 0$ is the union of the coordinate hyperplanes, and $G = (\mathbb{C}^*)^n$ acts on X by coordinate-wise multiplication. Then (X, D) is log-homogeneous.
(2) Suppose that X is a smooth projective toric variety for a torus $G = T$, so that T sits inside X as an open T-orbit. The boundary $D = X \backslash T$ is a normal crossing divisor, and X can be covered by open T-stable affine spaces $\mathbb{C}^n$ on which T acts by coordinate multiplication. It follows that (X, D) is log-homogeneous. (See [Ful].)

Remark 10.9. Suppose that (X, D) is log-homogeneous. Then the restriction

$$\underline{\mathrm{op}}_{X,D|X^\circ} = \underline{\mathrm{op}}_{X^\circ} : \mathcal{O}_{X|X^\circ} \otimes \mathfrak{g} \longrightarrow \mathcal{T}_X(-\log D)_{|X^\circ} = \mathcal{T}_{X^\circ}$$

is surjective, so X° is a homogeneous space.

Construct a stratification of the divisor D as follows: let

$$X_1 = D, X_2 = \mathrm{Sing}(D), \ldots, X_m = \mathrm{Sing}(X_{m-1}), \ldots,$$

and let the strata be the connected components of $X_m \backslash X_{m+1}$. They are smooth, locally-closed, and G-stable because G is connected.

Fix a stratum S and a point $x \in S$, and let $x_1, \ldots, x_n$ be coordinates at x such that the divisor D is given by

$$D = \{x_1 \cdot \ldots \cdot x_k = 0\}.$$

Then $X_m \backslash X_{m+1}$ is the locus where exactly m coordinates are zero, and

$$S = \{x_1 = \ldots = x_k = 0\}$$

has codimension k. The stratum S is the intersection of the stratum closures

$$\bar{S}_i = \{x_j = 0 \mid j \leq k, j \neq i\}.$$

The normal space of S in X at x

$$N_{S/X,x} = T_x X / T_x S$$

decomposes as a sum of lines

$$N_{S/X,x} = L_1 \oplus \ldots \oplus L_k, \tag{10.1}$$

where each L_i is the normal space to S in $\bar{S}_i$ at x.

The stabilizer $\mathrm{Stab}_G(x) = G^x$ of x in G acts on all these spaces, and its identity component preserves each line L_i. The action map

$$\rho_x : (G^x)^\circ \longrightarrow (\mathbb{C}^*)^k$$

has differential

$$d\rho_x : \mathfrak{g}^x \longrightarrow \mathbb{C}^k.$$

The following gives a criterion for log-homogeneity. (See [Bri1], Proposition 2.1.2.)

Proposition 10.10. *The following are equivalent:*

1. *The pair (X, D) is log-homogeneous.*
2. *Each stratum S is a single G-orbit and the differential $d\rho_x$ is surjective at every $x \in S$.*

If these conditions hold, there is a short exact sequence of Lie algebras

$$0 \longrightarrow \ker(d\rho_x) \longrightarrow \mathfrak{g} \xrightarrow{op_{X,D}} T_x X(-\log D) \longrightarrow 0.$$

Proof. Because $\mathcal{T}_X(-\log D)$ preserves the ideal sheaf of S, there is a morphism of sheaves

$$\mathcal{T}_X(-\log D)_{|S} \longrightarrow \mathcal{T}_S$$

that descends to a linear map on fibers

$$p_x : T_x X(-\log D) \longrightarrow T_x S.$$

In coordinates $x_1, \ldots, x_n$ at x, p_x is the projection

$$\{x_1 \partial_1, \ldots, x_k \partial_k, \partial_{k+1}, \ldots, \partial_n\} \longrightarrow \{\partial_{k+1}, \ldots, \partial_n\}.$$

Since

$$p_x \circ \mathrm{op}_{X,D} = \mathrm{op}_S : \mathfrak{g} \longrightarrow T_x S,$$

the composition $p_x \circ \mathrm{op}_{X,D}$ factors through the injection

$$\iota_x : \mathfrak{g}/\mathfrak{g}^x \longrightarrow T_x S.$$

We obtain a commutative diagram in which the rows are short exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{g}^x & \longrightarrow & \mathfrak{g} & \longrightarrow & \mathfrak{g}/\mathfrak{g}^x & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle d\rho_x} & & \downarrow{\scriptstyle \mathrm{op}_{X,D}} & & \downarrow{\scriptstyle \iota_x} & & \\
0 & \longrightarrow & \langle x_i\partial_i\rangle & \longrightarrow & T_xX(-\log D) & \xrightarrow{p_x} & T_xS & \longrightarrow & 0.
\end{array}$$

Because ι_x is injective, it follows by the Snake Lemma that (1) $\mathrm{op}_{X,D}$ is surjective if and only if (2) both $d\rho_x$ and ι_x are surjective. The latter is equivalent to the condition that S is a single G-orbit.

Moreover, we have

$$\ker(d\rho_x) = \ker(\mathrm{op}_{X,D}).$$

If $\mathrm{op}_{X,D}$ is surjective, this gives the short exact sequence

$$0 \longrightarrow ker(d\rho_x) \longrightarrow \mathfrak{g} \xrightarrow{\mathrm{op}_{X,D}} T_xX(-\log D) \longrightarrow 0. \qquad \square$$

11. The logarithmic cotangent bundle of $\overline{G}$

Now let G be a semisimple connected algebraic group with trivial center, and let X once again be the wonderful compactification of G. Write $D \subset X$ for the boundary divisor, which is a normal crossing divisor by Theorem 5.3.

