Method of Riemann surfaces in the study of supercavitating flow around two hydrofoils in a channel

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Abstract

In the framework of the Tulin model of supercavitating flow, the problem of reconstructing the free surface of a channel and the shapes of the cavities behind two hydrofoils placed in an ideal fluid is solved in closed form. The conformal map that transforms a parametric plane with three cuts along the real axis into the triple-connected flow domain is found by quadratures. The use of the theory of Riemann surfaces (the Schottky doubles) enables the non-linear model problem to be reduced to two separate Riemann–Hilbert problems on a hyperelliptic surface of genus two. The solution to the first problem is a rational function with certain zeros and poles on a Riemann surface. The second problem is solved in terms of singular integrals with the Weierstrass kernel. The essential singularities of the solution at the infinite points of the surface due to a pole of the kernel are removed by solving a real analogue of the Jacobi inversion problem on the surface. The unknown parameters of the conformal map are recovered from a system of certain algebraic and transcendental equations.

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1. Introduction

When an obstacle is placed stationary in a moving fluid the flow breaks away from the barrier along separating streamlines, and a wake (a “dead-zone”) forms behind the obstacle. In the case of high-speed flow the wake becomes a vapor-filled cavity. For water under atmospheric pressure for example, this occurs when the speed is 100 \( f/s \) or more (lower speeds under reduced pressure) \cite{1}. Cavitating flow has been studied intensively since marine engineers at the end of the 19th century became aware of the serious problem due to cavitation: an increase in pressure causes the cavities to collapse and release energy resulting in a force which may damage submarine propellers when they are operating at certain depths. Cavitation has long been of interest not only in the field of shipbuilding and hydraulic machinery, but also in chemical processing, nuclear physics and medicine (potential bioeffects of ultrasound caused by acoustic cavitation in blood vessels).

Modeling of cavitation is based on the results by Brillouin (1911) (see, e.g. \cite{1}) who proved that the maximum velocity must be attained on the free surface and also that the boundary of a cavity is convex. Good references to the theory of cavitating flow around obstacles are \cite{1–3}. Tulin \cite{4} proposed a model of the cavitating flow which admits the presence of the singularity of the solution at the point say, \( C \), where the two streamlines along the cavity attempt to close it, namely,

\[
\log(\frac{dw}{dz}) \sim K (w - w_0)^{-1/2},
\]

\[
z \to C, \quad K = \text{const},
\]

where \( w_0 = w(C), w = w(z) \) is the complex potential of the motion. This condition extends the class of solutions to the governing boundary-value problem which makes possible to reconstruct a flow that meets the condition

\[
\oint_L dz = 0,
\]

and is therefore single-valued. Here \( L \) is the boundary of the cavity combined with the boundary of the hydrofoil.
For model problems of cavitating flow, numerical and analytical methods are employed. The former technique requires the solution of associated integral equations with the contour to be recovered. The analytical methods for the study of cavitating flow in simply- and double-connected domains are well developed [1,3,5]. These methods are based on the use of a Schwarz–Christoffel transformation to map a circle, a quadrant, or a plane with a cut (simply-connected flow), or an annulus, a rectangular, or a plane with two cuts on the real axis (double-connected flow) into the flow domain. A closed-form solution to the model problem can then be found in terms of elementary and elliptic functions for a simply- and double-connected flow, respectively.

In the case of free boundary problems in a triple-connected flow domain, the problem of fluid mechanics can be formulated [6] as the Hilbert boundary-value problem with a piecewise constant coefficient on three cuts along the real axis. The actual problem solved in [6] concerned non-cavitating flow of a fluid around a single foil in a half-plane with a free surface. A method of Riemann surfaces for an elasticity problem \([\text{A method of Riemann surfaces for an elasticity problem }\] for the solution \([\text{of the Riemann–Hilbert problem on a Riemann surface. This}}\) is represented as a quotient of two functions, \(d_w/dz\), \(d_z\), \(w/\zeta\), where \(w\) is a complex potential of the flow. In Section 4 the function \(d_w/dz\) is found as a rational function with certain zeros and poles on a hyperelliptic surface of genus two. To define the function \(\omega(\zeta) = \log(V_{\infty}^{-1} d_w(z)/d\zeta)\) \((V_{\infty}\text{ is the speed at infinity), in Section 5 we reduce the Hilbert problem on the three cuts with a piecewise constant coefficient to the Riemann–Hilbert problem on the Riemann surface introduced in Section 4. Its solution is found by quadratures in terms of singular integrals with the Weierstrass kernel. Initially, it has an inadmissible exponential growth at infinity. The conditions which make the solution bounded at infinity are written as the Jacobi inversion problem for hyperelliptic integrals. This non-linear problem requires finding two points on the surface and four integers. The solution of the Jacobi problem is found in closed form by reducing it to a system of two algebraic equations with the right-hand side expressed through the Riemann \(\theta\)-function of the surface. Section 6 writes down additional conditions to be satisfied in order to fix 21 unknown real parameters. The system of equations for the unknowns consists of 8 linear and 13 non-linear relations which are algebraic and transcendental equations. Finally, equations of the free surface of the channel and the boundaries of the two cavities are found by quadratures.

2. Formulation

Let two hydrofoils \(B_1D_1\) and \(B_2D_2\) (Fig. 1) be placed in an incompressible gravity-free fluid which is moving steadily and irrotationally in a channel. The bottom \((-\infty < x < \infty, y = 0)\) of the channel is solid, and its upper boundary is a free surface. Far away from the hydrofoils, the flow is uniform with velocity \(\mathbf{v} = (V_{\infty}, 0)\) across the channel of depth \(h\). It is assumed that at the ends \(B_j\) and \(D_j\) \((j = 1, 2)\) the jets break away from the hydrofoils, and cavities \(B_jC_jD_j\) \((j = 1, 2)\) form behind the foils. The cavities are convex and bounded but not closed. The unknown boundaries of the cavities are streamlines. In the framework of the model considered, the velocity vector is constant and prescribed on the boundaries of the cavities (the constants are not necessarily the same). The loops \(A_jB_jC_jD_jA_j\) are smooth in a neighborhood of the points \(B_j\) and \(D_j\). At the stagnation points \(A_j\) (unknown \(a\)) priori), the flow branches and the velocity vector vanishes. Under these assumptions, the model problem of fluid mechanics is reduced to that of finding a complex potential of the motion \(w(z) = \phi + i \psi\) in the 3-connected domain \(\mathcal{D}\), occupied by the fluid (the physical domain), together with boundary conditions of the form

\[
\text{Im } w(z) = \begin{cases} W_0^\pm, & z \in E_1^\pm E_2^\pm, \\ W_j, & z \in L_j, \end{cases} \quad \text{for } j = 1, 2,
\]

\[
\frac{dw}{dz} = \begin{cases} V_{\infty}, & z \in E_1^\pm E_2^\pm, \\ V_j, & z \in B_jC_jD_j, \end{cases} \quad \text{for } j = 1, 2,
\]

\[
\arg \left( \frac{dw}{dz} \right) = \begin{cases} 0, & z \in E_1^- E_2^-, \\ -\alpha_j, & z \in A_jB_j, \end{cases} \quad \text{for } j = 1, 2.
\]

