ANALYSIS QUALIFYING EXAMS

Herein are solutions to many of the problems on old qualifying exams. Sometimes multiple solutions are given. Some solutions were written years ago when I knew much less (that may make the solutions better or worse, depending on your point of view—but it also might mean that the solutions are incorrect). Comments to myself appear in brackets, whose presence further indicates that you're peeking at a preliminary draft of something. Although I sometimes obtained solutions from listening to others or reading their work (and I try to give credit wherever this is the case), I typed up the solutions myself using style according to my whim and taste (or lack thereof); hence if anyone is to be blamed for incorrect solutions, typos, style, and such, it is I, me, myself, I, and no one else.

I would appreciate your help with this document; if you have solutions to unsolved problems or have corrections or comments on material that appears here, please let me know. Thanks.

SAMPLE PROBLEMS FROM LOUISIANA STATE UNIVERSITY

1. Let $\langle a_n \rangle$ be a sequence of positive real numbers such that

$$a_1 > a_2 > \cdots > a_n > \cdots$$

and for which $\sum_{n=1}^{\infty} a_n < \infty$.
(a) Prove that $f(x) = \sum_{n=1}^{\infty} a_n \cos(nx)$ is a continuous function.
(b) Prove that the equation $f(x) = 0$ has infinitely many solutions.

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2. Suppose $\langle f_n \rangle$ and $\langle g_n \rangle$ are sequences of functions on $\mathbb{R}$ such that $f_n \to f, g_n \to g$ uniformly on $\mathbb{R}$. Does $\langle f_n g_n \rangle \to fg$ uniformly on $\mathbb{R}$? Prove this or provide a counterexample.

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3. Let $f \in L^1(\mathbb{R})$. Prove that $\text{LIM}_{n\to\infty} \int_n^{n+1} f(x)\,\mathrm{d}x = 0$.

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4. Prove that $C[0,1]$ with the sup norm topology is a separable metric space; that is, prove that it has a countable dense subset.

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5. Let $f \in C[0,1]$ with the property $\int_0^1 f(x)x^n\,\mathrm{d}x = 0$ for $n = 0, 1, 2, \ldots$. Prove that $f = 0$.

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6. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^1$ be continuously differentiable. Show that f is not one-to-one.

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7. Let K be a subset of $\mathbb{R}^n$; let $C(K)$ be the continuous real-valued functions on K; let $\mathcal{F}$ be a set of functions defined on K.

(a) Define what is meant by "$\mathcal{F}$ is equicontinuous on K".

(b) State, for compact K, the Arzela-Ascoli theorem (giving a criterion for the compactness of $\mathcal{F} \subseteq C(K)$). [I assume that $\mathcal{F}$ is in $C(K)$ in the first place. Also, is the parenthetical part telling us what Arzela-Ascoli talks about, or are we also supposed to show this?]

(c) Let $\langle f_n \rangle$ be an equicontinuous sequence of functions from $\mathbb{R}$ to $\mathbb{R}^p$ $(p \geq 1)$ which converges at each rational number. Show that $\langle f_n \rangle$ converges at every point of $\mathbb{R}$ and the convergence is uniform.

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8. Do the following.

(a) Give a necessary and sufficient condition for a bounded function on $[a,b]$ to be Riemann integrable.

(b) Let $\langle r_n \rangle$ be an enumeration of the rationals in $[0,1]$. Define a function f on $[0,1]$ by

$$f(x) = \begin{cases} 1/n, & \text{if } x = r_n; \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Is f Riemann integrable? Explain!

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9. Give an example of a Lebesgue-integrable function f on $[0,1]$ with $\int_0^1 f(x)\,\mathrm{d}x = 1$ with the additional property that this function is not Riemann-integrable.

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10. Prove or disprove the statement: "If f is infinitely-differentiable in an open interval (a,b) and if there exists $x_0 \in (a,b)$ such that $f^{(k)}(x_0) = 0$ for all $k \geq 0$, then f is identically zero on (a,b)".

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11. Prove that a monotone function has at most countably-many discontinuities.

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12. Let $f_n(x) = ne^{nx}$.

(a) Determine $f(x) = \text{LIM}_{n\to\infty} f_n(x)$ for $x \in (-\infty, 0)$. Is the convergence uniform on $(-\infty, 0)$?

(b) Calculate $\int_{-\infty}^0 f_n(x)\,\mathrm{d}x$. Why does Lebesgue's Dominated Convergence Theorem not apply?

SAMPLE PROBLEMS FROM UNIVERSITY OF ARIZONA

1. State and prove the integral test for the absolute convergence of an infinite series $\sum_{j=1}^{\infty} f(j)$.

2. Suppose that $f\colon \mathbb{R}^2 \to \mathbb{R}$ is a smooth function and

$$\nabla f(x,y) = (2x - e^{-x} + 2xy^2 + 1, 2y + 2x^2y).$$

Show that f has a local minimum at $(x,y) = (0,0)$.

SOLUTION:

Note that the first partials f_x, f_y are zero at $(0,0)$; thus the possibility of a local extremum exists. The Taylor series of f at $(0,0)$ starts

$$f(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \tfrac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \tfrac{1}{2}f_{yy}(0,0)y^2 + \cdots$$

but removing the first partials (since they are zero) and moving the first term to the other side gives

$$f(x,y) - f(0,0) = \tfrac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \tfrac{1}{2}f_{yy}(0,0)y^2 + \cdots.$$

I would like to determine the sign of $f(x,y) - f(0,0)$ in a small neighborhood about $(0,0)$. The higher-order terms of the Taylor series don't interest me. To be conclusive about the sign of $f(x,y) - f(0,0)$, I need $f(x,y) - f(0,0)$ to remain strictly positive or strictly negative about $(0,0)$; that is, I need the quadratic portion of the Taylor series, when considered as a quadratic form, to be either positive definite or negative definite; this happens precisely when the discriminant

$$f_{xy}^2(0,0) - f_{xx}(0,0)f_{yy}(0,0)$$

is strictly negative, i.e., when

$$f_{xy}^2(0,0) < f_{xx}(0,0)f_{yy}(0,0).$$

Note that in this case $f_{xx} > 0$ and $f_{yy} > 0$ have the same sign, and that sign determines whether the paraboloid defined by the quadratic form opens upward or downward. That was the general theory; now let's consider the present problem.

Here, the mixed partials are

$$f_{xx} = 2 + e^{-x} + 2y^2 \qquad f_{xy} = f_{yx} = 4xy \qquad f_{yy} = 2 + 2x^2.$$

Evaluating at $(0,0)$, I see that the quadratic form under investigation is $\frac{3}{2}x^2 + y^2$, and without looking any further I recognize this is a paraboloid that opens upward, so that the point $(0,0)$ is indeed a local minimum, as was to be shown.