Proposition 11.1. *The pair (X, D) is log-homogeneous.*

Proof. The stratification of the divisor D given above is exactly the stratification of the boundary of X into $G \times G$-orbits from Section 5. It is enough to check the criterion in Proposition 10.10 at the orbits basepoints z_I, $I \subset \{1, \ldots, l\}$.

Recall from Theorem 4.4 that X is covered by $G \times G$-translates of the big cell

$$X_0 \cong U^- \times U \times Z,$$

where $U-$ and U are the unipotent radicals of a fixed pair of opposite Borels, and Z is the closure of the resulting maximal torus in X_0, isomorphic to $\mathbb{C}^l$ by Proposition 3.4. Moreover, each orbit basepoint z_I is contained in Z.

Keeping the notation of Section 5, assume without loss of generality that

$$I = \{1, \ldots, k\}.$$

Then the basepoints z_I is of the form

$$z_I = (0, \ldots, 0, 1, \ldots, 1)$$

and we have the following tangent spaces:

$$T_{z_I}X = T_{z_I}X_0 \cong \mathfrak{u}^- \oplus \mathfrak{u} \oplus \mathbb{C}^l$$
$$T_{z_I}(G \times G)z_I \cong \mathfrak{u}^- \times \mathfrak{u} \times \mathbb{C}^{l-k}.$$

By Proposition 3.4, the torus T acts on $Z \cong \mathbb{C}^l$ on the left via

$$(-\alpha_1, \ldots, -\alpha_l),$$

and so it acts on the normal space

$$T_{z_I}X_0/T_{z_I}(G\times G)z_I \cong \mathbb{C}^k$$

by $(-\alpha_1,\dots,-\alpha_k)$.

Recall from Proposition 6.3 that the stabilizer of z_I in $G\times G$ is

$$\mathrm{Stab}_{G\times G}(z_I) = \{(ux,vy)\in U_IL_I\times U_I^-L_I \mid xy^{-1}\in Z(L_I)\}.$$

It acts on the normal space by fixing each line in the decomposition (10.1), and it acts on the line L_i by the central character $-\alpha_i$. It follows that the map

$$\begin{aligned} d\rho_{z_I} : \mathrm{Lie}(\mathrm{Stab}_{G\times G}(z_I)) &\longrightarrow \mathbb{C}^k \\ (u+x,v+y) &\longmapsto (\alpha_1(y-x),\dots,\alpha_k(y-x)) \end{aligned}$$

is surjective, and so it follows by Proposition 10.10 the wonderful compactification X is log-homogeneous. □

Corollary 11.2. *The isotropy Lie algebra of the orbit basepoint z_I is*

$$\ker(d\rho_{z_I}) = \mathfrak{p}_I \times_{\mathfrak{l}_I} \mathfrak{p}_I^-.$$

Now consider the vector bundle R_X on X, with fiber at $x\in X$ given by

$$R_{X,x} = \ker(d\rho_{z_I}).$$

It is called the *bundle of isotropy subalgebras*. By Proposition 9.6, this vector bundle is isomorphic to the restriction to X of the tautological bundle on the Grassmannian

$$\mathrm{Gr}(n,\mathfrak{g}\times\mathfrak{g}).$$

Moreover, by Proposition 10.10, there is a short exact sequence of vector bundles on X:

$$0\longrightarrow R_X\longrightarrow X\times\mathfrak{g}\times\mathfrak{g}\longrightarrow TX(-\log D)\longrightarrow 0. \tag{11.1}$$

Proposition 11.3. *There is an isomorphism of vector bundles on X between the bundle of isotropy subalgebras and the logarithmic cotangent bundle of X:*

$$R_X \cong T^*X(-\log D).$$

Proof. (See [Bri2], Example 2.5.) Let β be a nondegenerate G-invariant symmetric bilinear form on $\mathfrak{g}$. The form $(\beta,-\beta)$ is a nondegenerate G-invariant symmetric bilinear form on $\mathfrak{g}\times\mathfrak{g}$, and the fiber

$$R_{X,e} = \mathfrak{g}_\Delta$$

is Lagrangian.

Then R_X is a Lagrangian subbundle of $X\times\mathfrak{g}\times\mathfrak{g}$, and from the short exact sequence (11.1) we get an isomorphism

$$R_X\cong R_X^\perp \cong (X\times\mathfrak{g}\times\mathfrak{g}/R_X)^*\cong T^*X(-\log D).$$ □

12. Cohomology of the wonderful compactification

We compute the cohomology of X by decomposing it into a union of affine cells using the Bialynicki-Birula decomposition (see Part 1 of these notes, Theorem 2.2). This section follows [EJ], Sections 4.1 and 4.2. See also [DS].

As before, let T be a maximal torus of G and let W be the associated Weyl group. For every element $w \in W$, choose a coset representative $\dot{w} \in N_G(T)$. Let

$$z_0 = z_{\{1,\ldots,l\}} \in X$$

be the basepoint of the unique closed $G \times G$-orbit in X.