Here \(W_0^\pm\) and \(W_j\) \((j = 1, 2)\) are some constants, \(d_w/dz = v_x - iv_y, v_x\) and \(v_y\) are the velocity components, \(V_j\) are positive constants defined by the Bernoulli equation

\[
\frac{1}{2} (V_j^2 - V_{\infty}^2) + \frac{p_j - p_{\infty}}{\rho} = 0, \quad \text{for } j = 1, 2,
\]

\(p_{\infty}\) and \(p_j\) are the pressure at infinity and in the cavities,
respectively, \( \alpha_j \) are the angles the foils make with the real axis, 
\( L_0 = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1} \) is the boundary of the channel, 
\( E_3^{-1} = \infty + i0, E_4^{-1} = \infty + ih \), \( E_1^{-1}E_2^{-1} \) is a free surface. The first equation in (2.1) reflects the fact that the stream function \( \psi \) is piecewise constant on the boundary of the flow that consists of \( L_0 \) and two loops \( L_j = A_j B_j C_j D_j A_j \) \((j = 1, 2)\).

The complex potential \( w \) is an analytic function in the flow domain \( D \). It becomes a single-valued function in the domain \( D \) cut along the two straight lines \( P_j A_j \) joining the critical points \( A_j \) \((j = 1, 2)\) with the infinite point \(-\infty + iy, 0 \leq y \leq h\).

The problem of interest is non-linear first, for the shapes of the free surface of the channel, the boundaries of the cavities, and the position of the stagnation points \( A_j \) being unknown and second, for the boundary conditions (2.1) being non-linear.

### 3. Conformal mapping

Let \( z = f(\zeta) \) be a conformal mapping of a parametric \( \zeta \)-plane cut along three segments of the real axis \( l_1 = [-1/k, 1], l_0 = [k_2, k_1], \) and \( l_2 = [1/k, 1/k] \) \((0 < k < 1, -1 < k_1 < k_2 < 1)\) onto the 3-connected flow domain \( D \). Such a map always exists [12]. It becomes unique if, in addition, it is required that the segments \( l_j \) are transformed into the loops \( L_j \) \((j = 0, 1, 2)\), and the images of the points \( a_j, b_j, c_j, d_j \) \((j = 1, 2)\), \( e_1, e_2 \) are the points of the physical domain \( A_j, B_j, C_j, D_j, -\infty + iy, and + \infty + iy \) \((0 \leq y \leq h)\), respectively.

None of the points of the real axis \( k_1, k_2, k \), or the points \( a_j, b_j, c_j, d_j, e_j \) \((j = 1, 2)\) can be prescribed arbitrary. They have to be found as a part of the solution to the problem. What is known however, is that these points lie on either side of the cuts: \( a_j, b_j, c_j, d_j \in l_j \) and \( e_1, e_2 \in l_0 \), and they follow each other in clockwise direction (the same direction is chosen for the contours \( L_j \) in the flow domain \( D \)). The image pattern of the flow domain by the inverse transform \( \zeta = f^{-1}(z) \) is shown in Fig. 2. The solid line corresponds to the solid boundary of the flow, and the broken line is the image of the unknown curves.

It will be convenient to introduce a new function \( \omega(\zeta) \),

\[
\omega(\zeta) = \log \frac{dw(\zeta)}{V_{\infty}dz}, \quad z = f(\zeta). \tag{3.1}
\]

Then

\[
\omega(\zeta) = \log \frac{V}{V_{\infty}} - i\theta, \tag{3.2}
\]

where \( V = |dw/dz| \) is the speed, and \( \theta = -\arg dw/dz \) is the angle that the velocity vector makes with the x-axis. The function \( \omega(f^{-1}(z)) \) is defined in the flow domain \( D \) cut along the streamlines joining the point \(-\infty + iy \) \((0 \leq y \leq h)\) with the points \( A_1 \) and \( A_2 \).

The boundary conditions (2.1) on the boundary of the flow transform into ones on the sides of the cuts. The free surface boundary condition becomes

\[
\text{Re } \omega(\zeta) = \begin{cases} 0, & \zeta \in e_2e_1 \\ \log(V_j/V_{\infty}), & \zeta \in b_jc_jd_j, \end{cases} \quad j = 1, 2, \tag{3.3}
\]

and the boundary condition on the solid part of the boundary can be written as

\[
\text{Im } \omega(\zeta) = \begin{cases} 0, & \zeta \in e_1e_2 \\ -\alpha_j, & \zeta \in a_jb_j \\ \pi - \alpha_j, & \zeta \in d_ja_j, \end{cases} \quad j = 1, 2. \tag{3.4}
\]

Since \( dw/dz = 0 \) at \( z = A_j \) \((j = 1, 2)\), the new function \( \omega(\zeta) \) has the logarithmic singularities at the points \( \zeta = a_j \).

In a neighborhood of the points \( e_j \), the function \( \omega(\zeta) \) has a singularity [4,5]

\[
\omega(\zeta) \sim M'_j(w - w_j)^{-1/2}, \quad z \to C_j, \quad M'_j \neq 0, \tag{3.5}
\]

where \( w_j = w(C_j) \), and a single branch of the square root is considered in the \( w \)-plane cut along the axis \( \arg(w - w_j) = \pi \). A neighborhood \(|w - w_j| < \varepsilon \) of the point \( w = w_j \) cut along the line \( \arg(w - w_j) = \pi \) corresponds to a neighborhood of the point \( z = C_j \) which locally is a semi-disc. Therefore, \( w - w_j \sim N'_j(z - C_j)^2, z \to C_j, N'_j \neq 0 \). Next, \( z - C_j \sim N''_j(\zeta - c_j), \)

\[
\zeta \to c_j, \quad N''_j \neq 0. \tag{3.6}
\]

Here \( M''_j \) are non-zero real constants and \( j = 1, 2 \).