3. Let f be a nonnegative Lebesgue measurable function on $\mathbb{R}$. Let $\mu(E) = \int_E f \,\mathrm{d}x$.

(a) Show that μ is a measure on the σ-algebra of Lebesgue measurable subsets of $\mathbb{R}$.

(b) Show that μ is σ-finite if and only if f is finite almost everywhere (with respect to Lebesgue measure).

SOLUTION:

(a) Let E be any Lebesgue measurable subset of $\mathbb{R}$. Certainly the definition of $\mu(E)$ makes sense for such subsets. After all, by definition $\int_E f\,\mathrm{d}x = \int f \cdot 1_E \,\mathrm{d}x$, and 1_E is measurable and the product of two measurable functions is measurable. The integral of a nonnegative function is always nonnegative. Thus μ certainly is a set function, defined on the Lebesgue measurable sets, and taking values in the nonnegative extended real numbers. Then μ is a measure if and only if it is countable additive. Let E_1, E_2, ... be a sequence of pairwise disjoint measurable sets. Then

$$\begin{aligned}
\mu(\bigcup E_k) &= \int_{\bigcup E_k} f \,\mathrm{d}x = \int f \cdot 1_{\bigcup E_k} \,\mathrm{d}x = \int f \cdot (\sum 1_{E_k}) \,\mathrm{d}x = \int (\sum f \cdot 1_{E_k}) \,\mathrm{d}x \\
&= \int (\operatorname*{LIM}_{n\to\infty} \sum_{k=1}^{n} f \cdot 1_{E_k}) \,\mathrm{d}x = \operatorname*{LIM}_{n\to\infty} \int (\sum_{k=1}^{n} f \cdot 1_{E_k}) \,\mathrm{d}x = \operatorname*{LIM}_{n\to\infty} \sum_{k=1}^{n} \int f \cdot 1_{E_k} \,\mathrm{d}x = \sum_{n=1}^{\infty} \int_{E_k} f \,\mathrm{d}x \\
&= \sum_{n=1}^{\infty} \mu(E_k).
\end{aligned}$$

Although quoting the theorem of Beppo Levi would have shortened the caculation just performed, I effectively reproved that theorem (that integrals of infinite series of nonnegative functions become infinite series of integrals) in the situation under consideration; the limit popped out of the integral because the hypotheses of the monotone convergence theorem (namely, that one has a monotone sequence of nonnegative measurable integrands) were satisfied. Since μ is a countably-additive set function, defined on the σ-algebra of Lebesgue measurable sets, that takes nonnegative extended real number values and is countable additive, by definition μ is a measure on that σ-algebra.

(b) Let μ be σ-finite. Then there is a countable collection U_1, U_2, ... of measurable subsets of $\mathbb{R}$ such that these cover $\mathbb{R}$ and the μ-measure of each is finite; that is, for each k, $\mu(U_k) = \int_{U_k} f \, dx < \infty$. Thus on each U_k, the measure of the set where f is infinite must be zero; else the integral would be infinite as well. The union over k of the null sets where f is infinite inside each U_k is the entire subset of $\mathbb{R}$ on which f is infinite; but the countable union of null sets is a null set, and thus f is infinite on a set of measure zero. Now assume that the set where f is infinite is a null set. I wish to show that μ is σ-finite. Define the doubly-indexed sets

$$U_{n,k} = [n, n+1] \cap \{\, x : f(x) \leq k \,\}.$$

The Lebesgue measure of each set is at most 1, and note that f is bounded on each set. Thus the μ-measure of each such set is finite, since the μ-measure of $U_{n,k}$ is bounded by k. The doubly-indexed union $\bigcup U_{n,k}$ covers all of $\mathbb{R}$ except where f is infinite; but the set where f is infinite, being a Lebesgue null set, has μ-measure zero. Thus the countable collection of sets $U_{n,k}$, together with the μ-null set where f is infinite, forms a countable cover of $\mathbb{R}$ such that the μ-measure of each set in the cover is finite. Thus μ is σ-finite.

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4. It is well-known that if $f, g \in l^2(\mathbb{Z})$ (complex-valued sequences indexed by the integers) then

$$\bigl|\langle f, g\rangle\bigr| \leq \bigl|\langle f, f\rangle\bigr|^{1/2} \cdot \bigl|\langle g, g\rangle\bigr|^{1/2}$$

where

$$\langle f, g\rangle = \sum_{n\in\mathbb{Z}} f(n)\overline{g(n)}.$$

This is the Cauchy-Bunyakowski-Schwarz inequality. Prove it.

SOLUTION:

The inequality holds whenever any real or complex inner product is defined; in particular it holds for the inner product given here. I shall prove the inequality in complete generality—then I shall verify that the inner product defined above satisfies the necessary inner product axioms, so that my argument applies to the situation demanded here.

Let V be a real or complex vector space. Call a map $\langle\cdot,\cdot\rangle \to \mathbb{C}$ an inner product if it is positive definite, Hermitian bilinear (also called sesquilinear), and Hermitian symmetric (also called conjugate symmetric). That is, $\langle 0, 0\rangle = 0$, $\langle f, f\rangle > 0$ when $f \neq 0$, $\langle g, f\rangle = \overline{\langle f, g\rangle}$, $\langle af + g, h\rangle = a\langle f, h\rangle + \langle g, h\rangle$.

Let $f, g \in V$ be fixed. Now $\langle f, g\rangle$ is some complex number; let $c \in \mathbb{C}$ be such that $|c| = 1$ and $c\langle f, g\rangle$ is real; namely, let $c = \overline{\langle f, g\rangle}/|\langle f, g\rangle|$. Notice that then $c\langle f, g\rangle = \overline{c\langle f, g\rangle} = \overline{c}\langle g, f\rangle$. Let x be any real number. Then

$$\begin{aligned} 0 &\leq \langle cf + xg, cf + xg\rangle = c\langle f, cf + xg\rangle + x\langle g, cf + xg\rangle \\ &= c\overline{c}\langle f, f\rangle + xc\langle f, g\rangle + x\overline{c}\langle g, f\rangle + x^2\langle g, g\rangle \\ &= \langle f, f\rangle + 2xc\langle f, g\rangle + x^2\langle g, g\rangle. \end{aligned}$$

The last line in the display is a quadratic polynomial in x with real coefficients that is never negative. Thus its discriminant is zero or negative; that is,

$$(2c\langle f, g\rangle)^2 - 4\langle f, f\rangle\langle g, g\rangle \leq 0.$$

Using $c\langle f, g\rangle = \overline{c\langle f, g\rangle} = |\langle f, g\rangle|^2$, this becomes

$$4\bigl|\langle f, g\rangle\bigr|^2 \leq 4\langle f, f\rangle\langle g, g\rangle.$$

Dividing both sides by 4 and taking square-roots (note that both sides are positive) gives

$$\bigl|\langle f, g\rangle\bigr| \leq \sqrt{\langle f, f\rangle}\sqrt{\langle g, g\rangle},$$

as was to be proved.