Proposition 12.1. *The $T \times T$-fixed points in X are exactly the points*

$$\{z_{y,w} = (\dot{y}, \dot{w}) \cdot z_0 \mid y, w \in W\}.$$

Proof. Decompose

$$X = \coprod_{I \subseteq \{1,\ldots,l\}} (G \times G) \cdot z_I$$

and suppose

$$x \in (G \times G) \cdot z_I$$

is fixed by $T \times T$. Then the stabilizer of x in $G \times G$ contains a torus of dimension $2l$, and so does the stabilizer of z_I. But the maximal torus of

$$\mathrm{Stab}_{G\times G}(z_I) = \{(ux, vy) \in U_I L_I \times U_I^- L_I \mid xy^{-1} \in Z(L_I)\}$$

is the subgroup

$$\{(x, y) \in T \times T \mid xy^{-1} \in Z(L_I)\},$$

which has dimension $l + |I|$. It follows that $|I| = l$ and that

$$I = \{1, \ldots, l\},$$

so x is contained in the $G \times G$-orbit of minimal dimension.

By Remark 6.4, this orbit if $G \times G$-isomorphic to the product of two copies of the flag variety. By Theorem 2.1 in Part 1, the $T \times T$-fixed points in

$$G/B \times G/B^-$$

are exactly the point $(\dot{y}B, \dot{w}B^-)$. The Proposition follows. □

Proposition 12.2. *The $T \times T$-weights on $T_{z_0}X$ are*

(1) $(-\alpha, 0), \quad \alpha \in \Phi^+$.
(2) $(0, \alpha), \quad \alpha \in \Phi^+$.
(3) $(-\alpha_i, \alpha_i), \quad \alpha_i \in \Delta$.

Proof. Recall once again that the point z_0 is contained in the big cell $X_0 \cong U^- \times U \times Z$, and that this isomorphism is $U^-T \times U$-equivariant. Then

$$T_{z_0}X_0 \cong \mathfrak{u}^- \oplus \mathfrak{u} \oplus \mathbb{C}^l$$

and $T\times T$ acts on the first summand by the weights $\{(-\alpha,0)\mid\alpha\in\Phi^+\}$ and on the second summand by the weights $\{(0,\alpha)\mid\alpha\in\Phi^+\}$.

To see how $T\times T$ acts on the tangent space of Z, recall that

$$X\subseteq\mathbb{P}(\mathrm{End}\ V)$$

and choose as in Section 3 a basis $v_0,\dots,v_n$ for V such that v_i has weight λ_i with

$$\lambda_0=\lambda$$
$$\lambda_i=\lambda-\alpha_i\quad\text{for } i=1,\dots,l.$$

Then the isomorphism $Z\cong\mathbb{C}^l$ is given by

$$(z_1,\dots,z_l)\longmapsto\left[v_0\otimes v_0^*+\sum_{i=1}^{l}z_iv_i\otimes v_i^*+\sum_{i>l}\clubsuit v_i\otimes v_i^*\right]$$

and the action of $(t_1,t_2)\in T\times T$ at the identity element in $\mathbb{P}(\mathrm{End}\ V)$ is given by

$$\begin{aligned}(t_1,t_2)\cdot[\textstyle\sum v_i\otimes v_i^*]&=[\lambda_i(t_1)v_i\otimes\lambda_i(t_2^{-1})v_i^*]\\&=\left[v_0\otimes v_0^*+\sum_{i=1}^{l}\frac{\alpha_i(t_2)}{\alpha_i(t_1)}v_i\otimes v_i^*+\sum_{i>l}\clubsuit v_i\otimes v_i^*\right].\end{aligned}$$

So the weights of $T\times T$ on the tangent space of Z are

$$\{(-\alpha_i,\alpha_i)\mid\alpha_i\in\Delta\}.$$ □

Corollary 12.3. *The $T\times T$-weights on $T_{z_{yw}}X$ are*

(1) $(-y\alpha,0),\quad\alpha\in\Phi^+$.
(2) $(0,w\alpha),\quad\alpha\in\Phi^+$.
(3) $(-y\alpha_i,w\alpha_i),\quad\alpha_i\in\Delta$.

Remark 12.4. Recall that if Y is a toric variety for a torus S, it is associated to a union of cones

$$\mathrm{Fan}(Y)=\left\{C_y\subset X_*(S)\otimes_{\mathbb{Z}}\mathbb{R}\mid y\in Y^S\right\}$$

indexed by S-fixed points in the following way: Let $y\in Y$ be fixed by S, and let $\mu_1,\dots,\mu_l$ be the weights of S on the tangent space T_yY. Then the cone C_y is defined by

$$C_y=\{x\in X_*(S)\otimes_{\mathbb{Z}}\mathbb{R}\mid\mu_i(x)\geq 0\,\forall i=1,\dots,l\}.$$

Moreover, the toric variety Y is complete if and only if its fan covers the entire cocharacter space—in other words, if and only if

$$\bigcup_{y\in Y^S}C_y=X_*(S)\otimes_{\mathbb{Z}}\mathbb{R}.$$

For details on toric varieties, see [Ful].

Let $\overline{T}\subset X$ be the closure of the maximal torus T inside the wonderful compactification.

Proposition 12.5. *The fan of $\overline{T}$ is the fan of Weyl chambers.*

Proof. Define the intermediate variety

$$\widetilde{Z} = \bigcup_{w\in W} \dot{w}Z\dot{w}^{-1} \subseteq \overline{T}.$$

This is a smooth toric variety for the torus $e \times T$ and its T-fixed points are

$$\widetilde{Z}^T = \{z_{ww} \mid w \in W\}.$$

By Corollary 12.3, the weights of T on the tangent space $T_{z_{ww}}\widetilde{Z}$ are

$$w\alpha_1, \dots, w\alpha_l,$$

and the corresponding cone is

$$C_w = \{x \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{R} \mid w\alpha_i(x) \geq 0\} = w \cdot C_0,$$

where C_0 is the dominant Weyl chamber.

