GEOMETRIC STRUCTURES OF SEMISIMPLE LIE ALGEBRAS

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1. Introduction

The aim of this paper is to explore the geometry of a Lie algebra $\mathfrak{g}$ through the action of its nilpotent elements and through the structure of the algebra $\mathbb{C}[\mathfrak{g}]^G$ of polynomial invariants. These two approaches will converge to give broad results about the geometry of the G-orbits in $\mathfrak{g}$ and about the closure relations between them.

When we inspect the adjoint action of nilpotent elements we will primarily be concerned with a distinguished class of nilpotents called principal nilpotents. These give rise to a copy of $\mathfrak{sl}_2$ inside $\mathfrak{g}$ that is, up to conjugacy, unique. The invariant polynomials in $\mathbb{C}[\mathfrak{g}]^G$ behave nicely along planes determined by this subalgebra, and the adjoint action of the subalgebra itself reflects the structure of $\mathbb{C}[\mathfrak{g}]^G$.

Moreover, there is a surjection $\mathfrak{g} \twoheadrightarrow \mathbb{C}^l$ obtained via some classical results by mapping each element to its image under the generators of $\mathbb{C}[\mathfrak{g}]^G$. This surjection encodes much of the orbit structure of $\mathfrak{g}$, since its fibers are closures of G-orbits, and the "most common" fibers are in fact single semisimple G-orbits. The orbit structure in these fibers mirrors and generalizes the orbit structure in the set of nilpotent elements.

2. The Jacobson-Morozov Theorem

Throughout this paper, let G be a complex semisimple connected algebraic group of rank l, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ the Cartan subalgebra and $\mathfrak{n}$ the nilradical of $\mathfrak{g}$. Let $\mathbb{C}[\mathfrak{g}]^G$ be the algebra of G-invariant polynomials on $\mathfrak{g}$. For an element $x \in \mathfrak{g}$, denote its centralizers in G and $\mathfrak{g}$ by

$$Z_G(x) = \{g \in G \mid Ad(g) \cdot x = x\}$$
$$\mathfrak{g}^x = \{y \in \mathfrak{g} \mid ad(y) \cdot x = 0\}$$

respectively. Notice that $\mathfrak{g}^x$ is precisely the kernel of the $\mathrm{ad}(x)$-action on $\mathfrak{g}$, and that the Lie algebra of $Z_G(x)$ is precisely $\mathfrak{g}^x$.

Theorem 2.1. (Jacobson-Morozov)[Bou05, 11.1.2] *For any nilpotent element $e \in \mathfrak{g}$ there are elements $h, f \in \mathfrak{g}$ such that*

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h,$$

and moreover h is semisimple and f is nilpotent.

Henceforth such a triple (e, h, f) will be called an $\mathfrak{sl}_2$-*triple*.

Example 2.2. Let $\mathfrak{g} = \mathfrak{sl}_\mathfrak{n}$. Every nilpotent element is conjugate to a matrix of nilpotent Jordan blocks, so we only need to consider the case where e is a single block of size $m \times m$. An $\mathfrak{sl}_2$-triple in this case is given by the following:

$$e = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & 1 & & \\ & & \ddots & 1 & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} m-1 & 0 & & & \\ 0 & m-3 & & & \\ & & \ddots & & \\ & & & -m+3 & 0 \\ & & & 0 & -m+1 \end{pmatrix}$$

$$f = \begin{pmatrix} 0 & 0 & & & \\ m-1 & 0 & & & \\ & 2(m-2) & \ddots & & \\ & & 2(-m+2) & 0 & 0 \\ & & & -m+1 & 0 \end{pmatrix}$$

When $m = 3$, this becomes

$$e = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad h = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}.$$

Remark 2.3. It is natural to ask to what extent the $\mathfrak{sl}_2$-triples of Jacobson-Morozov are determined by the choice of e. In fact, it can be shown (see [CG97, 3.7.3]) that any two triples (e, h, f) and (e, h', f') containing the same nilpotent e are conjugate in $Z_G(e)$—thus, there is a bijective correspondence between the nilpotent G-orbits in $\mathfrak{g}$ and the conjugacy classes of $\mathfrak{sl}_2$-triples.