To finish the problem, I shall show that the inner product defined on l^2 in the problem statement satisfies the definition of inner product used in my solution above. Indeed, if f is a sequence of complex numbers, then

$$\langle f, f\rangle = \sum_{n\in\mathbb{Z}} f(n)\overline{f(n)} = \sum_{n\in\mathbb{Z}} |f(n)|^2,$$

and thus $\langle f, f\rangle \geq 0$ and certainly $\langle f, f\rangle = 0$ if and only if every term of the sequence f is zero. Since complex conjugation is continuous, conjugation can be brought in and out of the summation limit, giving $\langle f, g\rangle = \overline{\langle g, f\rangle}$. Finally, let f, g, h be sequences and let a be an arbitrary complex number. Then

$$\begin{aligned}\langle af + g, h\rangle &= \sum_{n\in\mathbb{Z}} \big(af(n) + g(n)\big)\overline{h(n)}\\ &= \sum_{n\in\mathbb{Z}} af(n)\overline{h(n)} + g(n)\overline{h(n)}.\end{aligned}$$

Let me show that the last series is absolutely convergent.

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} \big|af(n)\overline{h(n)} + g(n)\overline{h(n)}\big|\\ &\qquad\qquad \leq \sum_{n\in\mathbb{Z}} |a|\big|f(n)\big|\big|h(n)\big| + \big|g(n)\big|\big|h(n)\big|\\ &\qquad\qquad oops\end{aligned}$$

[I'M STUCK]

Although the Cauchy-Bunyakowski-Schwarz inequality holds in general for any inner product, I can prove it directly for the particular inner product given in this problem. Notice that both sides of the inequality are positive; thus the inequality is equivalent to its squared form

$$\big|\langle f, g\rangle\big|^2 \leq \langle f, f\rangle \cdot \langle g, g\rangle.$$

Since complex conjugation is continuous and therefore may be brought inside and out of the convergent infinite sums, notice that $\langle f, g\rangle = \overline{\langle g, f\rangle}$. Thus the inequality is equivalent to the form

$$\langle f, g\rangle \cdot \langle g, f\rangle \leq \langle f, f\rangle \cdot \langle g, g\rangle.$$

Now this is just

$$\sum_{n\in\mathbb{Z}} f(n)\overline{g(n)} \cdot \sum_{n\in\mathbb{Z}} g(n)\overline{f(n)} \leq \sum_{n\in\mathbb{Z}} \big|f(n)\big|^2 \cdot \sum_{n\in\mathbb{Z}} \big|g(n)\big|^2.$$

And is this true?

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5. Suppose that f is continuous and compactly supported on $\mathbb{R}$. Show that

$$\hat{f}(xi) = \frac{1}{2\pi}\int_R f(x)e^{-ix\xi}\,\mathrm{d}x$$

is an entire function of ξ. If $\big|f(x)\big| \leq Ce^{-B|x|}$ for some constants $B, C > 0$, show that $\hat{f}(\xi)$ is analytic in the strip $|\xi| < B$.

SOLUTION:

[This doesn't make sense. If I show the function is entire, then this means it is analytic everywhere. Why would I need to show it is analytic in some region knowing additional information about the function? I already showed it to be entire!]

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6. Let $I_n = \int_0^1 \big|f_n(x) - f(x)\big|\,\mathrm{d}x$. Give an example illustrating the following situations, or prove that no such example can be found.

(a) $f_n \to f$ almost everywhere with respect to Lebesgue measure on $\mathbb{R}$ but I_n does not tend to 0.

(b) I_n tends to 0 as $n \to \infty$ but $f_n(x)$ does not tend to $f(x)$ almost everywhere with respect to Lebesgue measure.

(c) I_n tends to 0 as $n \to \infty$ but it is not true that

$$\text{measure}\big\{\, x : \big|f_n(x) - f(x)\big| > \alpha \,\big\} \to 0 \quad \text{for all } \alpha > 0.$$

7. Let f_n be a sequence of continuous functions on $[0,1]$. We say that $f_n \to 0$ weakly if $\int f_n \, \mathrm{d}\mu \to 0$ for each regular Borel measure on $[0,1]$. Prove that $f_n \to 0$ weakly if and only if $f_n(x) \to 0$ for each $x \in [0,1]$ and $\sup \bigl|f_n(x)\bigr| < \infty$, where the supremum is taken over all n and all $x \in [0,1]$.

8. Let f, g be Lebesgue integrable on $[0,1]$. Let

$$F(x) = \int_0^x f(t)\,\mathrm{d}t, \qquad G(x) = \int_0^x g(t)\,\mathrm{d}t.$$

Prove the integration-by-parts formula:

$$\int_0^1 F(x)g(x)\,\mathrm{d}x = F(1)G(1) - \int_0^1 f(x)G(x)\,\mathrm{d}x.$$

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN AUGUST 1992

1. Determine (with a proof) whether or not the following series converge:

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$;

(b) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^2}$.

SOLUTION:

One of the canned methods (a.k.a. recipes) for determining whether series converge is the so-called "integral test". I find it difficult to correctly memorize the various hypotheses and steps that go with each convergence test. However, I am familiar with the general (and highly applicable) techniques that form the foundation of the integral test; these I use below.

(a) I shall prove that the series in part (a) diverges to infinity. Define the step-like function $f(x)\colon [2,\infty) \to \mathbb{R}$ via the formula $f(x) = 1/(n \ln n)$ when $x \in [n, n+1)$. Then $\sum_{n=2}^{\infty} 1/(n \ln n) = \int_2^{\infty} f(x)\,\mathrm{d}x$, by which I mean that the series converges if and only if the integral converges (i.e., is finite) and furthermore, if both converge, then they are equal. The function $g(x) = 1/(x \ln x)$ is monotone decreasing (since both denominators are monotone increasing) and agrees with $f(x)$ when $x \geq 2$ is an integer; but $f(x)$ is constant between consecutive integers; thus $g(x) \leq f(x)$ for $x \in [2,\infty)$. I shall presently verify that $\int_2^{\infty} g(x)\,\mathrm{d}x$ diverges; thus $\int_2^{\infty} f(x)\,\mathrm{d}x = \sum_{n=2}^{\infty} 1/(n \ln n)$ diverges. The remaining task is a calculation:

$$\begin{aligned}\int_2^{\infty} g(x)\,\mathrm{d}x &= \int_2^{\infty} \frac{\mathrm{d}x}{x \ln x} \\ &= \lim_{n\to\infty} \int_2^{n} \frac{\mathrm{d}x}{x \ln x} \qquad \text{(substitute } u = \ln x\text{)} \\ &= \lim_{n\to\infty} \int_{\ln 2}^{\ln n} \frac{\mathrm{d}u}{u} = \lim_{n\to\infty} \ln(\ln n) - \ln 2;\end{aligned}$$

since the logarithm is unbounded as its input increases, the limit diverges to infinity.