It follows that the fan of $\widetilde{Z}$ is the fan of Weyl chambers. But this fan covers the entire cocharacter space, so $\widetilde{Z}$ is complete. Since it is also dense in $\overline{T}$, equality must hold:

$$\widetilde{Z} = \overline{T}.$$

□

Now fix an element $h \in \mathfrak{h}$ such that $\alpha_i(h) = 1$ for all $i = 1, \dots, l$, and let $n \in \mathbb{Z}$ be an integer such that

$$n > \beta(h) \quad \text{for all } \beta \in \Phi^+.$$

Define a one-parameter subgroup

$$\gamma : \mathbb{C}^* \longrightarrow T \times T$$

such that

$$\text{Lie } \gamma(\mathbb{C}^*) = \mathbb{C}(nh, -h).$$

Proposition 12.6.

$$X^{T\times T} = X^{\mathbb{C}^*}.$$

Proof. Let X' be a connected component of the fixed point set $X^{\mathbb{C}^*}$. Because the action of $T \times T$ commutes with the $\mathbb{C}^*$-action, X' is $T \times T$-stable. But then $T \times T$ is a solvable group acting on the projective variety X', and this action must have a fixed point.

Suppose $z_{yw} \in X'$ is a $T \times T$-fixed point. By Corollary 12.3, the eigenvalues of $(nh, -h)$ on the tangent space $T_{z_{yw}}X'$ are

(1) $-n(y\alpha(h)), \quad \alpha \in \Phi^+$.
(2) $-w\alpha(h), \quad \alpha \in \Phi^+$.
(3) $-n(y\alpha_i(h)) - w\alpha_i(h), \quad \alpha_i \in \Delta$.

The first two are non-zero by the choice of h, and the third is nonzero by the choice of n. It follows that z_{yw} is an isolated fixed point of the $\mathbb{C}^*$-action, so

$$X' = \{z_{yw}\} \subseteq X^{T\times T}.$$

□

Recall that for any $y \in W$, the *length* of y is

$$l(y) = \#\{\alpha \in \Phi^+ \mid y\alpha \in -\Phi^+\}.$$

Define the *simple length* of y to be

$$m(y) = \#\{\alpha_i \in \Delta \mid y\alpha \in -\Phi^+\}.$$

Theorem 12.7. *Let*

$$X_{yw} = \{x \in X \mid \lim_{t\to 0} \gamma(t) \cdot x = z_{yw}\}.$$

Then the set

$$\{[X_{yw}] \mid y, w \in W\}$$

forms an additive basis for $H_*(X)$, and the degree of the basis element $[X_{yw}]$ is

$$2(l(y) + l(w) + m(y)).$$

Proof. The first part of the theorem follows from the Bialynicki-Birula decomposition, which also states that there is a $\mathbb{C}^*$-equivariant isomorphism

$$X_{yw} \cong T^+_{z_{yw}} X,$$

where $T^+_{z_{yw}} X$ is the subspace of the tangent space $T_{z_{yw}} X$ on which $\mathbb{C}^*$ acts with positive weights. It follows that

$$\deg[X_{yw}] = 2 \dim X_{yw} = \dim T^+_{z_{yw}} X.$$

From the proof of Proposition 12.6,

(1) $-n(y\alpha(h)) > 0$ if and only if $y\alpha \in -\Phi^+$.
(2) $-w\alpha(h) > 0$ if and only if $w\alpha \in -\Phi^+$.
(3) $-n(y\alpha_i(h)) - w\alpha_i(h)$ if and only if $n(y\alpha_i(h)) < -w\alpha_i(h)$, which is if and only if $y\alpha_i \in -\Phi^+$.

It follows that $\mathbb{C}^*$ has

$$l(y) + l(w) + m(y)$$

positive eigenvalues on $T_{z_{yw}} X$. □

13. The Picard group

This section follows the exposition in [BK], where the structure of the wonderful compactification is developed more generally over fields of arbitrary characteristic.

Because the wonderful compactification X is smooth, the Picard group parametrizes both equivalence classes of divisors on X and isomorphism classes of invertible sheaves on X. As before, write W for the Weyl group and $X_0 \subset X$ for the big cell of the wonderful compactification. Let $s_1, \ldots, s_l \in W$ be the simple reflections, and for any element $w \in W$ let $\dot{w} \in N_G(T)$ be a preimage,

Lemma 13.1. *The boundary $X \backslash X_0$ is the union of the divisors*

$$\overline{B\dot{s}_i B^-},$$

and these freely generate the Picard group Pic(X)*.*

Proof. Since X_0 is an affine open subset of X, the complement $X\backslash X_0$ is of pure codimension 1 by Lemma 5.1. Moreover, X_0 intersects every $G \times G$-orbit, so $X\backslash X_0$ contains no $G \times G$-orbits and therefore $G\backslash X_0$ is dense in $X\backslash X_0$.