The recovering of the mapping function \( z = f(\zeta) \) is the key step in the definition of the free surfaces. Its derivative can be found from the relation

\[
\frac{dw}{dz} = V_{\infty}e^{\omega(\zeta)}, \tag{3.7}
\]

to find the derivative \( dw/dz \), one needs to solve the Hilbert problem (3.3) and (3.4) for the function \( \omega(\zeta) \).

As for the function \( dw/d\zeta \), since \( \text{Im } w(f(\zeta)) \) is a piecewise constant function on the sides of the cuts \( l_j \) \((j = 0, 1, 2)\), the imaginary part of the derivative \( dw/d\zeta \) is equal to zero on \( l_j \)

\[
\text{Im } \frac{dw}{d\zeta} = 0, \quad \zeta \in l_0 \cup l_1 \cup l_2. \tag{3.9}
\]

At the points \( a_j \) and \( c_j \) \((j = 1, 2)\), the function \( dw/d\zeta \) has simple zeros if these points do not coincide with the ending points of the cuts. Indeed, according to the model, \( \text{Im } w(\zeta) \) has constant values \( W_j \) on the curves \( P_jA_j, A_jB_jC_j, A_jD_jC_j, \) and \( C_jQ_j \). These lines are the images of the lines \( p_ja_j \), the corresponding parts of the cuts \( l_j \), and the lines \( c_jq_j \) in the parametric \( \zeta \)-plane (Fig. 2). Since the curves \( P_jA_j \) and \( C_jQ_j \) are orthogonal to the curves \( A_jB_jC_j \) and \( A_jD_jC_j \), the curves
$p_ja_j$ and $q_jc_j$ are orthogonal to the cuts $l_j$ ($j = 1, 2$). Therefore, in a neighborhood of the points $\zeta = a_j$ and $c_j$,

$$w(f(\zeta)) = w(A_j) + K_j'(\zeta - a_j)^2 + \cdots, \quad \zeta \to a_j,$$

$$K_j' \neq 0,$$

$$w(f(\zeta)) = w(C_j) + K_j''(\zeta - c_j)^2 + \cdots, \quad \zeta \to c_j,$$

$$K_j'' \neq 0,$$

(3.10)

and $dw/d\zeta = 0$ at $\zeta = a_j$ and $\zeta = c_j$, $j = 1, 2$. When the points $a_j$ or $c_j$ coincide with one of the branch points, the function $dw/d\zeta$ is bounded and not equal to zero.

The point $\zeta = \infty$ corresponds to a finite point of the flow domain. Therefore, in a neighborhood of the point $\zeta = \infty$, $w = w_0 + c\zeta^{-1} + \cdots$, $c = \text{const}$. Clearly, the function $dw/d\zeta$ has a zero of the second order at the infinite point of the parametric $\zeta$-plane.

Analyze next the behavior of the function $dw/d\zeta$ at the points $e_1$ and $e_2$. The function $z = f(\zeta)$ maps the $\zeta$-plane with three cuts onto the domain $D$. At $z \to \mp\infty + iy$ ($0 \leq y \leq h$), the flow domain $D$ looks like a strip of width $h$. Therefore, locally,

$$f(\zeta) = \frac{(-1)^{j-1}h}{\sigma \pi} \log(\zeta - e_j) + O(1), \quad \zeta \to e_j,$$

$$j = 1, 2,$$

(3.11)

where $\sigma = 1$ if $e_j$ does not coincide with the ends of the cut $l_0$, and $\sigma = 2$ otherwise. Then the function $dw/d\zeta$ has simple poles at the points $e_1$ and $e_2$.

$$\frac{dw}{d\zeta} = \frac{df}{dz} \frac{dz}{d\zeta} = \frac{(-1)^{j-1}hV_{\infty}}{\sigma \pi} \frac{1}{\zeta - e_j} + O(1),$$

$$\zeta \to e_j, \quad j = 1, 2.$$

(3.12)

In a neighborhood of any ending point (call it $\zeta$) of the cuts, if it coincides with none of the critical points $a_j$, $c_j$, or $e_j$, $f(\zeta) \sim K(\zeta - \zeta)^{1/2} (K \neq 0)$. Therefore, in this case, $dw/d\zeta \sim K(\zeta - \zeta)^{-1/2}$, $\zeta \to \zeta$, $K$ is a non-zero constant.

Then, the functions $dw/d\zeta$ and $w(\zeta)$ solve the two Hilbert boundary-value problems (3.9), (3.3) and (3.4) in the classes of functions prescribed. The solution of these problems will be found in Sections 4 and 5 by a method that exploits the symmetry of the parametric $\zeta$-plane, the fact that the imaginary part of the unknown function is equal to zero on the whole boundary (in the case of $dw/d\zeta$) or on its portion (in the case of $\omega(\zeta)$) and the theory of Riemann surfaces of algebraic functions. After the solution of the problems the mapping function $z = f(\zeta)$ is found by integration of the derivative $df/d\zeta$.

4. Function $dw/d\zeta$

Let $R$ be the hyperelliptic surface given by the algebraic equation

$$u^2 = p(\zeta), \quad p(\zeta) = (\zeta^2 - 1)(k_2\zeta^2 - 1)(\zeta - k_1)(\zeta - k_2).$$

(4.1)

The surface is formed by gluing two copies $C_1$ and $C_2$ of the extended complex $\zeta$-plane $C \cup \{\infty\}$ cut along the segments $l_j$, $(j = 0, 1, 2)$. The positive sides $l_j^+$ of the cuts $l_j \subset C_1$ are glued with the negative sides $l_j^-$ of the cuts $l_j \subset C_2$, and the sides $l_j^+ \subset C_1$ are glued with $l_j^- \subset C_2$. The function $u(\zeta)$ defined by (4.1) is single-valued on $\mathcal{R}$:

$$u = \begin{cases} 
\frac{1}{\sqrt{2}}(\zeta), & \zeta \in C_1 \\
-\frac{1}{\sqrt{2}}(\zeta), & \zeta \in C_2. 
\end{cases}$$

(4.2)

Here $p^{1/2}(\zeta)$ is a branch fixed by the condition

$$p^{1/2}(\zeta) \sim k\zeta^3, \quad \zeta \to \infty.$$

(4.3)

The pair $(\zeta, p^{1/2}(\zeta))$ denotes a point on $C_1$ with affix $\zeta$, whilst the notation $(\zeta, -p^{1/2}(\zeta))$ will be used for its counterpart on the second sheet.