(b) I shall prove that the series in part (b) converges. Similar to before, define the step-like function $f(x)\colon [2,\infty) \to \mathbb{R}$ via the formula $f(x) = 1/\bigl(n(\ln n)^2\bigr)$ when $x \in [n, n+1)$. Then the series in question has the same convergence behavior as $\int_2^\infty f(x)\,\mathrm{d}x$ and they are equal when they converge. The function $g(x) = 1/\big(x(\operatorname{LN} x)^2\big)$ is monotone decreasing on $[2,\infty)$ and agrees with $f(x)$ when $x \geq 2$ is an integer; but $f(x)$ is constant between consecutive integers; thus $g(x) \leq f(x)$ for $x \in [2,\infty)$. This is not so useful; however, a simple shift to the right makes it useful: the function $h(x) = g(x-1)$ satisfies $f(x) \leq h(x)$ for $x \in [3,\infty)$. I shall presently verify that $\int_3^\infty h(x)\,\mathrm{d}x$ converges; thus $\int_3^\infty f(x)\,\mathrm{d}x = \sum_{n=3}^\infty 1/\big(n(\operatorname{LN} n)^2\big)$ converges; thus the series in question, which consists of one additional term, also converges. The remaining task is a calculation:

$$\begin{aligned}
\int_3^\infty h(x)\,\mathrm{d}x &= \int_3^\infty \frac{\mathrm{d}x}{(x-1)\big(\operatorname{LN}(x-1)\big)^2} \\
&= \operatorname*{LIM}_{n\to\infty} \int_3^n \frac{\mathrm{d}x}{(x-1)\big(\operatorname{LN}(x-1)\big)^2} \qquad \text{(substitute } y = x-1\text{)} \\
&= \operatorname*{LIM}_{n\to\infty} \int_2^{n-1} \frac{\mathrm{d}y}{y(\operatorname{LN} y)^2} \qquad \text{(substitute } u = \operatorname{LN} y\text{)} \\
&= \operatorname*{LIM}_{n\to\infty} \int_{\operatorname{LN} 2}^{\operatorname{LN}(n-1)} \frac{\mathrm{d}u}{u^2} = \operatorname*{LIM}_{n\to\infty} \Big(-\frac{1}{\operatorname{LN}(n-1)} + \frac{1}{\operatorname{LN} 2}\Big) \\
&= \frac{1}{\operatorname{LN} 2} - \operatorname*{LIM}_{n\to\infty} \frac{1}{\operatorname{LN}(n-1)};
\end{aligned}$$

since the logarithm is unbounded as its input increases, the denominator diverges to infinity and thus the limit converges to 0.

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2. Prove that a continuous real-valued function on $[0,1]$ is uniformly continuous. State any theorems that you use.

SOLUTION:

The Heine-Borel theorem characterizes the compact sets in $\mathbb{R}^n$ as the sets that are closed and bounded; thus the interval $[0,1]$ is compact in $\mathbb{R}^1$. I shall presently prove that a continuous real-valued function on any compact set in $\mathbb{R}^1$ is uniformly continuous.

Let $f(x)$ be a continuous real-valued function on the compact set $C \subset \mathbb{R}^1$. Let $\varepsilon > 0$ be given. To show that $f(x)$ is uniformly continuous on C, I must show the existence of a single $\delta > 0$ such that $\big|f(x) - f(y)\big| < \varepsilon$ whenever $|x-y| < \delta$. Corresponding to each $x \in C$, continuity guarantees the existence of a $\delta_x > 0$ such that $\big|f(x) - f(y)\big| < (\varepsilon/2)$ whenever $|x-y| < \delta_x$. Naturally I should like to let the single δ be the smallest of these various δ_x; the problem is that there are infinitely many of these δ_x and thus their infimum might well be zero; I must use the compactness of C to throw away all but a finite number of these δ_x. The sets $\{\, y \in \mathbb{R} : |x-y| < (\delta_x/2) \,\}$, indexed by x, are open and cover C; thus a finite number of them cover C; call the corresponding indices $x_1, \ldots, x_n$. Let $\delta = \operatorname{MIN}_{1\leq i\leq n} \delta_{x_i}$; notice that $\delta > 0$. I shall next show that this δ turns the trick.

Pick $x, y \in C$ with $|x-y| < \delta$. The point x, since it lies in C, is covered by my finite cover; that is, there is an index i (and fix it for the rest of the proof) such that $|x_i - x| < (\delta_{x_i}/2)$. In particular, also $|x_i - x| < \delta_{x_i}$ and thus $\big|f(x_i) - f(x)\big| < (\varepsilon/2)$. Furthermore, $|x_i - y| = |x_i - x + x - y| \leq |x_i - x| + |x-y| < (\delta_{x_i}/2) + \delta \leq (\delta_{x_i}/2) + (\delta_{x_i}/2) = \delta_{x_i}$, and thus $\big|f(x_i) - f(y)\big| < (\varepsilon/2)$. Combining the pieces, $\big|f(x) - f(y)\big| = \big|f(x) - f(x_i) + f(x_i) - f(y)\big| \leq \big|f(x) - f(x_i)\big| + \big|f(x_i) - f(y)\big| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$. To summarize, I have proved the existence of a single $\delta > 0$ such that $\big|f(x) - f(y)\big| < \varepsilon$ whenever $|x-y| < \delta$; that is, the function $f(x)$ is uniformly continuous on C.

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3. Is every function on $\mathbb{R}$ that is almost everywhere equal to a continuous function also almost everywhere continuous? Prove the result or give a counterexample with proof.

SOLUTION:

The answer is "no". Define the function $\psi\colon [0,1] \to \mathbb{R}$ to take the value 0 on rational inputs and 1 on irrational inputs. The rationals form a null set and thus the function ψ equals the continuous function $g(x) = 1$ almost everywhere. Yet, as I shall presently prove, the function ψ is continuous nowhere.

Choose any $\varepsilon < 1$ (and $\varepsilon = 1/2$ will do just fine). An arbitrarily small δ-neighborhood about either a rational or irrational number contains both rational and irrational numbers. Thus the function outputs can be separated to 1 (thus exceeding ε) no matter how small any δ confines the inputs.