But $G \cap X_0 = BB^-$ by Proposition 3.2, so by the Bruhat decomposition the complement is

$$G\backslash X_0 = \coprod_{1\neq w\in W} B\dot{w}B^-.$$

It follows that

$$X\backslash X_0 = \bigcup_{s_i\in W} \overline{B\dot{s}_i B^-}.$$

Now suppose $D \subset X$ is a divisor. Because X_0 is an affine space, the intersection $D \cap X_0$ is principal, so D is equivalent in Pic(X) to a linear combination

$$\sum_{i=1}^{l} a_i \overline{B\dot{s}_i B^-}.$$

These coefficients are unique—if

$$D \sim \sum_{i=1}^{l} b_i \overline{B\dot{s}_i B^-},$$

then

$$\sum_{i=1}^{l} (a_i - b_i)\overline{B\dot{s}_i B^-} \sim 0$$

is principal, so it is cut out by a regular function on X that is nonvanishing on X_0. But X_0 is an affine space, so any such function is constant. □

Definition 13.2. Let $w_0 \in W$ be the longest word of the Weyl group. The divisors

$$D_i = \overline{B\dot{s}_i\dot{w}_0 B^-} = \overline{B\dot{s}_i B^-}\dot{w}_0$$

are called the *Schubert divisors* of X.

Remark 13.3. Because Pic(X) is discrete, the action of G on Pic(X) is trivial, and the divisor D_i is equivalent to $\overline{B\dot{s}_i B^-}$. The Schubert divisors $D_1, \dots, D_l$ form a basis for the Picard group Pic(X).

Consider the unique closed $G \times G$-orbit of minimal dimension in $Y \subset X$. By Remark 6.4,

$$Y \cong G/B \times G/B$$

is isomorphic to a product of two copies of the flag variety.

We will classify the invertible sheaves on X by restricting them to Y and using the Borel-Weil theorem. Let Λ be the weight lattice of the maximal torus $\widetilde{T}$, and let Λ^+ be the cone of dominant weights.

Theorem 13.4 (Borel-Weil)**.** *There is an isomorphism of abelian groups*

$$\mathrm{Pic}(G/B) \cong \Lambda.$$

The weight $\lambda \in \Lambda$ corresponds to a line bundle

$$G \times_B \mathbb{C}_\lambda$$

with sheaf of sections $\mathcal{L}(\lambda)$, and the global sections of this sheaf are

$$\Gamma(G/B, \mathcal{L}(\lambda)) = \begin{cases} V_\lambda^*, & \lambda \in \Lambda^+ \\ 0, & \textit{else.} \end{cases}$$

Moreover, $\mathcal{L}(\lambda)$ is ample if and only if λ is regular dominant.

Let

$$\mathcal{L}_Y(\lambda) = \mathcal{L}(-w_0\lambda) \boxtimes \mathcal{L}(\lambda).$$

be the invertible sheaf on Y corresponding to the weights $(-w_0\lambda, \lambda)$.

Proposition 13.5. *The restriction*

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(Y)$$

is injective with image

$$\{[\mathcal{L}_Y(\lambda)] \mid \lambda \in \Lambda\}.$$

Proof. Recall that by the Peter-Weyl theorem, the regular functions on $\widetilde{G}$ are given by

$$\mathbb{C}[\widetilde{G}] = \bigoplus_{\mu \in \Lambda^+} V_\mu \otimes V_\mu^*,$$

where V_μ is the irreducible $\widetilde{G}$-representation of highest weight μ, V_μ^* is its dual of highest weight $-w_0\mu$, and the functions are

$$v \otimes w^*(g) = w^*(g \cdot v).$$

Let $\chi_1, \dots, \chi_l$ be the fundamental weights, and let $v_i \in V_{\chi_i}$ and $w_i \in V_{\chi_i}^*$ be highest weight vectors. In the simply connected cover $\widetilde{G}$ of G, the intersection

$$\widetilde{D}_i \cap \widetilde{G} = \widetilde{B} s_i w_0 \widetilde{B}$$

is a principal divisor, cut out by the function $v_i \otimes w_i$. This function is a $\widetilde{B} \times \widetilde{B}$-weight vector with weight

$$(\chi_i, -w_0\chi_i),$$

so the canonical section τ_i of the invertible sheaf $\mathcal{O}_X(D_i)$ is a $\widetilde{B} \times \widetilde{B}$-weight vector of the same weight.

It follows that

$$\mathcal{O}_X(D_i)_{|Y} = \mathcal{L}_Y(\chi_i).$$

Since the isomorphism classes $[\mathcal{O}_X(D_i)]$ generate $\mathrm{Pic}(X)$, and since the invertible sheaves $\mathcal{L}_Y(\chi_i)$ form a linearly independent set in the Picard group $\mathrm{Pic}(Y)$, the proposition is proved. □

Denote by $\mathcal{L}_X(\lambda)$ the unique invertible sheaf on X that restricts to $\mathcal{L}_Y(\lambda)$ along Y. As in Section 5, let

$$S_1, \ldots, S_l$$

be the irreducible components of the boundary divisor $X \backslash G$, and write σ_i for the canonical section of the invertible sheaf $\mathcal{O}_X(S_i)$. Because S_i is $G \times G$-stable, the section σ_i is $\widetilde{G} \times \widetilde{G}$-invariant.

Lemma 13.6. (1) $\mathcal{O}_X(D_i) = \mathcal{L}_X(\chi_i)$
(2) $\mathcal{O}_X(S_i) = \mathcal{L}_X(\alpha_i)$

Proof. Part (1) is already contained in the proof of Proposition 13.5.