Notice that the sides of the cuts $l_j$ ($j = 0, 1, 2$) form the symmetry line for the surface $\mathcal{R}$ which splits the surface into two symmetrical halves, and the surface itself is therefore the Schottky double. Since $1m dw/d\zeta = 0$ on $l_j$ ($j = 0, 1, 2$) the function $dw/d\zeta$ can be analytically continued by symmetry on the whole Riemann surface. Then

$$(dw/d\zeta)^+ - (dw/d\zeta)^- = 0, \quad (\zeta, u) \in (l_0 \cup l_1 \cup l_2) \subset \mathcal{R}.$$  

(4.4)

By the generalized Liouville theorem, the solution of this simplest form of the Riemann–Hilbert problem is a rational function say, $R(\zeta, u)$, on the Riemann surface $\mathcal{R}$. If the points $a_j$, $c_j$ do not coincide with the branch points of the surface $\mathcal{R}$, this function has simple zeros at the points $(a_j, u(a_j))$ and $(c_j, u(c_j))$ which simultaneously belong to the first and second sheet of the surface. Notice that the function $dw/d\zeta$ is bounded and not equal to zero at any point of the set $(a_1, a_2, e_1, e_2)$ if that point coincides with one of the branch points of the surface. The rational function $R(\zeta, u)$ on the surface has zeros of the second order at the infinite points $(\infty, +\infty)$ and $(\infty, -\infty)$ of the surface $\mathcal{R}$. At the points $e_j$ (but not at the points $e_j$ ($j = 1, 2$) this function has simple poles and

$$\text{res}_{\zeta=e_j} R(\zeta, u) = \frac{(-1)^{j-1}hV_{\infty}}{\sigma \pi}.$$  

(4.5)

Notice that the number of poles and zeros (counting multiplicities) of this function is the same.

Consider the most interesting and realistic case when none of the points $a_j$, $c_j$, $e_j$ ($j = 1, 2$) coincide with the branch points of the surface. Then $\sigma = 1$ and the function $R(\zeta, u)$ must have simple zeros at the points $(a_j, u(a_j))$ and $(c_j, u(c_j))$. The most general form of the rational function $R(\zeta, u)$ on the surface $\mathcal{R}$ is

$$R(\zeta, u) = R_1(\zeta) + \frac{R_2(\zeta)}{u},$$

(4.6)

where the functions $R_1(\zeta)$ have simple poles at the points $e_1$ and $e_2$. At infinity, $R_1(\zeta) = O(\zeta^{-2})$ and $R_2(\zeta) = O(\zeta)$. The functions with these properties have the form

$$R_1(\zeta) = \left(\frac{1}{\zeta - e_1} - \frac{1}{\zeta - e_2}\right) N_0.$$
\[ R_2(\zeta) = iN_1 + iN_2 \zeta + \frac{N_3}{\zeta - e_1} + \frac{N_4}{\zeta - e_2}, \tag{4.7} \]

where \( N_j \) \((j = 0, 1, \ldots, 4)\) are arbitrary constants. It is necessary to impose the following four conditions

\[
\begin{align*}
\lim_{\zeta\to e_j} \left[ R_1(\zeta) + \frac{R_2(\zeta)}{u(\zeta)} \right] &= (-1)^{j-1} \frac{hV_\infty}{\sigma\pi}, \\
\lim_{\zeta\to e_j} \left[ R_1(\zeta) + \frac{R_2(\zeta)}{u(\zeta)} \right] &= 0,
\end{align*}
\]

which define the constants \( N_0, N_3, \) and \( N_4 \)

\[
N_0 = \frac{hV_\infty}{2\pi}, \quad N_3 = \frac{hV_\infty}{2\pi} u(e_1), \quad N_4 = -\frac{hV_\infty}{2\pi} u(e_2).
\tag{4.9}
\]

Substitute now (4.7) and (4.9) into (4.6) and take \((\zeta, u)\) as a point on the first sheet \( \mathbb{C}_1 \). This defines the expression of the derivative \( dw/d\zeta \) on the parametric \( \zeta \)-plane

\[
\frac{dw}{d\zeta} = \frac{i[N_1 + N_2 \zeta + r(\zeta)]}{p^{1/2}(\zeta)}, \quad \zeta \in \mathbb{C},
\tag{4.10}
\]

where

\[
r(\zeta) = \frac{-ihV_\infty}{2\pi} \left( \frac{p^{1/2}(\zeta) + p^{1/2}(e_1)}{\zeta - e_1} - \frac{p^{1/2}(\zeta) + p^{1/2}(e_2)}{\zeta - e_2} \right).
\tag{4.11}
\]

Now, the function \( dw/d\zeta \) has to be equal to zero at the point \( a_j \) and \( c_j \) \((j = 1, 2)\). The first two conditions define the real constants \( N_1 \) and \( N_2 \)

\[
\begin{align*}
N_1 &= \frac{a_2 r(a_1) - a_1 r(a_2)}{a_1 - a_2}, \\
N_2 &= \frac{r(a_2) - r(a_1)}{a_1 - a_2}.
\end{align*}
\tag{4.12}
\]

The other two conditions \( dw/d\zeta(c_j) = 0 \) \((j = 1, 2)\) give two real non-linear equations for the parameters \( a_j, c_j, \) and \( e_j \) \((j = 1, 2)\)

\[
N_1 + N_2 c_j + r(c_j) = 0, \quad j = 1, 2.
\tag{4.13}
\]

5. Riemann–Hilbert problem on a Riemann surface

5.1. Formulation

The Hilbert problem (3.3) and (3.4) for the function \( \omega(\zeta) \) may be formulated as a boundary-value problem on the Riemann surface of the algebraic function (4.1). Introduce the function

\[
\Phi(\zeta, u) = \begin{cases} 
\omega(\zeta), & (\zeta, u) \in \mathbb{C}_1 \\
\bar{\omega}(\bar{\zeta}), & (\zeta, u) \in \mathbb{C}_2.
\end{cases}
\tag{5.1}
\]