3. The Nilpotent Cone

Let $\mathcal{N}$ be the subvariety of nilpotent elements in $\mathfrak{g}$. Since any scalar multiple of a nilpotent element is also nilpotent, $\mathcal{N}$ is an algebraic cone. The following proposition gives some properties of $\mathcal{N}$ and, more importantly, a criterion for nilpotency that is independent of the notion of representation:

Proposition 3.1. [Dix96, 8.1.3] *The nilpotent cone $\mathcal{N}$ is an irreducible subvariety of $\mathfrak{g}$ of dimension* $2\dim(\mathfrak{n})$. *An element x is in $\mathcal{N}$ if and only if $P(x) = 0$ for every polynomial $P \in \mathbb{C}[\mathfrak{g}]^G_+$, the subalgebra of $\mathbb{C}[\mathfrak{g}]^G$ of polynomials with no constant term.*

By Remark 2.3, to gain a complete picture of the conjugacy classes of $\mathfrak{sl}_2$-triples we need to understand the nilpotent orbits in $\mathfrak{g}$.

Definition 3.2. An element $x \in \mathfrak{g}$ is *regular* if $\dim Z_G(x) = \dim \mathfrak{g}^x = \operatorname{rk} \mathfrak{g}$, and the set of regular elements in $\mathfrak{g}$ is denoted by $\mathfrak{g}^{reg}$. A nilpotent element that is also regular is *principal*, and an $\mathfrak{sl}_2$-triple whose nilpotent element is principal is a *principal $\mathfrak{sl}_2$-triple.*

Theorem 3.3. [Dix96, 8.1.3] *The nilpotent cone $\mathcal{N}$ is a union of finitely many G-orbits, and the set of principal nilpotents $\mathcal{N} \cap \mathfrak{g}^{reg}$ forms a single, open, dense G-orbit in $\mathcal{N}$.*

Proof. The idea of the proof is to embed $\mathfrak{g}$ into some matrix algebra $\mathfrak{gl}_n$. The Lie algebra $\mathfrak{gl}_n$ has finitely many nilpotent GL_n-conjugacy classes since any nilpotent matrix is conjugate to a matrix of nilpotent Jordan blocks. It can be proved by a simple analysis of tangent spaces (see [Dix96, 8.1.2]) that for any GL_n-orbit $\mathcal{O}$ in $\mathfrak{gl}_n$, the irreducible components of the intersection $\mathcal{O} \cap \mathfrak{g}$ are G-orbits, completing the proof of the first part.

To show the second part, note that since $\mathcal{N}$ is irreducible, it contains a unique open orbit $G \cdot x$ of maximal dimension

$$2\dim(\mathfrak{n}) = \dim(G \cdot x) = \dim(G) - \dim(Z_G(x)).$$

Then $\dim(Z_G(x))=l$ and x is a principal nilpotent, conjugate to all other principal nilpotents by the uniqueness of the orbit. □

Example 3.4. Consider once again $\mathfrak{g} = \mathfrak{sl}_n$. The ring of invariants is generated by functions of the form $x \mapsto \mathrm{tr}(x^n)$ (see [Dix96, 7.3.5(ii)]), so Proposition 3.1 says that an element of $\mathfrak{sl}_n$ is nilpotent if and only if the trace of all its powers vanishes.

The first part of Theorem 3.3 is clear for $\mathfrak{sl}_n$ by Jordan normal form. The second part tells us that the orbit of the $n \times n$-dimensional nilpotent Jordan block is dense and open in the nilpotent cone of $\mathfrak{sl}_n$.

4. The Principal $\mathfrak{sl}_2$-Triple

As shown in Theorem 3.3, the principal nilpotent elements form a single G-orbit in $\mathfrak{g}$, so all the principal $\mathfrak{sl}_2$-triples are conjugate to each other. Thus when we speak of a principal $\mathfrak{sl}_2$-triple acting on $\mathfrak{g}$, we are discussing an action that is, up to conjugation, unique.