4. Let μ be a measure on the Borel subsets of $\mathbb{R}$ such that $\mu([0,1]) = 1$ and $\mu(E + x) = \mu(E)$ for all Borel sets E. Prove that μ agrees with Lebesgue measure on the Borel sets.

SOLUTION:

Translation invariant measures that assign finite measure to an infinite set must be nonatomic; that is, they must assign zero measure to points. Although the general proof is almost identical, I shall avoid introducing superfluous notation by focusing on the present problem: if μ assigns positive measure to the set $\{0\}$, then by countable additivity μ must assign infinite measure to the set of rationals in $[0,1]$, and thus by monotonicity μ would have to assign infinite measure to the entire interval $[0,1]$; since, however, that is not the case, we must have $\mu(\{0\}) = 0$ and thus, by translation invariance, $\mu(\{x\}) = 0$ for all x. Therefore, throughout the rest of this solution, whether endpoints of intervals are included is immaterial.

I will use the following theorem, which I assume without proof. If one has, on a collection of subsets called a semiring or semialgebra, a measure-like function that is σ-finite, then this function extends uniquely to a measure on the smallest σ-algebra containing the semialgebra. By a "measure-like" function I mean a set function that assigns nonnegative extended real numbers to sets in some collection of subsets with the additional properties of assigning zero to the empty set and being countably additive (for those countable collections whose union is in the domain of the function); a measure-like function is a measure precisely when the domain is a σ-algebra. A measure-like function is σ-finite if the entire space can be covered by a countable collection of sets in the domain of the function such that the function is finite on each member of this collection. My impression of reading various authors is that there is no general agreement on the terms "semiring" and "semialgebra"; for some they are equivalent, while for others the latter must contain the entire space as an element; the theorem quoted above is true for the less restrictive notion of "semiring", which I now define. A collection of subsets of some space is a semiring if it contains the empty set, if it is closed under intersection, and if the complement of any member of the collection can be written as a finite union of members of the collection; note that the latter condition is equivalent to any complement being written as a finite union of *pairwise disjoint* members of the collection.

The σ-algebra of Borel sets is often defined as the smallest σ-algebra that contains the open sets. Since every open set can be written as a countable union of pairwise disjoint open intervals, the σ-algebra of Borel sets is in fact the smallest σ-algebra containing all open intervals. Since σ-algebras contain complements, the Borel sets include all closed sets, and thus the σ-algebra of Borel sets can also be classified as the smallest σ-algebra containing all open or closed intervals; this is a technicality but useful because the collection of all open or closed intervals, together with the empty set, forms a semiring. **[NO: Need all intervals, or half-open ones?]**

Note that μ, restricted to the semiring of open or closed intervals (and the empty set), is a σ-finite measure-like function; it inherits the measure-like properties from its larger original domain of definition (the Borel sets), and it is σ-finite on the semiring because the line can be covered by the countable collection of intervals of the form $[n, n+1]$, where n is an integer, and by translation invariance the μ-measure of each such interval is $\mu([0,1]) = 1$. Thus the theorem applies and μ extends uniquely to a measure on the Borel sets; but Lebesgue measure is also an extension of a measure-life function (namely its restriction) on this semiring to the Borel sets; thus, if μ agrees with Lebesgue measure on the semiring, then μ must agree with Lebesgue measure on the Borel sets.

The remaining task is to show that μ agrees with Lebesgue measure on the semiring of open and closed intervals (and the empty set). They certainly agree on the empty set (measures always assign zero to the empty set); if an interval has infinite length, then one can easily translate infinitely disjoint copies of $[0,1]$ into that interval, and thus μ, just like Lebesgue measure, assigns infinite measure to such an interval. As stated before, μ doesn't concern itself with single points, and neither does Lebesgue measure; thus I may ignore closed intervals and now must only show that μ agrees with Lebesgue measure on open bounded intervals, i.e., intervals of the form (a,b). Since both μ and Lebesgue measure are translation invariant, I may reduce the consideration to intervals of the form $(0,b)$. (Although it should be clear, let me avoid misunderstanding by emphasizing that b is positive.) I shall break the matter into two cases.

If b is rational, say $b = c/d$, then the interval $(0,c)$ can be covered (except for finitely-many points) by d disjoint copies of $(0,b)$; the interval $(0,c)$ can be covered (except for finitely-many points) by c disjoint copies of $(0,1)$. Thus μ, just like Lebesgue measure, assigns measure c to $(0,c)$, and therefore μ, just like Lebesgue measure, assigns measure $c/d = b$ to $(0,c/d) = (0,b)$. That is, μ agrees with Lebesgue measure on intervals of the form $(0,b)$, where b is rational.

If b is irrational, then let b_n be a monotone increasing sequence of rational numbers whose limit is b. The intervals $(0,b_n)$ form an increasing nested family of sets whose union is $(0,b)$. Measures are "continuous" for increasing nested families; that is,

$$\operatorname{LIM}_{n\to\infty} \mu\big((0,b_n)\big) = \mu\Big(\bigcup_{n=1}^{\infty}(0,b_n)\Big) = \mu\big((0,b)\big).$$

Since each b_n is rational, the previously considered case tells us that $\mu\big((0,b_n)\big) = b_n$; thus $\mu\big((0,b)\big) = \operatorname{LIM}_{n\to\infty} b_n = b$. That is, μ agrees with Lebesgue measure on intervals of the form $(0,b)$, where b is irrational.

Let me summarize. The measure μ agrees with Lebesgue measure on all intervals of the form $(0,b)$, where b is any positive real number. By translation invariance, μ agrees with Lebesgue measure on all intervals of the form (a,b). Since μ is nonatomic, it agrees with Lebesgue measure on all intervals of the form $[a,b]$. Furthermore, they agree on the empty set and on intervals of infinite length. Thus they agree on a semiring for which both are σ-finite. Thus they extend uniquely to a measure on the smallest σ-algebra containing that semiring, and that σ-algebra is the Borel sets. But they each already are measures on the Borel sets. Since the extension is unique, they must be equal on the Borel sets.

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5. Find, with a proof:

$$\operatorname{LIM}_{r\to\infty} \int_0^r \left(1-\frac{x}{r}\right)^r \mathrm{d}x.$$

SOLUTION:

If one substitutes $u = 1-(x/r)$, then $\mathrm{d}u = -(\mathrm{d}x/r)$. Thus

$$\begin{aligned}\operatorname{LIM}_{r\to\infty} \int_0^r \left(1-\frac{x}{r}\right)^r \mathrm{d}x &= \operatorname{LIM}_{r\to\infty} -r\int_1^0 u^r\,\mathrm{d}u = \operatorname{LIM}_{r\to\infty} r\int_0^1 u^r\,\mathrm{d}u \\ &= \operatorname{LIM}_{r\to\infty} r\left(\frac{u^{r+1}}{r+1}\Big|_0^1\right) = \operatorname{LIM}_{r\to\infty} r\left(\frac{1}{r+1}\right) = 1.\end{aligned}$$

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6. Let $1 \le p < r < q \le \infty$. Prove that if $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$, then $f \in L^r(\mathbb{R})$.