The intersection $S_i \cap X_0$ is a principal divisor, cut out by a nonzero regular function on X_0 as follows: recall from Theorem 3.5 the $U^- T \times U$-equivariant isomorphism

$$X_0 \cong U^- \times U \times \mathbb{C}^l.$$

The intersection

$$S_i \cap X_0 = U^- \times U \times \{(z_1, \ldots, z_l) \mid z_i = 0\}$$

is cut out by the regular function $z_i = 0$, so the canonical section of the invertible sheaf $\mathcal{O}_X(S_i)$ has $T \times T$-weight $(\alpha_i, -w_0\alpha_i)$. (Cf. Proposition 12.2. In this section we are working with the Borel $B \times B$ instead of $B \times B^-$, so the second factor is always twisted by the longest element w_0 of W.) Part (2) follows. □

Proposition 13.7. *The invertible sheaf $\mathcal{L}_X(\lambda)$ is generated by global sections if and only if the weight λ is dominant, and it is ample if and only if λ is regular dominant.*

Proof. If $\mathcal{L}_X(\lambda)$ is globally generated (respectively ample), then its restriction $\mathcal{L}_Y(\lambda)$ is globally generated (resp. ample), so by Borel-Weil the weight λ is dominant (resp. regular dominant.)

For the converse, because the divisor D_i contains no $G \times G$-orbits, the $G \times G$-translates of the canonical section τ_i have no common zeros. It follows that the invertible sheaf $\mathcal{L}_X(\chi_i)$ is globally generated.

If λ is a dominant weight, then

$$\lambda = \sum_{i=1}^{l} \langle \lambda, \check{\alpha}_i \rangle \chi_i$$

with non-negative coefficients $\langle \lambda, \check{\alpha}_i \rangle$. It follows that

$$\mathcal{L}_X(\lambda) = \bigotimes_{i=1}^{l} \mathcal{L}_X(\chi_i)^{\otimes \langle \lambda, \check{\alpha}_i \rangle}$$

is also generated by global sections.

If λ is regular and dominant, fix a very ample invertible sheaf $\mathcal{L} = \mathcal{L}_X(\mu)$. For a sufficiently large $N \in \mathbb{Z}$, the weight

$$N\lambda - \mu$$

is dominant, so the invertible sheaf

$$\mathcal{L}_X(\lambda)^{\otimes N} \otimes \mathcal{L}^{-1}$$

is generated by global sections. But then by tensoring with $\mathcal{L}$,

$$\mathcal{L}_X(\lambda)^{\otimes N}$$

is very ample. □

14. The total coordinate ring

Consider the sheaf of $\mathcal{O}_X$-modules

$$\bigoplus_{\lambda \in \Lambda} \mathcal{L}_X(\lambda).$$

Taking its relative spec gives a scheme $\hat{X}$ with a morphism

$$\hat{X} \xrightarrow{\pi} X.$$

The scheme $\hat{X}$ has a $\widetilde{G} \times \widetilde{G}$-action that is inherited from the action on X and a commuting $\widetilde{T}$-action along the fibers of π, which make the morphism π a $\widetilde{G} \times \widetilde{G}$-equivariant principal $\widetilde{T}$-bundle.

In particular, because the wonderful compactification X is spherical for the action of the Borel subgroup

$$\widetilde{B} \times \widetilde{B} \subset \widetilde{G} \times \widetilde{G},$$

the scheme $\hat{X}$ is spherical for the action of the Borel subgroup

$$\widetilde{B} \times \widetilde{B} \times \widetilde{T} \subset \widetilde{G} \times \widetilde{G} \times \widetilde{T}.$$

Proposition 14.1. *The scheme $\hat{X}$ is a quasi-affine variety.*

Proof. Fix a very ample invertible sheaf $\mathcal{L}$ on X. Then the invertible sheaves

$$\mathcal{L} \otimes \mathcal{L}_X(\chi_i)$$

are very ample and their classes form a basis of the Picard group $\mathrm{Pic}(X)$. Each one gives a projective embedding

$$X \hookrightarrow \mathbb{P}_i.$$

Let $\hat{\mathbb{P}}_i$ be the tautological bundle over $\mathbb{P}_i$. The commutative diagram

$$\begin{array}{ccc} \hat{X} & \longrightarrow & \prod_{i=1}^{l} \hat{\mathbb{P}}_i \\ \downarrow{\scriptstyle \pi} & & \downarrow \\ X & \hookrightarrow & \prod_{i=1}^{l} \mathbb{P}_i, \end{array} \tag{14.1}$$

is actually a pullback square. Since each $\hat{\mathbb{P}}_i$ is quasi-affine, being the complement of a point in an affine space, so is the pullback $\hat{X}$. □

Definition 14.2. The *total coordinate ring* of X is

$$R[X] = \bigoplus_{\lambda \in \Lambda} \Gamma(X, \mathcal{L}_X(\lambda)).$$

For a detailed introduction to total coordinate rings, see [ADHL]. In the case of wonderful varieties, they are discussed more generally in [Bri1], whose exposition we follow here in the special case of the wonderful compactification of G.

Remark 14.3. The ring $R[X]$ is the ring of regular functions on the spherical quasi-affine variety $\hat{X}$. By Section 30.5 in [Tim], it follows that $R[X]$ is finitely-generated and normal.