With respect to this action, the Lie algebra $\mathfrak{g}$ decomposes as a direct sum of irreducible $\mathfrak{sl}_2$-representations, the structure of which is well-known (see [FH72]). Since each of these irreducible representations contributes one dimension to the kernel of e, the nilpotent e is principal if and only if there are exactly l such irreducibles in the decomposition. In this case the centralizer of h has dimension at most l, so h is also regular and each of the irreducible representations contains a 0-eigenspace for h. Thus $\mathfrak{g}$ decomposes as a sum of l odd-dimensional irreducible (e, h, f)-representations, and all the eigenvalues of h on $\mathfrak{g}$ are even integers.

From now on, let (e, h, f) be a principal $\mathfrak{sl}_2$-triple.

Theorem 4.1. [Kos59, 8.7] (cf. also [Dix96, 8.1.1(iii)]) *Let $\mathfrak{g} = \oplus_{i=1}^{l} V_i$ be the decomposition of $\mathfrak{g}$ into irreducible representations of (e, h, f), with $\dim(V_i) = 2\lambda_i + 1$ ordered such that $\lambda_1 \leq \ldots \leq \lambda_l$. Let $d_1 \leq \ldots \leq d_l$ be the degrees of the homogeneous generators $f_1, \ldots, f_l$ of the invariant polynomial algebra $\mathbb{C}[\mathfrak{g}]^G$. Then $d_i = \lambda_i + 1$.*

Proof. Using the decomposition $\mathfrak{g} = [\mathfrak{g}, e] \oplus \mathfrak{g}^f$, take a basis $y_1, \ldots, y_l$ of $\mathfrak{g}^f$ such that $h \cdot y_i = -2\lambda_i y_i$, and define the function

$$\begin{aligned} \psi : G \times \mathbb{C}^l &\longrightarrow \mathfrak{g} \\ (g, \zeta_1, \ldots, \zeta_l) &\longmapsto g \cdot (e + \zeta_1 y_1 + \ldots + \zeta_l y_l). \end{aligned}$$

The differential of ψ is surjective, so its image $G\cdot(e+\mathfrak{g}^f)$ is dense in $\mathfrak{g}$, and the restriction

$$\begin{aligned}\mathbb{C}[\mathfrak{g}]^G &\longrightarrow \mathbb{C}[e+\mathfrak{g}^f]\\ f &\longmapsto f_{|e+\mathfrak{g}^f}\end{aligned}$$

is an injection.

By applying the Euler formula (where x is viewed as a tangent vector to $\mathfrak{g}$ at x)

$$\langle x, \nabla f\rangle = (\deg\, f) f(x),$$

we obtain for every invariant homogeneous generator f_j

$$d_j(f_j\circ\psi)(\zeta_1,\dots,\zeta_l) = \sum_{i=1}^{l}(1+\lambda_i)\zeta_i\frac{\partial(f_j\circ\psi)}{\partial\zeta_i}(\zeta_1,\dots,\zeta_l).$$

Thus if c_{ij} is the sum of the exponents of ζ_i in $f_j\circ\psi$,

$$\sum_i(1+\lambda_i)c_{ij} = d_j.$$

If there is some index j_0 such that $(1+\lambda_{j_0}) > d_{j_0}$, the polynomials $f_1\circ\psi,\dots,f_{j_0}\circ\psi$ are algebraically dependent, contradicting the injectivity of $f\mapsto f_{|e+\mathfrak{g}^f}$. So

$$d_1+\ldots+d_l \;\geq\; l+\lambda_1+\ldots+\lambda_l \;=\; \frac{1}{2}(l+\dim\mathfrak{g}).$$

This equation is an equality, by the equality (see [CG97, 6.7.17]) of Poincaré polynomials

$$\sum_{w\in W} t^{l(w)} = \frac{\prod(1-t^{d_i})}{(1-t)^l}.$$

Thus equality holds for each degree and $d_j = 1+\lambda_j$. □

Example 4.2. When $\mathfrak{g} = \mathfrak{sl}_n$, it is known that the degrees of the homogeneous invariant generators are $2, 3, \dots, n$, so according to Theorem 4.1 the Lie algebra $\mathfrak{sl}_n$ decomposes as a sum of $n-1$ irreducible representations of dimensions $3, 5, \dots, 2n-1$.