SOLUTION:

Define $A = \{\, x \in \mathbb{R} : |f(x)| > 1 \,\}$. Note $\mathbb{R} = A \cup A^{\complement}$. Furthermore, A must have finite measure, since the measure is just $\mu A = \int_A 1 \le \int_A |f|^p \le \int_{\mathbb{R}} |f|^p < \infty$. Note $\int_{\mathbb{R}} |f|^r = \int_A |f|^r + \int_{A^{\complement}} |f|^r$; I will show that each term of the sum is finite, thus establishing that $f \in L^r(\mathbb{R})$.

On $A^{\complement}$, we have $|f| \le 1$ and thus $|f|^r \le |f|^p$. Hence

$$\int_{A^{\complement}} |f|^r \le \int_{A^{\complement}} |f|^p \le \int_{\mathbb{R}} |f|^p < \infty,$$

and thus $\int_{A^{\complement}} |f|^r$ is finite.

If $q = \infty$, then $f \in L^\infty(\mathbb{R})$, meaning that there is a constant M such that $|f| \le M$ except on a null set; then $\int_A |f|^r \le \int_A M^r = M^r \cdot \mu A < \infty$. Otherwise, if $q < \infty$, then on A we have $|f| > 1$ and thus $|f|^r \le |f|^q$; then

$$\int_A |f|^r \le \int_A |f|^q \le \int_{\mathbb{R}} |f|^q < \infty,$$

and thus $\int_A |f|^r$ is finite.

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7. Let $D = \{\, x \in \mathbb{R}^n : \|x\| \le 1 \,\}$ denote the unit ball in $\mathbb{R}^n$. Find the values of $\alpha \in \mathbb{R}$ so that $L^2(D)$ contains the function

$$f(x) = \big(1 - \|x\|\big)^{\alpha}.$$

SOLUTION:

Note that $\int_D |f|^2 = \int_D \big|\big(1-\|x\|\big)^{\alpha}\big|^2 = \int_D \big(1-\|x\|\big)^{2\alpha}$. I wish to determine for which $\alpha \in \mathbb{R}$ this integral is finite. Note that D has finite measure. Since $1 - \|x\| \le 1$ on D, the integrand is bounded by 1 when $\alpha \ge 0$; thus, for $\alpha \ge 0$, certainly $\int_D \big(1-\|x\|\big)^{2\alpha} \le \mu D < \infty$. The remaining task is to determine for which negative α the square of the function is integrable.

I shall first explore low-dimensional cases so that I can form some reasonable conjectures and then attempt to answer them. For $n = 1$, the unit ball is the interval $D = [-1, 1]$, and

$$\int_D \left(1 - \|x\|\right)^{2\alpha} = \int_{-1}^{1} \left(1 - |x|\right)^{2\alpha} \mathrm{d}x = 2\int_0^1 (1-x)^{2\alpha} \mathrm{d}x = 2\int_0^1 u^{2\alpha} \mathrm{d}u.$$

If $\alpha = -\frac{1}{2}$, then the integral is infinite; else, if $\alpha \neq -\frac{1}{2}$, then the integral is finite so long as $2\alpha + 1 > 0$, or simply $\alpha > -\frac{1}{2}$. For $n = 2$, the unit ball is the solid unit circle, and

$$\int_D \left(1 - \|x\|\right)^{2\alpha} = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \left(1 - \sqrt{x^2+y^2}\right)^{2\alpha} \mathrm{d}y\, \mathrm{d}x.$$

Changing to polar coordinates gives

$$\begin{aligned}\int_0^{2\pi}\int_0^1 (1-r)^{2\alpha} r \,\mathrm{d}r\,\mathrm{d}\theta &= \int_0^{2\pi} \mathrm{d}\theta \int_0^1 r(1-r)^{2\alpha}\,\mathrm{d}r = 2\pi \int_0^1 (1-u)u^{2\alpha}\,\mathrm{d}u \\ &= 2\pi\int_0^1 u^{2\alpha}\,\mathrm{d}u - 2\pi\int_0^1 u^{2\alpha+1}\,\mathrm{d}u.\end{aligned}$$

I already saw that the first integral is finite precisely when $\alpha > -\frac{1}{2}$. A similar analysis shows that the second integral is finite precisely when $\alpha > -1$. Since the two conditions must hold simultaneously, I again have the simple condition $\alpha > \frac{1}{2}$. Although I now have a reasonable conjecture, let me consider one more case. For $n = 3$, the unit ball is the solid unit sphere. Switching to spherical coordinates gives

$$\begin{aligned}\int_D \left(1 - \|x\|\right)^{2\alpha} &= \int_0^{\pi}\int_0^{2\pi}\int_0^1 (1-r)^{2\alpha} r^2 \sin\phi \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = \int_0^{\pi}\int_0^{2\pi} \sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi \int_0^1 r^2(1-r)^{2\alpha}\,\mathrm{d}r \\ &= \int_0^{\pi}\int_0^{2\pi} \sin\phi, \mathrm{d}\theta\,\mathrm{d}\phi \int_0^1 (1-u)^2 u^{2\alpha}\,\mathrm{d}u = V\cdot\int_0^1 (u^{2\alpha} - 2u2\alpha + 1 + u^{2\alpha+2})\,\mathrm{d}u.\end{aligned}$$

Note that $V = \int_0^{\pi}\int_0^{2\pi} \sin\phi\,\mathrm{d}\theta\,\mathrm{d}\phi$ is finite. As before, the conditions on α for the last few integrals to converge combine to give $\alpha > -\frac{1}{2}$. Now I am prepared to answer the general case.

