Line Integral. Work Done by a Force

Calculate \( \int_C \mathbf{F}(r) \cdot dr \) for the following data. (If \( \mathbf{F} \) is a force, this gives the work in the displacement along \( C \).) Show the details of your work.

1. \( \mathbf{F} = [y^2, -x^2], \quad C \) the straight-line segment from \((0, 0)\) to \((1, 4)\)
2. \( \mathbf{F} \) as in Prob. 1, \quad \( C: y = 4x^2 \) from \((0, 0)\) to \((1, 4)\)
3. \( \mathbf{F} = [xy, x^2y^2], \quad C \) the quarter-circle from \((2, 0)\) to \((0, 2)\) with center at \((0, 0)\)
4. \( \mathbf{F} \) as in Prob. 3, \quad \( C \) the straight-line segment from \((2, 0)\) to \((0, 2)\)
5. \( \mathbf{F} = [(x - y)^2, (y - x)^2], \quad C: xy = 1, 1 \leq x \leq 4 \)
6. \( \mathbf{F} = [\exp(y^{2/3}), -\exp(x^{2/3})], \quad C \) the semicubical parabola \( y = x^{3/2} \) from \((0, 0)\) to \((1, 1)\)
7. \( \mathbf{F} = [2z, x, -y], \quad C: \mathbf{r} = [\cos t, \sin t, 2t] \) from \((0, 0, 0)\) to \((1, 0, 4\pi)\)
8. \( \mathbf{F} = [x - y, y - z, z - x], \quad C: [2 \cos t, t, 2 \sin t] \) from \((2, 0, 0)\) to \((2, 2\pi, 0)\)
9. \( \mathbf{F} = [e^x, e^{-y}, e^t], \quad C: \mathbf{r} = [t, r^2, r^3] \) from \((0, 0, 0)\) to \((1, 1, 1)\)
10. \( \mathbf{F} = [\cosh x, \sinh y, e^t], \quad C: \mathbf{r} = [t, r^2, r^3] \) from \((0, 0, 0)\) to \((2, 2, 8)\)

11. WRITING PROJECT. Line Integral Generalizes Definite Integral. Write a short essay on this topic. Geometrically, the definite integral gives the area under the curve of the integrand. Explain the corresponding interpretation for a line integral. Include examples.

12. PROJECT. Independence of Representation. Dependence on Path. Consider the integral \( \int_C \mathbf{F}(r) \cdot dr \), where \( \mathbf{F} = [-x^2, xy] \).

(a) One path, several representations. Find the value of the integral when \( \mathbf{r} = [\cos t, \sin t], \) \( 0 \leq t \leq \pi \). Show that the value remains the same if you set \( t = -p \) or \( t = p^2 \) or apply two other parametric transformations of your own choice.

(b) Several paths. Evaluate the integral when \( C: y = x^n \), thus \( \mathbf{r} = [t, t^n], 0 \leq t \leq 1 \), where \( n = 1, 2, 3, \ldots \). Note that these infinitely many paths have the same endpoints.

(c) Limit. What is the limit in (b) as \( n \to \infty \)? Can you confirm your result by direct integration without referring to (b)?

(d) Show path dependence with a simple example of your own choice involving two paths.

Integrals \( \int_C f(r) \, ds \) with Arc Length as Parameter

Evaluate this integral with \( f \) and \( C \) as follows. (Show the details.)

13. \( f = x^2 + y^2, \quad C: y = 3x \) from \((0, 0)\) to \((2, 6)\)
14. \( f = x^3y, \quad C: \mathbf{r} = [2 \cos t, 2 \sin t], 0 \leq t \leq \pi/2 \)
15. \( f = x^2 + y^2 + z^2, \quad C: [\cos t, \sin t, 2t], 0 \leq t \leq 4\pi \)
16. \( f = \sqrt{2 + x^3 + 3y^2}, \quad C: \mathbf{r} = [t, t^2, 0], 0 \leq t \leq 3 \)
17. \( f = 1 + y^2 + z^2, \quad C: \mathbf{r} = [t, \cos t, \sin t], 0 \leq t \leq \pi \)
18. \( f = 1 - \sinh^2 x, \quad C \) the catenary \( \mathbf{r} = [t, \cosh t], 0 \leq t \leq 2 \)
19. \( f = x^2 + (xy)^{1/2}, \quad C \) the hypocycloid \( \mathbf{r} = [\cos^3 t, \sin^3 t], 0 \leq t \leq \pi \)
20. \( f = \sqrt{16x^2 + 81y^2}, \quad C: \mathbf{r} = [3 \cos t, 2 \sin t], 0 \leq t \leq \pi \)