Proposition 4.3. [Dix96, 8.1.1(iv)] *The differentials $df_1,\dots,df_l$ are linearly independent at every point of $e+\mathfrak{g}^f$.*

Proof. By the result of the previous theorem, $f_j\circ\psi$ is independent of ζ_i whenever $d_i > d_j$, and c_{ij} is either 0 or 1 when $d_i = d_j$. Then $f_j\circ\psi$ can be written

$$f_j\circ\psi(\zeta_1,\dots,\zeta_l) = \sum_{d_k=d_j}\alpha_{kj}\zeta_k + g_j$$

where the α_{kj}'s are scalars and g_j depends only on the ζ_i's such that $d_i < d_j$. The Jacobian of the $f_j\circ\psi$'s is then block-upper-triangular and the diagonal blocks are constant, so

$$\det\left(\frac{\partial(f_j\circ\psi)}{\partial\zeta_i}\right) = \prod_{d_k=d_j}\det\left(\alpha_{kj}\right).$$

Since the $f_j \circ \psi$'s are algebraically independent, this constant determinant is non-zero. The function ψ is linear when restricted to $\{1\} \times \mathbb{C}^l$, and since the f_j's are G-invariant we can apply this restriction without loss of generality, so the matrix $(\partial_{y_i} f_j)$ is also nonsingular. $\square$

5. The Principal Slice

Let (e, h, f) be a principal $\mathfrak{sl}_2$-triple, and $\mathfrak{g}^f$ the centralizer of f in $\mathfrak{g}$. Then the plane $e + \mathfrak{g}^f$ will be called the *principal slice*. It is distinguished by the fact that all its elements are regular and every G-orbit on $\mathfrak{g}^{reg}$ intersects it in exactly one point.

Theorem 5.1. [Kos63, 4.7.8] *Every element of $e + \mathfrak{g}^f$ is regular, and the composition*

$$e + \mathfrak{g}^f \hookrightarrow \mathfrak{g}^{reg} \twoheadrightarrow \mathfrak{g}^{reg}/G$$

gives an isomorphism of $e + \mathfrak{g}^f$ with the G-orbit space of $\mathfrak{g}^{reg}$.

Proof. Define a function

$$\begin{aligned} f : \mathfrak{g} &\to \mathbb{C}^l \\ x &\mapsto (f_1(x), \ldots, f_l(x)) \end{aligned}$$

The function f restricts to a function on the principal slice, and since the f_i's are G-invariant it also restricts to a function f' on the orbit space. We have the following commutative diagram:

$$\begin{array}{ccc} e + \mathfrak{g}^f & \longrightarrow & \mathfrak{g}^{reg}/G \\ & \searrow^{f} & \downarrow{\scriptstyle f'} \\ & & \mathbb{C}^l \end{array}$$

By Propositions 5.2 and 5.3 that follow, f' is injective and f is an isomorphism, so the commutative diagram gives an isomorphism $e + \mathfrak{g}^f \xrightarrow{\sim} \mathfrak{g}^{reg}/G$. $\square$

Proposition 5.2. [Kos63, 3.5.2] *The map $f' : \mathfrak{g}^{reg}/G \longrightarrow \mathbb{C}^l$ is an injection.*

Proof. For any $x \in \mathfrak{g}$, let $x = s + n$ be its Jordan decomposition. Since $s + n$ is conjugate to $s + cn$ for any $c \in \mathbb{C}^*$, the value of any G-invariant polynomial on x is determined by its semisimple part. Thus, it suffices to show that f' is injective on semisimple orbits. This follows from an application of the Chevalley Restriction Theorem (see 6.1), by an averaging under the action of the (finite) Weyl group W. $\square$

Proposition 5.3. [Kos63, 4.7.7] *The map $f : e + \mathfrak{g}^f \longrightarrow \mathbb{C}^l$ is an isomorphism.*

Proof. Recall the injective homomorphism $\gamma : \mathfrak{sl}_2 \longrightarrow \mathfrak{g}$ given by the Jacobson-Morozov theorem. This homomorphism induces $\gamma : SL_2 \longrightarrow G$, which gives an embedding of the torus $\mathbb{C}^*$ of SL_2 into G.