My conjecture is that $f(x) \in L^2(D)$ precisely when $\alpha > -\frac{1}{2}$. I shall need to use higher-dimensional spherical coordinates. Given a point $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, I can repeatedly project the point into hyperplanes to obtain

$$\begin{aligned}x_1 &= \rho\cos\theta \\ x_2 &= \rho\sin\theta\cos\phi_1 \\ x_3 &= \rho\sin\theta\sin\phi_1\cos\phi_2 \\ \vdots &= \quad\vdots \\ x_{n-1} &= \rho\sin\theta\sin\phi_1\sin\phi_2\cdots\sin\phi_{n-3}\cos\phi_{n-2} \\ x_n &= \rho\sin\theta\sin\phi_1\sin\phi_2\cdots\sin\phi_{n-3}\sin\phi_{n-2};\end{aligned}$$

here $0 < \rho < \infty$, $0 \le \theta < 2\pi$, and for each ϕ_k, $0 \le \phi_k < \pi$. To effect a change-of-variable into these coordinates, I need to know how the volume form $\mathrm{d}x_1\,\mathrm{d}x_2\,\cdots\,\mathrm{d}x_n$ is written in terms of the new coordinates. I ask you to take it on faith (I have figured this out before but don't wish to get bogged down with this now) that the new volume form is $\rho^{n-1}g(\theta, \phi_1, \ldots, \phi_{n-2})\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi_1\,\cdots\,\mathrm{d}\phi_{n-2}$, where the function g is expressed using various sines and cosines of the angles. Then

$$\begin{aligned}\int_D \left(1 - \|x\|\right)^{2\alpha} &= \int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi}\int_0^1 (1-\rho)^{2\alpha}\rho^{n-1}g(\theta,\phi_1,\ldots,\phi_{n-2})\,\mathrm{d}\rho\,\mathrm{d}\theta\,\mathrm{d}\phi_1\,\cdots\,\mathrm{d}\phi_{n-2} \\ &= \int_0^{\pi}\cdots\int_0^{\pi}\int_0^{2\pi} g(\theta,\phi_1,\ldots,\phi_{n-2})\,\mathrm{d}\theta\,\mathrm{d}\phi_1\,\cdots\,\mathrm{d}\phi_{n-2}\int_0^1 (1-\rho)^{2\alpha}\rho^{n-1}\,\mathrm{d}\rho \\ &= V\cdot\int_0^1 u^{2\alpha}(1-u)^{n-1}\,\mathrm{d}u \\ &= V\cdot\int_0^1 u^{2\alpha}\sum_{k=0}^{n-1}\binom{n-1}{k}(-u)^k\,\mathrm{d}u \\ &= V\cdot\sum_{k=0}^{n-1}(-1)^k\binom{n-1}{k}\int_0^1 u^{2\alpha+k}\,\mathrm{d}u\end{aligned}$$

and as before, I know that all integrals are finite if and only if $\alpha > -\frac{1}{2}$. Thus the answer to the problem is that $f(x) \in L^2(D)$ precisely when $\alpha > -\frac{1}{2}$. Note that the answer is independent of the dimension n in which the question is posed.

8. Suppose that f is a non-constant periodic differentiable function with period 1 and mean 0. Use Fourier series to show that
$$\int_0^1 \left|f(x)\right|^2 \,\mathrm{d}x \le \frac{1}{4\pi^2}\int_0^1 \left|f'(x)\right|^2 \,\mathrm{d}x.$$

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN JANUARY 1993

1. What is the Lebesgue measure of the set of numbers in $[0,1]$ such that in their decimal expansion the string "1993" does not appear as four consecutive digits.

SOLUTION:

We can rethink this problem in terms of probabilities. Imagine a process (such as an abstract pin being dropped at random on the unit interval) that chooses numbers in $[0,1]$ at random (using a uniform distribution); then the probability that the result of this process is a number without "1993" is precisely the Lebesgue measure of the set in question. Ask yourself what the probability is that a random number contains the string "1993"; I claim that this happens with probability 1. After all, think of the production of a random number in $[0,1]$ as a sequential process of producing countably many digits chosen at random but uniformly distributed. At any point in this process, the probability that the next four digits are "1993" is $1/10000$; but the process goes on forever and thus with probability 1 we can say that "1993" will actually occur. Therefore the Lebesgue measure of set of numbers in $[0,1]$ such that in their decimal expansion the string "1993" does not appear as four consecutive digits has measure zero.

2. Let $f(x)$ be a Lebesgue integrable function on $[0,\infty)$ such that $\left|f(x)\right| < \pi/2$ almost everywhere.
(a) Prove that
$$\lim_{n\to\infty}\int_0^\infty \big(\sin f(x)\big)^n \,\mathrm{d}x = 0.$$
(b) Suppose that in place of the assumption that $f \in L^1\big([0,\infty)\big)$, we instead assume that $f \in L^p\big([0,\infty)\big)$ for some p with $1 < p < \infty$. Is the conclusion still true? Justify your answer.

SOLUTION:

(a) I will need the fact that $|\sin x| \le |x|$ for all x. By symmetry and appealing to a graph, I must merely show $\sin x \le x$ for $0 < x < \pi/2$; in fact, I shall show $\sin x \le x$ for $x \ge 0$; I give two proofs. The derivative of $x - \sin x$ is $1 - \cos x$; the latter is nonnegative and thus $x - \sin x$ is nondecreasing; at 0, the value of $x - \sin x$ is 0; thus, for $x \ge 0$ we have $0 \le x - \sin x$ or simply $\sin x \le x$. The second proof uses the mean value theorem. For $x > 0$, there is a constant c such that $(\sin x - \sin 0)/(x - 0) = \cos c$; that is, $\sin x = x\cos c \le x$.

Since $\left|f(x)\right| < \pi/2$ almost everywhere, we have $\left|\sin f(x)\right| < 1$ almost everywhere. Thus almost everywhere we have $\left|\big(\sin f(x)\big)^n\right| = \left|\sin f(x)\right|^n \le \left|\sin f(x)\right| \le \left|f(x)\right|$; that is, the integrand in question is absolutely bounded by the integrable function $\left|f(x)\right|$; the dominated convergence theorem applies and allows the limit to slide inside the integral. Since $-1 < \sin f(x) < 1$, we have $\lim_{n\to\infty}\big(\sin f(x)\big)^n = 0$. Thus
$$\lim_{n\to\infty}\int_0^\infty \big(\sin f(x)\big)^n \,\mathrm{d}x = \int_0^\infty \lim_{n\to\infty}\big(\sin f(x)\big)^n \,\mathrm{d}x = \int_0^\infty 0 \,\mathrm{d}x = 0.$$

(b) If $1 < p < \infty$, then the conclusion is still true. Eventually $n > p$ in the limit. From this point onward, we have $\left|\big(\sin f(x)\big)^n\right| = \left|\sin f(x)\right|^n \le \left|\sin f(x)\right|^p \le \left|f(x)\right|^p$; that is, the integrand in question is absolutely bounded by the integrable function $\left|f(x)\right|^p$; the dominated convergence theorem applies as before and the conclusion holds.

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3. Let μ be a finite measure on X, and let f be a measurable real-valued function on X. Define a function F on $\mathbb{R}$ by
$$F(a) = \mu\Big(\big\{\, x \in X : f(x) \le a \,\big\}\Big).$$

(a) Prove that F is continuous from the right; i.e., $\mathrm{LIM}_{y\to a^+} F(y) = F(a)$.
(b) Find a condition involving μ and f that is necessary and sufficient for F to be continuous.