This function satisfies the symmetry condition

\[
\Phi(\zeta, u_s) = \Phi(\zeta, u),
\tag{5.2}
\]

where \((\zeta_s, u_s) = (\bar{\zeta}, -u(\bar{\zeta}))\) is the point symmetrical to a point \((\zeta, u(\zeta))\) with respect to the lines \( l_0 \cup l_1 \cup l_2 \) along which the two halves of the surface are glued. Thus, \((\zeta, u) \in \mathbb{C}_1, (\zeta_s, u_s) \in \mathbb{C}_2.\) On the symmetry lines, the boundary values of the function \( \Phi(\zeta, u) \)

\[
\Phi^+(\zeta, v) = \omega(\zeta), \quad \Phi^-(\zeta, v) = \bar{\omega}(\bar{\zeta}), \quad (\zeta, v) \in l_1 \cup l_0 \cup l_2, \quad v = u(\bar{\zeta}).
\tag{5.3}
\]

It is clear from inspection of the boundary conditions (3.3) and (3.4) that the function \( \Phi(\zeta, u) \) is continuous through the curve \( e_1 e_2 \) (a part of the contour \( l_0 \)) and is discontinuous through the contours \( l_1, l_2 \) and the rest \( e_2 e_1 \) of the contour \( l_0 \). Thus, the function \( \Phi(\zeta, u) \) solves the following Riemann–Hilbert problem on the Riemann surface \( \mathbb{R} \).

Find all the functions \( \Phi(\zeta, u) \) analytic in \( \mathbb{R} \setminus (l_0 \cup l_1 \cup l_2) \), Hölder-continuous up to the boundary \( l_0 \cup l_1 \cup l_2 \) with boundary values satisfying the relations

\[
\begin{align*}
\Phi^+(\zeta, v) - \Phi^-(\zeta, v) &= \begin{cases} 
0, & (\zeta, v) \in e_1 e_2 \\
-2i\omega_j & \zeta = \omega_j, \\
2i(\pi - \alpha_j) & \zeta = \alpha_j
\end{cases}, \\
\Phi^+(\zeta, v) + \Phi^-(\zeta, v) &= \begin{cases} 
0, & (\zeta, v) \in e_2 e_1 \\
2\log(V_\infty/V_\infty) & \zeta \to \infty
\end{cases}.
\end{align*}
\tag{5.4}
\]

and the symmetry condition (5.2). The function \( \Phi(\zeta, u) \) has logarithmic singularities at the points \((a_j, u(a_j))\) and poles of the first order at the points \((c_j, u(c_j))\) \((j = 1, 2)\). It is bounded at the points \((b_j, u(b_j)), (d_j, u(d_j))\), both infinite points \((\infty, \pm \infty)\), and \( \Phi(e_j, u(e_j)) = 0, j = 1, 2 \).

5.2. Factorization

Let \( \mathbb{L} = b_1 c_1 d_1 \cup e_2 e_1 \cup b_2 c_2 d_2 \). To solve the problem (5.4), consider first the homogeneous Riemann–Hilbert problem

\[
X^+(\zeta, v) = -X^-(\zeta, v), \quad (\zeta, v) \in \mathbb{L}.
\tag{5.5}
\]

By the Sokhotski–Plemelj formulas the function

\[
\chi_0(\zeta, u) = \exp \left\{ \frac{1}{2\pi i} \int_{\mathbb{L}} \log(-1)dw \right\}
\]

\[
\chi_0(\zeta, u) = \left[ \frac{\zeta - d_1(\zeta - d_2)(\zeta - e_1)}{(\zeta - b_1)(\zeta - b_2)(\zeta - e_2)} \right]^{1/4}
\]

\[
\times \exp \left\{ \frac{u}{4} \int_{\mathbb{L}} \frac{d\xi}{\zeta - \bar{\xi}} \right\}
\tag{5.6}
\]

satisfies the boundary condition (5.5). Here \( dW \) is the Weierstrass kernel

\[
dW = \frac{1}{2} \left( 1 + \frac{u}{v} \right) \frac{d\xi}{\zeta - \bar{\xi}} \quad u = u(\zeta), \quad v = v(\bar{\zeta}).
\tag{5.7}
\]

Also, it is directly verified that the function \( \chi_0(\zeta, u) \) meets the symmetry condition (5.2). However, at the points \((\infty, \pm \infty)\) it has essential singularities. This is because of the second order
pole of the kernel $dW$ at both infinite points of the surface. To remove these essential singularities, analyze the function

$$
\chi_0(\zeta, u) \chi_1(\zeta, u) \chi(\zeta_*, u_*) \chi(\zeta, u),
$$

where

$$
\chi_1(\zeta, u) = \exp\left\{ - \sum_{j=1}^{2} \int_{r_j} + m_j \oint_{a_j} + n_j \oint_{b_j} dW \right\}.
$$

Here $a_j$ and $b_j$ $(j = 1, 2)$ form a system of canonical cross-sections of the surface $R$. The contours $a_1$ and $a_2$ are smooth loops which lie on both sheets of the surface and coincide with the banks of the cuts $l_0$ and $l_2$, respectively (Fig. 3). The positive direction is that which leaves the sheet $C_1$ on the left. The loop $b_1$ consists of the segments $[k_1, -1] \subset C_1$ and $[-1, k_1] \subset C_2$. The contour $b_2$ consists of the segments $[1, -1] \subset C_1$ and $[-1, 1] \subset C_2$. In Fig. 3, the part of the contours $b_j$ which lies on the first sheet is shown as the solid lines. The broken line corresponds to the part lying on $C_2$. The loops $a_j$ and $b_j$ $(\nu \neq j)$ do not intersect. The cross-section $a_j$ intersects $b_j$ from left to the right, and there is only one point of intersection.

The contours $\gamma_1$ and $\gamma_2$ are continuous curves with the same starting point $(-1/k, 0)$. The ending points $q_j = (\zeta_j, u_j) \in R$ $(u_j = u(\zeta_j))$ are to be determined. These points may lie on either sheet of the surface. The contours $\gamma_j$ cannot intersect the canonical cross-sections. The parameters $m_j$ and $n_j$ are integers and will be found later on.