We equip both the principal slice and the vector space $\mathbb{C}^l$ with $\mathbb{C}^*$-actions as follows:

$$t \cdot (e + x) = t^2(\gamma(t^{-1}) \cdot (e + x))$$
$$t \cdot (\zeta_1, \ldots, \zeta_l) = (t^{2d_1}\zeta_1, \ldots, t^{2d_l}\zeta_l).$$

The coordinate rings of $e + \mathfrak{g}^f$ and $\mathbb{C}^*$ are graded by $\mathbb{C}^*$-weights,

$$\mathbb{C}[e + \mathfrak{g}^f] = \oplus A_i$$
$$\mathbb{C}[x_1, \ldots, x_l] = \oplus B_i,$$

and since f is $\mathbb{C}^*$-equivariant, the pullback preserves the grading:

$$f^* : B_i \longrightarrow A_i.$$

To show f is an isomorphism of varieties, it suffices to show that f^* is an isomorphism of rings. Since A_i and B_i are finite-dimensional, for this it is enough to show that $f^* : B_i \longrightarrow A_i$ is injective and that $\dim(B_i)$=$\dim(A_i)$. Both of these facts follow from inspecting the differential

$$df : T_x(e + \mathfrak{g}^f) \longrightarrow T_0\mathbb{C}^l,$$

which is bijective since the differentials $df_1, \ldots, df_l$ are linearly independent at any point of the principal slice by Proposition 4.3. □

6. The Ring of Polynomial Invariants

Now we are interested in the ring $\mathbb{C}[\mathfrak{g}]^G$ itself. Let $\mathbb{C}[\mathfrak{h}]^W$ denote the algebra of polynomials on the Cartan subalgebra $\mathfrak{h}$ invariant under the action of the Weyl group W. Since G-conjugate elements in $\mathfrak{h}$ are in fact conjugate under W, the usual restriction gives a map

$$\mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}[\mathfrak{h}]^W.$$

Theorem 6.1. (Chevalley Restriction Theorem)[Hum72, 23.1] *The restriction map $\mathbb{C}[\mathfrak{g}]^G \to \mathbb{C}[\mathfrak{h}]^W$ is an isomorphism.*

This isomorphism produces an embedding $\mathbb{C}[\mathfrak{h}]^W \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}]^G \hookrightarrow \mathbb{C}[\mathfrak{g}]$ that induces a morphism of algebraic varieties $\rho : \mathfrak{g} \longrightarrow \mathfrak{h}/W$. We are interested in the fiber above a point $\chi \in \mathfrak{h}/W$:

$$\mathcal{V}_\chi = \{x \in \mathfrak{g} \mid \rho(x) = \chi\}.$$

The fiber above 0 is precisely the nilpotent cone $\mathcal{N}$, and many of the properties of $\mathcal{N}$ generalize to the fibers $\mathcal{V}_\chi$.

Theorem 6.2. [Kos63, 4.8.10] (cf. also [CG97, 6.7.2]) *Let $N = \dim(\mathfrak{n})$. The map $\rho : \mathfrak{g} \to \mathfrak{h}/W$ is a surjective morphism with $2N$-dimensional irreducible fibers $\mathcal{V}_\chi$ such that:*

(i) *$\mathcal{V}_\chi$ is a G-stable closed subvariety of $\mathfrak{g}$ and consists of finitely many G-orbits.*
(ii) *$\mathcal{V}_\chi \cap \mathfrak{g}^{reg}$ is the unique open dense G-conjugacy class in $\mathcal{V}_\chi$.*
(iii) *$\mathcal{V}_\chi \cap \mathfrak{g}^{ss}$ is the unique closed G-conjugacy class of minimal dimension.*
(iv) *$\mathcal{V}_\chi$ is a single G-orbit if and only if it contains a regular semisimple element.*

Proof. Let $\mathcal{V}_h$ be the fiber above the orbit of h, and let $\mathcal{N}^h$ be the set of nilpotent elements inside $\mathfrak{g}^h$. Every element $x \in \mathcal{V}_h$ has Jordan decomposition $x = h + n$ for some $n \in \mathcal{N}^h$, so that $\mathcal{V}_h = G \cdot (h + \mathcal{N}^h)$.