By the previous remark, we can define the normal affine variety

$$\widetilde{X} = \text{Spec } R[X].$$

It is the affine closure of $\hat{X}$, so it is equipped with an open embedding

$$\iota : \hat{X} \hookrightarrow \widetilde{X}.$$

Proposition 14.4. *The group $\widetilde{G} \times \widetilde{G} \times \widetilde{T}$ acts on $\widetilde{X}$ with open orbit*

$$\widetilde{X}_0 \cong \widetilde{G} \times_{\widetilde{Z}} \widetilde{T},$$

where $\widetilde{Z}$ is the center of $\widetilde{G}$.

Proof. The open $\widetilde{G} \times \widetilde{G} \times \widetilde{T}$-orbit on $\widetilde{X}$ is exactly the preimage under π of the open $\widetilde{G} \times \widetilde{G}$-orbit on the wonderful compactification X.

This open orbit is a homogeneous $\widetilde{G} \times \widetilde{G}$-space isomorphic to the group G, and from diagram (14.1) its preimage is

$$\widetilde{X}_0 = \pi^{-1}(G) \cong (\widetilde{G} \times \widetilde{G}) \times_{\text{Stab}_{\widetilde{G}\times\widetilde{G}}(e)} \widetilde{T},$$

where the torus $\widetilde{T}$ is recovered as the torus corresponding to the character group generated by $\chi_1, \ldots, \chi_l$. The stabilizer of the identity $e \in G$ is

$$\text{Stab}_{\widetilde{G}\times\widetilde{G}}(e) = \widetilde{G}_\Delta \times \widetilde{Z}_1,$$

where $\widetilde{G}_\Delta$ is the diagonal embedding of $\widetilde{G}$ into $\widetilde{G} \times \widetilde{G}$, and $\widetilde{Z}_1$ is the embedding of $\widetilde{Z}$ into the first coordinate of $\widetilde{Z} \times \widetilde{Z}$.

The factor $\widetilde{G}_\Delta$ acts on $\widetilde{T}$ trivially, and the factor $\widetilde{Z}_1$ acts on $\widetilde{T}$ by the fundamental weights $\chi_1, \ldots, \chi_l$. It follows that

$$\widetilde{X}_0 \cong (\widetilde{G} \times \widetilde{G}) \times_{\text{Stab}_{\widetilde{G}\times\widetilde{G}}(e)} \widetilde{T} \cong \widetilde{G} \times_{\widetilde{Z}} \widetilde{T}.$$ □

Let $R \subset \Lambda$ be the root lattice of G, and for every weight λ denote by

$$t^\lambda : \widetilde{T} \longrightarrow \mathbb{C}$$
$$z \longmapsto \lambda(z)$$

the corresponding character.

Proposition 14.5. *There is an isomorphism of $\widetilde{G} \times \widetilde{G} \times \widetilde{T}$-algebras*

$$\mathbb{C}[\widetilde{X}_0] \cong \bigoplus_{\lambda \in \Lambda} \left(\bigoplus_{\substack{\mu \in \Lambda^+ \\ \lambda - \mu \in R}} V_\mu \otimes V_\mu^* \right) t^\lambda,$$

where the right-hand side is viewed as a subalgebra of $\mathbb{C}[\widetilde{G} \times \widetilde{T}]$.

Proof. By Proposition 14.4, there is an isomorphism of $\widetilde{G} \times \widetilde{G} \times \widetilde{T}$-algebras

$$\mathbb{C}[\widetilde{X}_0] \cong \mathbb{C}[\widetilde{G} \times_{\widetilde{Z}} \widetilde{T}] \cong \left(\mathbb{C}[\widetilde{G}] \otimes \mathbb{C}[\widetilde{T}] \right)^{\widetilde{Z}}.$$

By the Peter-Weyl theorem, the first factor is

$$\mathbb{C}[\widetilde{G}] \cong \bigoplus_{\mu \in \Lambda^+} V_\mu \otimes V_\mu^*.$$

The second factor is

$$\mathbb{C}[\widetilde{T}] \cong \bigoplus_{\lambda \in \Lambda} \mathbb{C} t^\lambda.$$

Invariance under $\widetilde{Z}$ means exactly that

$$\mu_{|\widetilde{Z}} = \lambda_{|\widetilde{Z}},$$

which is to say that $\lambda - \mu \in R$. ☐

Theorem 14.6. *There is an isomorphism of $\widetilde{G} \times \widetilde{G} \times \widetilde{T}$-algebras*

$$\mathbb{C}[\widetilde{X}] \cong \bigoplus_{\lambda \in \Lambda} \left(\bigoplus_{\substack{\mu \in \Lambda^+ \\ \mu \leq \lambda}} V_\mu \otimes V_\mu^* \right) t^\lambda,$$

where the right-hand side is viewed as a subalgebra of $\mathbb{C}[\widetilde{G} \times \widetilde{T}]$, and the ordering $\mu \leq \lambda$ is the usual ordering on the weight lattice.