Now, formulas (5.6) and (5.9) can be combined to give an expression for the function (5.8)

$$
\chi_0(\zeta, u) \chi_1(\zeta, u) \chi(\zeta, u) = \frac{(\zeta + 1/k)^2 \chi(\zeta, u)}{\sqrt{(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)(\zeta - \zeta_4)}},
$$

where

$$
\chi(\zeta, u) = \exp\left\{ u \int_\mathcal{L} \frac{d\xi}{\xi - \zeta} - \sum_{j=1}^{2} \int_{r_j} \frac{d\xi}{2v \xi - \zeta} + m_j \int_{a_j} \frac{d\xi}{p^{1/2}(\xi)(\xi - \zeta)} \right\}.
$$

Here the contours $\Gamma_j$ are curves symmetrical with respect to the real line, lying on one of the sheets of the surface, starting at the points $(\zeta_j, u_j)$ and ending at $(\zeta_j, u_j)$. The contours $a_j^+$ and $a_j^-$ are the upper banks of the cuts $l_0$ and $l_2$, respectively. Notice that the integrals over the loops $b_j$ vanish in formula (5.11) because of the symmetry.

It will be convenient to take the solution of the factorization problem (5.5) as

$$
X(\zeta, u) = \frac{(\zeta - d_1)^{1/4}(\zeta - d_2)^{1/4}(\zeta - e_1)^{1/4}(\zeta - e_2)^{1/4}}{\sqrt{(\zeta - \zeta_1)(\zeta - \zeta_2)(\zeta - \zeta_3)(\zeta - \zeta_4)}} \chi(\zeta, u).
$$

(5.12)

Let $\beta$ be one of the points $b_j, d_j, e_j$. Analysis of the singular integrals in (5.11) in a neighborhood of that point indicates that

$$
X(\zeta, u) \sim A_\beta(\zeta - \beta)^{1/2}, \quad \zeta \to \beta, \quad A_\beta = \text{const} \neq 0,
$$

(5.13)

while at the point $\zeta = \bar{\beta}$ that lies on the opposite side of the corresponding cut,

$$
X(\zeta, u) \sim \begin{cases} A_{\bar{b}_j}, & \zeta \to \bar{b}_j \cr A_{\bar{d}_j}, & \zeta \to \bar{d}_j \cr A_{\bar{e}_1}, & \zeta \to \bar{e}_1 \cr A_{\bar{e}_2}, & \zeta \to \bar{e}_2. \end{cases} \quad \text{for } j = 1, 2.
$$

(5.14)

At the points $q_j = (\zeta_j, u_j)$, the factorization function $X(\zeta, u)$ has simple poles and it is bounded at the points $(\zeta_j^*, u_j)$.

At present, the function $X(\zeta, u)$ and therefore the function $X(\zeta, u)$ have an inappropriate exponential growth at infinity caused by the pole of the second order at infinity of the kernel $dW$. In order to get a solution of a finite order at infinity, expand the function $X(\zeta, u)$ as $\zeta \to \infty$,

$$
X(\zeta, u) = \chi \exp\left\{ - \sum_{j=1}^{2} \int_{r_j} \frac{1}{4} \int_\mathcal{L} \frac{\xi^{v-1} d\xi}{v} \right\}
$$

$$
- \sum_{j=1}^{2} \left( \int_{r_j} \frac{\xi^{v-1} d\xi}{2v} + 2m_j \int_{a_j} \frac{\xi^{v-1} d\xi}{p^{1/2}(\xi)} \right) + O(\zeta), \quad \zeta \to \infty.
$$

(5.15)

Because of the symmetry of the contours $\Gamma_j$, the requirement of the canonical function to have a finite order at infinity enables us to write the following system of non-linear equations

$$
\text{Im} \left\{ \frac{1}{4} \int_\mathcal{L} \frac{\xi^{v-1} d\xi}{v} - \sum_{j=1}^{2} \int_{r_j} \frac{\xi^{v-1} d\xi}{v} \right\} = 0, \quad v = 1, 2.
$$

(5.16)

This system presents a real analogue of the Jacobi inversion problem on the Riemann surface $R$. To reduce the system (5.16) to the classical inversion problem, notice that

$$
\text{Re} \left( \int_\mathcal{L} \frac{\xi^{v-1} d\xi}{v} \right) = 0,
$$

$$
2 \int_{a_j} \frac{\xi^{v-1} d\xi}{p^{1/2}(\xi)} = \oint_{a_j} d\omega_v = A_v, \quad \text{Re} A_v = 0,
$$

$$
\oint_{b_j} d\omega_v = B_v, \quad \text{Im} B_v = 0.
$$

(5.17)
where \( d\omega_v \) are abelian differentials of the first kind of the surface \( \mathcal{R} \):

\[
d\omega_v = \frac{\xi^{v-1} \, d\xi}{v}, \quad v = 1, 2, \tag{5.18}
\]

and \( A_{vj} \) and \( B_{vj} \) are the \( A \)- and \( B \)-periods of the abelian integrals

\[
\omega_v = \omega_v(\xi, u) = \int_{(-1/k,0)}^{(\xi, u)} \frac{\xi^{v-1} \, d\xi}{v}, \quad v = 1, 2. \tag{5.19}
\]

Now, the imaginary part of the system

\[
\frac{1}{4} \int_{-\mathcal{L}} \frac{\xi^{v-1} \, d\xi}{v} - \sum_{j=1}^{2} \left( \int_{\gamma_j} \frac{\xi^{v-1} \, d\xi}{v} + m_j A_{vj} + n_j B_{vj} \right) = 0,
\]

\( v = 1, 2 \) \tag{5.20}

coincides with the system (5.16). On the other hand, it can be written as the classical Jacobi problem

\[
\sum_{j=1}^{2} \left( \omega_v(q_j) + m_j A_{vj} + n_j B_{vj} \right) = \eta_v, \quad v = 1, 2, \tag{5.21}
\]

where

\[
\eta_v = \frac{1}{4} \int_{-\mathcal{L}} \frac{\xi^{v-1} \, d\xi}{v}. \tag{5.22}
\]

The solution to the inversion problem (5.21) with respect to the two points \( q_j = (\zeta_j, u_j) \in \mathcal{R} \) and the four integers \( m_j \) and \( n_j \) (\( j = 1, 2 \)) (the function \( \chi(\xi, u) \) in (5.11) is independent of the constants \( n_j \)) makes the function \( \chi(\xi, u) \) bounded at infinity.