If h is regular, $\mathcal{V}_h$ is a single orbit of dimension $2N$, so the generic fiber is $2N$-dimensional. If h is not regular, we have

$$\dim(\mathcal{V}_h) \geq 2N$$
$$\dim(\mathcal{V}_h) \leq \dim(G \times_{Z_G(h)} (h + \mathcal{N}^h)) \leq 2N$$

and again $\dim(\mathcal{V}_h)$=2N.

Since $\mathcal{N}^h$ is the nilpotent cone of $\mathfrak{g}^h$, it has the properties of Theorem 3.3, and from this (i) and (ii) follow.

Part (iii) follows from the fact that every nilpotent orbit contains 0 in its closure—thus, the semisimple conjugacy class $G \cdot (h+0)$ forms a unique closed orbit of minimal dimension in $\mathcal{V}_h$. Parts (ii) and (iii) immediately imply (iv). □

We can view a point $\chi \in \mathfrak{h}/W = \operatorname{Specm} \mathbb{C}[\mathfrak{g}]^G$ as a maximal ideal and thus as a homomorphism $\chi : \mathbb{C}[\mathfrak{g}]^G \longrightarrow \mathbb{C}$.

Theorem 6.3. [Kos63, 5.1.16] (cf. also [CG97, 6.7.3]) *For each* $\chi \in \operatorname{Specm} \mathbb{C}[\mathfrak{g}]^G = \mathfrak{h}/W$

(i) $f \in \mathbb{C}[\mathfrak{g}]$ *vanishes on the fiber* $\mathcal{V}_\chi$ *if and only if* $f \in \mathbb{C}[\mathfrak{g}]{\cdot}\ker(\chi)$. *The ring of regular functions on* $\mathcal{V}_\chi$ *is*
$$\mathcal{O}(\mathcal{V}_\chi) = \frac{\mathbb{C}[\mathfrak{g}]}{\mathbb{C}[\mathfrak{g}] \cdot \ker(\chi)}.$$
(ii) *The ring* $\mathcal{O}(\mathcal{V}_\chi)$ *is normal.*
(iii) *The natural restriction* $\mathcal{O}(\mathcal{V}_\chi) \longrightarrow \mathcal{O}(\mathcal{V}_\chi^{reg})$ *is an isomorphism.*

Proof. The proof of (i) follows from observing that the algebra $\mathbb{C}[\mathfrak{g}]$ is free of rank l over $\mathbb{C}[\mathfrak{g}/\mathfrak{h}] \otimes_{\mathbb{C}} \mathbb{C}[\mathfrak{g}]^G$. Then the quotient $\mathbb{C}[\mathfrak{g}]/\mathbb{C}[\mathfrak{g}] \cdot \ker\chi$ is free of rank l over $\mathbb{C}[\mathfrak{g}/\mathfrak{h}]$, so it is Cohen-Macaulay. This and the linear independence of the differentials $df_1, \ldots, df_l$ on $\mathcal{V}_\chi^{reg}$ imply (i).

Since any G-conjugacy class in $\mathcal{V}_\chi$ is a symplectic manifold, it is even-dimensional, and thus
$$\dim(\mathcal{V}_\chi \backslash \mathcal{V}_\chi^{reg}) \leq \dim(\mathcal{V}_\chi) - 2.$$
Parts (ii) and (iii) follow from this and from the irreducibility of $\mathcal{V}_\chi$ (see [CG97, 2.2.11]). □

Definition 6.4. A polynomial $f \in \mathbb{C}[\mathfrak{g}]$ is called *G-harmonic* if whenever D is a G-invariant differential operator with zero constant term, we have $D \cdot f = 0$.