Proof. The regular functions on $\widetilde{X}$ form a subalgebra of the regular functions on $\widetilde{X}_0$, and in view of Proposition 14.5 this gives an embedding

$$R[X] = \mathbb{C}[\widetilde{X}] \overset{\iota}{\hookrightarrow} \bigoplus_{\lambda \in \Lambda} \left(\bigoplus_{\substack{\mu \in \Lambda^+ \\ \lambda - \mu \in R}} V_\mu \otimes V_\mu^* \right) t^\lambda.$$

The canonical section

$$\sigma_i \in \Gamma(X, \mathcal{L}_X(\alpha_i))$$

is $\widetilde{G} \times \widetilde{G}$-invariant and a $\widetilde{T}$-eigenfunction with weight α_i. In the target there is a unique $\widetilde{T}$-eigenspace of weight α_i, so up to scalars

$$\iota(\sigma_i) = t^{\alpha_i}.$$

The canonical section

$$\tau_i \in \Gamma(X, \mathcal{L}_X(\alpha_i))$$

is a $\widetilde{B} \times \widetilde{B}$-eigenfunction with weight $(\chi_i, -w_0\chi_i)$, and a $\widetilde{T}$-eigenfunction of weight χ_i. It follows that

$$\iota(\tau_i) = (v_i \otimes w_i)t^{\chi_i},$$

where v_i is a highest weight vector of the fundamental representation V_{χ_i}, and w_i is a highest weight vector of its dual.

Because of this, the only degrees (μ, λ) that appear in the image of ι are those for which

$$\mu \leq \lambda,$$

and the theorem is proved. □

Corollary 14.7. *There is an isomorphism of $\widetilde{G} \times \widetilde{G}$-modules*

$$\Gamma(X, \mathcal{L}_X(\lambda)) \cong \bigoplus_{\substack{\mu \in \Lambda^+ \\ \mu \leq \lambda}} V_\mu \otimes V_\mu^*.$$

Remark 14.8. In particular, and unlike for the flag variety G/B, some line bundles on the wonderful compactification that correspond to non-dominant weights have global sections. For instance, any simple root α_i is greater than the 0-weight, and so

$$\Gamma(X, \mathcal{L}_X(\alpha_i)) \cong V_0 \otimes V_0^* \cong \mathbb{C}.$$

Remark 14.9. The affine variety $\widetilde{X}$ has the structure of a monoid. Let $V_1, \ldots, V_l$ be the fundamental representations of $\widetilde{G}$, let $V_1^*, \ldots, V_l^*$ be their duals, and for each $i = 1, \ldots, l$ let

$$\rho_i : \widetilde{G} \longrightarrow V_i^* \otimes V_i$$

be the representation map.

The ring of regular functions

$$\mathbb{C}[\widetilde{X}_0] \cong \bigoplus_{\lambda \in \Lambda} \left(\bigoplus_{\substack{\mu \in \Lambda^+ \\ \lambda - \mu \in R}} V_\mu \otimes V_\mu^* \right) t^\lambda,$$

from Proposition 14.5 gives an embedding

$$\begin{aligned} \psi : \widetilde{G} \times_{\widetilde{Z}} \widetilde{T} &\hookrightarrow \mathbb{C}^l \times \prod_{i=1}^{l} (V_i^* \otimes V_i) \\ (g, t) &\longmapsto (\alpha_1(t), \ldots, \alpha_l(t), \chi_1(t)\rho_1(g), \ldots, \chi_l(t)\rho_l(g)) \,. \end{aligned}$$

The variety $\widetilde{X}$ is nothing but the closure of the image of ψ, and in view of the proof of Proposition 14.1 the quasi-affine variety $\hat{X}$ is the closure of the image of ψ in

$$\mathbb{C}^l \times \prod_{i=1}^{l} ((V_i^* \otimes V_i) \backslash \{0\}) \,.$$

In fact, the variety $\widetilde{X}$ is the enveloping monoid studied by Vinberg in [Vin]. It sits above the wonderful compactification X as a multi-cone, and taking the quotient of the semistable locus $\hat{X}$

by the action of $\widetilde{T}$ gives an isomorphism

$$X \cong \hat{X}/\widetilde{T}.$$

Remark 14.10. The Vinberg monoid is universal in the following sense. Suppose that S is a monoid whose group of units $G(S)$ is a reductive algebraic group—such monoids are called *reductive.* Let

$$G_S = [G(S), G(S)]$$

be the derived subgroup of the groups of units of S, and let

$$A(S) = \mathrm{Spec}\ \mathbb{C}[S]^{G_S \times G_S}$$

be the invariant-theoretic quotient of S by the two-sided action of G_S.

The variety $A(S)$ is called the *abelianization* of S, it is normal if S is normal, and there is a canonical surjective morphism

$$\alpha : S \longrightarrow A(S).$$

(See [PV].) The monoid S is called *flat* if α is flat. Moreover, any homomorphism

$$\varphi : S' \longrightarrow S$$

of reductive monoids descends to a homomorphism of their abelianizations:

$$\begin{array}{ccc} S' & \xrightarrow{\ \varphi\ } & S \\ {\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\alpha} \\ A(S') & \longrightarrow & A(S). \end{array} \tag{14.2}$$

Now fix a connected semisimple algebraic group G_0, and consider the category $\mathcal{C}(G_0)$ of flat reductive monoids S which are normal, contain a zero, and such that

$$G_S \cong G_0.$$

There is a distinguished monoid $S \in \mathcal{C}(G_0)$—the *enveloping monoid* of G_0—with the property that for any $S' \in \mathcal{C}(G_0)$ and any isomorphism

$$\varphi_0 : G_{S'} \longrightarrow G_S$$

there is a unique homomorphism

$$\varphi : S' \longrightarrow S$$

extending φ_0 and such that the diagram (14.2) is a pullback square—that is,

$$S' \cong A(S') \times_{A(S)} S.$$

The Vinberg monoid $\widetilde{X}$ from above is the enveloping monoid of $\widetilde{G}$.

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