### 5.3. Jacobi inversion problem

Following [9,11] normalize first the basis of abelian integrals

\[
\hat{\omega} = T \omega, \quad T = A^{-1}, \tag{5.23}
\]

where \( A = \{A_{vj}\} \) (\( v, j = 1, 2 \)). The \( A \)- and \( B \)-periods of the canonical basis \( \hat{\omega} \) are

\[
A_{vj} = \oint_{a_j} d\hat{\omega}_v = \delta_{vj}, \quad B_{vj} = \oint_{b_j} d\hat{\omega}_v = \sum_{m=1}^{2} T_{vm} B_{mj}. \tag{5.24}
\]

Here \( \delta_{vj} \) is the Kronecker symbol. The matrix \( B = \{B_{vj}\} \) (\( v, j = 1, 2 \)) is symmetric, and \( \text{Im} \, B \) is positive definite. For the canonical basis, the problem (5.21) becomes

\[
\sum_{j=1}^{2} \hat{\omega}_v(q_j) = g_v - \kappa_v - \sum_{j=1}^{2} n_j B_{vj} - m_v \equiv g_v - \kappa_v, \tag{5.25}
\]

(modulo the periods), \( v = 1, 2, \)

where

\[
g_v = \sum_{j=1}^{2} T_{vj} \eta_j + \kappa_v, \quad v = 1, 2, \tag{5.26}
\]

and \( \kappa_1 \) and \( \kappa_2 \) are the Riemann constants for the chosen system of \( a \) and \( b \) cross-sections computed in [11]

\[
\kappa_v = \frac{1}{2} (-v + B_{v1} + B_{v2}). \tag{5.27}
\]

The auxiliums \( \zeta_1 \) and \( \zeta_2 \) of the points \( q_1 \) and \( q_2 \) on the surface \( \mathcal{R} \) coincide with the two zeros of the Riemann \( \theta \)-function

\[
\hat{\mathcal{G}}(q) = \hat{\theta}(\hat{\omega}_1(q) - g_1, \hat{\omega}_2(q) - g_2) = \sum_{t_1, t_2 = -\infty}^{\infty} \exp \left\{ \pi i \sum_{\mu=1}^{2} \sum_{\nu=1}^{2} B_{\mu \nu} t_{\mu} t_{\nu} \right\} + 2 \pi i \sum_{\nu=1}^{2} t_{\nu} \hat{\omega}_v(q) - g_v. \tag{5.28}
\]

Since the matrix \( \text{Im} \, B \) is positive definite the double series (5.28) converges exponentially. The auxiliums of the points \( q_1 \) and \( q_2 \) solve the system of two equations [13]

\[
\zeta_1^v + \zeta_2^v = \varepsilon_v, \quad v = 1, 2, \tag{5.29}
\]

where

\[
\varepsilon_v = 2 \sum_{j=1}^{2} \sum_{m=1}^{2} T_{jm} \int_{a_j} \frac{e^{m+n-1} \, d\tau}{p^{1/2}(\tau)} - \sum_{\mu=1}^{2} \text{res}_{\nu=\infty} \zeta_v^v \hat{\mathcal{G}}(q). \tag{5.30}
\]

Here \( \infty_{\mu} = (\infty, (1)^{\mu-1} \infty) \in C_{\mu} (\mu = 1, 2) \) are the two infinite points of the Riemann surface.

By solving (5.29),

\[
\zeta_j = \frac{\varepsilon_1 - (1)^{j} \sqrt{2 \varepsilon_2 - \varepsilon_1^2}}{2}, \quad j = 1, 2. \tag{5.31}
\]

Since \( \text{Re} \, B_{vj} = 0 \), the integers \( m_v \) are given by [13,11]

\[
m_v = \text{Re} (\hat{g}_v), \quad \hat{g}_v = g_v - \kappa_v - \sum_{j=1}^{2} \hat{\omega}_v(q_j), \tag{5.32}
\]

and the integers \( n_v \) solve the system of linear algebraic equations

\[
\sum_{j=1}^{2} n_v B_{vj} = \text{Im} (\hat{g}_v), \quad v = 1, 2. \tag{5.33}
\]

This completes the solution to the Jacobi problem (5.25) and therefore (5.21). We emphasize that the function \( \chi(\xi, u) \) is independent of the integers \( n_v \).

### 5.4. Function \( dw/dz \)

The definition of the function \( dw/dz \) requires solving the Riemann–Hilbert problem (5.4). Using the factorization (5.5) write the boundary condition (5.4) as

\[
\frac{\Phi^+(\xi, v)}{X^+(\xi, v)} - \frac{\Psi^+(\xi, v)}{X^-(\xi, v)} = \frac{\Phi^-(\xi, v)}{X^-(\xi, v)} - \frac{\Psi^-(\xi, v)}{X^-(\xi, v)}, \quad (\xi, v) \in \mathcal{L}, \tag{5.34}
\]
where $\Psi^\pm(\xi, \nu)$ are the boundary values of the function

$$
\Psi(\xi, u) = \Psi_0(\xi) + u \Psi_1(\xi),
$$

$$
\Psi_\nu(\xi) = \frac{1}{2} \sum_{j=1}^2 \left( \frac{\alpha_j}{\pi} \int_{d_j a_j b_j} \right. + \int_{d_j a_j} \right) + \frac{\log(V_j / V_{\infty})}{\pi i} \int_{b_j c_j d_j} \frac{d\xi}{X^+(\xi, \nu)(\xi - \xi)\nu},
$$

$$
v = 0, 1, 2.
$$

By the generalized Liouville theorem, the general solution of the problem (5.34) has the form

$$
\Phi(\xi, u) = X(\xi, u)[\Psi_0(\xi) + \Omega_0(\xi) + u[\Psi_1(\xi) + \Omega_1(\xi)]],
$$

(5.36)

where $\Omega_j(\xi)(j = 0, 1)$ are certain rational functions on the complex $\xi$-plane. The function $X(\xi, u)$ has simple zeros at the points $b_j (j = 1, 2, \ldots)$ and the solution itself has simple poles at the points $c_1$ and $c_2$. The functions $X(\xi, u)$ and $u(\xi)$ grow at infinity as $\xi$ and $\xi^3$, respectively. Therefore, the functions $\Omega_0(\xi)$ and $\Omega_1(\xi)$ are given by

$$
\Omega_0(\xi) = \sum_{j=0}^2 M_j \xi^j + \sum_{j=1}^2 \left( \sum_{k=0}^j \left( \frac{i M_{j+k+4} u(c_j)}{\xi - c_j} - \frac{i M_{j+k+2} u(b_j)}{\xi - b_j} \right) \right) - \frac{i M_7 u(\xi)}{\xi - e_2},
$$

$$
\Omega_1(\xi) = \sum_{j=1}^2 \left( \frac{i M_{j+2} u(c_j)}{\xi - c_j} + \frac{i M_{j+4} u(b_j)}{\xi - b_j} \right) + \frac{i M_7 u(\xi)}{\xi - e_2},
$$

(5.37)

where $M_j (j = 0, 1, \ldots, 7)$ are arbitrary real constants.