Theorem 6.5. [Kos63, 4.8.11, 5.1.16] (cf. also [CG97, 6.7.4]) *Let* $\mathcal{H}$ *denote the subalgebra of G-harmonic polynomials in* $\mathbb{C}[\mathfrak{g}]$.

(i) $\mathbb{C}[\mathfrak{g}]$ *is a free* $\mathbb{C}[\mathfrak{g}]^G$*-module, and multiplication gives an isomorphism*
$$\mathbb{C}[\mathfrak{g}]^G \otimes_{\mathbb{C}} \mathcal{H} \xrightarrow{\sim} \mathbb{C}[\mathfrak{g}].$$

(ii) *For every fiber, the G-module $\mathcal{O}(\mathcal{V}_\chi)$ decomposes as a direct sum of finite-dimensional simple G-modules. If V is one of these simple components, it appears with multiplicity equal to the dimension of V_0, the 0-weight space for the action of the torus $T \subseteq G$.*

Proof. Freeness follows from the proof of 6.3(i). To see that the multiplication map is an isomorphism, it suffices to check surjectivity inductively on the degree of elements in $\mathbb{C}[\mathfrak{g}]$ and then to show that the Poincaré polynomials of the two sides are equal.

To prove (ii), one checks that the composition

$$\mathcal{H} \hookrightarrow \mathbb{C}[\mathfrak{g}] \twoheadrightarrow \frac{\mathbb{C}[\mathfrak{g}]}{\mathbb{C}[\mathfrak{g}] \cdot \ker(\chi)} = \mathcal{O}(\mathcal{V}_\chi)$$

is a G-equivariant isomorphism. Then the G-module structure of $\mathcal{O}(\mathcal{V}_\chi)$ is independent of χ, and we may choose χ to be regular and semisimple, so that $\mathcal{V}_\chi \cong G/T$.

An analogue of the Peter-Weyl theorem gives us

$$\mathcal{O}(G) = \bigoplus (E \otimes \check{E})$$

where the sum is over all simple G-modules and $\check{E}$ is the contragredient representation of E. Then for any irreducible summand V,

$$[\mathcal{O}(\mathcal{V}_\chi) : V] = [\mathcal{O}[G/T] : V] = \sum [E \otimes \check{E}^T : V] = \dim \check{V}^T = \dim V_0$$

and (ii) is proved. ☐

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An Outline of the Cited Literature

The contents of this proposal have been collected from a variety of sources. Section 2 describes the Jacobson-Morozov theorem and some of its consequences. This theorem appears in many standard texts, as well as in [Kos59, 3.4], and most of these texts give an elementary, but tricky and lengthy, proof. An alternative proof can be found in [CG97, 3.7.1]—it relies on more representation-theoretic machinery, but produces a more streamlined and intuitive proof.

Section 3 gives a description of the nilpotent cone of a Lie algebra—this is necessary in order to understand the G-orbit structure of the nilpotent elements, which in turn sheds light on the conjugacy classes of $\mathfrak{sl}_2$-triples. The statements here are collected from [Dix96, 8.1], but many of the proofs come from [CG97, 3.2], where a more detailed discussion of the nilpotent cone can also be found.

Section 4 explicitly outlines the action of a principal $\mathfrak{sl}_2$-triple on $\mathfrak{g}$. This is one of the main results of [Kos59], wherein it is proved using Coxeter-Killing transformations. Coxeter observed [Cox51] and Coleman proved [Col58] that this type of transformation exhibits in its eigenvalues the exponents of the corresponding Lie algebra. However, this method of proof is quite cumbersome, so we outline instead the proof found in [Dix96].

Section 5, on the other hand, follows [Kos63] closely, and the structure of the main result is the same. It is only in Proposition 5.3 that we take a less geometric and more algebraic approach to the proof.

Section 6 is the one that requires the heaviest machinery and the most algebro-geometric background. The results originate in [Kos63], but we follow the proofs of [CG97]. These proofs rely heavily on the structure of the nilpotent cone and on some results on algebraic varieties [CG97, 2.2] and some analysis of the space of harmonic polynomials [CG97, 6.3].