The factorization function $X(\xi, u)$ has simple poles at the points $q_j = (\xi_j, u_j)$. To make these points removable singularities for the function $\Phi(\xi, u)$, the following two complex conditions need to be satisfied

$$
\Psi_0(\xi_j) + \Omega_0(\xi_j) + u_j[\Psi_1(\xi_j) + \Omega_1(\xi_j)] = 0, \quad j = 1, 2.
$$

(5.38)

At infinity, after the solution of the Jacobi inversion problem, since the factorization function $X(\xi, u)$ has a simple pole, the function $\Phi(\xi, u)$ has a pole of the third order. To write down the conditions which eliminate these singularities, expand the functions $u(\xi), \Omega_0(\xi), \Omega_1(\xi)$, and $\Psi_1(\xi)$ at infinity

$$
u = 0, 1, 2.
$$

Next, we specify the two distances $\mu_1$ and $\mu_2$ from the ending points $D_j$ to the bottom of the channel by

$$
\int_{e_0}^{d_j} \frac{df}{d\xi} d\xi = \mu_j, \quad j = 1, 2.
$$

(6.4)
where $e_0$ is an arbitrary fixed point on the portion $e_1 e_2$ of the cut $l_0$. If it turns out that $k_2 \in e_1 e_2$, then $k_2$ can be chosen as the point $e_0$.

The fifth condition describes the distance $\mu$ between the two ending points $D_1$ and $D_2$ of the hydrofoils

$$\text{Re} \int_{D_1}^{D_2} \frac{df}{d\zeta} d\zeta = \mu. \quad (6.5)$$

We emphasize that, in general, the conformal map $z = f(\zeta)$ is a multi-valued function. It becomes a one-to-one map if

$$\oint_{1_j} \frac{df}{d\zeta} d\zeta = 0, \quad j = 1, 2. \quad (6.6)$$

These two complex equations add four real conditions for the unknown parameters. Thus, in total we have 21 real conditions for the same number of real constants. Note that the system of these equations can be split into a set of eight linear equations for the constants $M_j$ $(j = 0, 1, \ldots, 7)$ and a system of 13 non-linear (algebraic and transcendental) equations for the parameters $a_j, b_j, c_j, d_j, e_j$ $(j = 1, 2), k, k_1,$ and $k_2$. These non-linear equations can be solved numerically.

Finally, we determine the equations for the unknown a priori boundaries of the cavities behind the hydrofoils and the free surface of the channel. Integrating the function (6.3) gives the boundaries of the cavities

$$z(\tau) = \int_{b_j}^{\tau} \frac{dw(f(\zeta))/d\zeta}{dw/d\zeta} d\zeta + B_j, \quad \tau \in b_j c_j, \quad z \in B_j C_j, \quad j = 1, 2, \quad (6.7)$$

and

$$z(\tau) = \int_{d_j}^{\tau} \frac{dw(f(\zeta))/d\zeta}{dw/d\zeta} d\zeta + D_j, \quad \tau \in c_j d_j, \quad z \in C_j D_j, \quad j = 1, 2, \quad (6.8)$$

whilst letting $\tau \in e_2 e_1$ by integration we recover the free surface of the channel

$$z(\tau) = \int_{b}^{\tau} \frac{dw(f(\zeta))/d\zeta}{dw/d\zeta} d\zeta + B, \quad \tau \in e_2 e_1, \quad z \in E_2^+ E_1^+. \quad (6.9)$$

The integrals (6.7) and (6.8) are taken either over the banks of the corresponding cuts, or over smooth curves on the cut $\zeta$-plane $C \setminus (l_0 \cup l_1 \cup l_2)$. The contour of integration $b \tau$ in (6.9) is a smooth curve on $C \setminus (l_0 \cup l_1 \cup l_2)$ which do not intersect the images of the cuts $P_j A_j$ (the complex potential $w$ is a single-valued analytic function in the flow domain $D$ cut along the streamlines $P_j A_j$ joining the stagnation points $A_j$ and the infinite point $-\infty + iy, 0 \leq y \leq h$), and $b$ is the image of a point $B \in D$. In the case shown in Fig. 1 $B_2$ can be taken as the the point $B$.

7. Conclusions

In the present paper, in the framework of the Tulin’s model of cavitating flow the steady streaming flow in a channel with a free surface around two hydrofoils (plates) with cavities of unknown shape behind them has been analyzed. By a conformal map of a parametric complex plane cut along three cuts on the real axis onto the triple-connected flow domain, the model problem has been reduced to two Riemann–Hilbert problems on a Schottky double which is a hyperelliptic surface of genus two. The solution of these problems was found to be necessary in order to recover the actual form of the conformal map. The solution to the first problem turned out to be a rational function on the Riemann surface and was constructed explicitly without quadratures. The second problem required the use of singular integrals with the Weierstrass kernel. Due to the second order pole of the kernel, in general, the solution had an essential singularity at the infinite points of the surface. The procedure of elimination of the essential singularities of the solution led to the classical Jacobi inversion problem of genus two which was solved explicitly.

Final equations for the velocity, the free surface of the channel, cavities and the conformal mapping itself possessed 21 unknown real parameters. To recover them the same number of real conditions (13 non-linear and 8 linear equations) have been found. The non-linear system of algebraic and transcendental equations can be separated from the linear system.

It is worth pointing out some limitations and possible extensions to the work done in the paper. First, for the method presented it is essential to be able to map a parametric plane cut along $n$ segments on the same straight line into the $n$-connected flow domain. If $1 \leq n \leq 3$, then such a map always exists [12]. If $n \geq 4$, in general, it does not. For this more general case, it is hoped to develop another technique based on the theory of the Riemann–Hilbert problem for symmetric automorphic functions. Second, the paper concerns the supercavitating steady flow around hydrofoils. In the transient case for not too rapidly varying flows, it is possible (at least approximately) to assume that the free boundaries of the cavities are streamlines [14] and modify the procedure of the paper accordingly.

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References