SOLUTION:

I will need two facts about measures. If A_n is an increasing family of nested sets whose union is A, then $\mathrm{LIM}_{n\to\infty} \mu A_n = \mu A$. If A_n is a decreasing family of nested sets whose intersection is A with the additional property that at least one of the sets, say the first one, has finite measure, then $\mathrm{LIM}_{n\to\infty} \mu A_n = \mu A$. One often quotes these facts verbally by saying that measures are always "continuous from below", and are "continuous from above" when infinity is avoided. In particular, finite measures are continuous from below and from above.

(a) One way to prove $\mathrm{LIM}_{y\to a^+} F(y) = F(a)$ is to prove $\mathrm{LIM}_{n\to\infty} F(y_n) = F(a)$ for *any* monotone decreasing sequence y_n whose limit is a. Fix such a sequence. Let $A_n = \big\{\, x \in X : f(x) \le y_n \,\big\}$. The sets A_n form a decreasing family of nested sets whose intersection is $A = \big\{\, x \in X : f(x) \le a \,\big\}$. Thus (since the measure is finite) $\mathrm{LIM}_{n\to\infty} F(y_n) = \mathrm{LIM}_{n\to\infty} \mu A_n = \mu A = F(a)$.

(b) We know that F is continuous from the right; to determine when F is continuous, I must determine when F is continuous from the left. Letting y_n be any monotone *increasing* sequence whose limit is a, the sets A_n form an increasing family of nested sets whose *union* is $\hat{A} = \big\{\, x \in X : f(x) < a \,\big\}$. Thus $\mathrm{LIM}_{n\to\infty} F(y_n) = \mu\hat{A}$. Let $P = \big\{\, x \in X : f(x) = a \,\big\}$. Then $\mu A = \mu\hat{A} + \mu P$. Thus $\mathrm{LIM}_{n\to\infty} F(y_n)$ will equal μA precisely when $\mu P = 0$. That is, F is continuous at a point a if and only if $\mu f^{-1}[a] = 0$.

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4. Suppose that c_n is a sequence of numbers such that $\sum_{n=-\infty}^{\infty} |c_n| < \infty$. Prove that the series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ is the Fourier series of a continuous periodic function.

SOLUTION:

Define $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ for $x \in [0, 2\pi]$. Note that f is well-defined on $[0, 2\pi]$ since the calculation
$$\sum_{n\in\mathbb{Z}} |c_n e^{inx}| = \sum_{n\in\mathbb{Z}} abs c_n < \infty$$
shows that the series is absolutely convergent and thus convergent. Since $f(0) = f(2\pi)$, the function f may be extended to all of $\mathbb{R}$ by periodic continuation, namely by defining $f(x + 2k\pi) = f(x)$. The function f will be continuous if and only if it is continuous on $[0, 2\pi]$. The sequence of numbers $|c_n|$ is a suitable sequence for the Weierstrass M-Test; namely, each term of the series satisfies $|c_n e^{inx}| \le |c_n|$ for all x, and the series of $|c_n|$ converges; thus the conditions of the M-Test are satisfied and thus the series defining f converges uniformly. Since continuity is preserved under uniform convergence and each partial sum, being the finite sum of continuous functions, is continuous, also the limit function f is continuous.

A theorem states that if a Fourier series converges uniformly, then it is the Fourier series of the function to which it converges. Thus one sees that the series given in the problem statement is the series of the continuous periodic function f; the problem is therefore solved. However, without relying on the theorem, one can easily compute the Fourier series of f and arrive back at the series given in the problem statement. The orthonormal basis of Fourier functions comprises the functions e^{inx}, where $n \in \mathbb{Z}$. The Fourier series of a function f is the series
$$\sum_{n\in\mathbb{Z}} \langle f, e^{inx} \rangle e^{inx}.$$
Thus the Fourier series of the function f defined in this problem is
$$\sum_{n\in\mathbb{Z}} \langle \sum c_n e^{inx}, e^{inx} \rangle e^{inx} = \sum_{n\in\mathbb{Z}} c_n e^{inx},$$
as was to be shown.

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5. Explain why the following function $f(x, y)$ is Lebesgue integrable on $[0, 1] \times [0, 1]$ and find the value of the integral:
$$f(x, y) = \sum_{n=1}^{\infty} a_n \frac{n \,\mathrm{SIN}\, n\pi x}{2^n};$$

where $y = \sum_{n=1}^{\infty}(a_n/2^n)$ is the binary expansion of $y \in [0,1]$. You may assume that $f(x,y)$ is measurable.

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6. Let $f(x) \in L^2\big([0,\infty)\big)$.
(a) Prove that if $t > 0$ then
$$e^{-t(x+1)} f(x) \in L^1\big([0,\infty)\big).$$
(b) Let
$$g(t) = \int_0^\infty e^{-t(x+1)} f(x)\,\mathrm{d}x;$$
prove that $g \in L^1\big([0,\infty)\big)$.

---

7. Let $\mathcal{P} \subset \mathcal{C}[0,1]$ be the subspace of polynomials. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$. Consider the following functionals on $\mathcal{P}$; for each, determine if it has a continuous extension to $\mathcal{C}[0,1]$.
(a) $\Phi(P) = a_0$;
(b) $\Phi(P) = a_0 + a_1$.

SOLUTION:

I am not sure what is meant by "continuous extension". That is, do I merely seek an extension that happens to be continuous, or does the extension-process itself need to be continuous in some way? Perhaps that doesn't make all that much sense. At any rate, I shall assume that the question is asking about the existence of an extension that is continuous. That is, given the functional Φ, I ask whether there exists a functional $\bar{\Phi}$ on $\mathcal{C}[0,1]$ such that $\bar{\Phi}$ restricted to $\mathcal{P}$ is just Φ, and furthermore, if there is a sequence of continuous functions f_n converging to a continuous function f (in the sup norm), then does $\bar{\Phi}(f_n)$ converge to $\bar{\Phi}(f)$?

(a) Define $\bar{\Phi}(f) = f(0)$. Clearly $\bar{\Phi} = \Phi$ when restricted to polynomials. Furthermore, if $f_n \to f$ in the sup norm, then in particular $f_n(0) \to f(0)$, and thus $\bar{\Phi}$ is continuous.

(b) The attempt $\bar{\Phi}(f) = f(0) + f'(0)$ is not well-defined since many functions in $\mathcal{C}[0,1]$ are not differentiable. However, since $\mathcal{P}$ is dense in $\mathcal{C}[0,1]$, I can obtain any continuous function f as a limit (in the sup norm) of polynomials. Thus I am tempted to define $\bar{\Phi}(f) = \mathrm{LIM}_{n\to\infty} \Phi(P_n)$, where P_n is a sequence of polynomials with $P_n \to f$; but I am not convinced that this limit always converges. I conjecture that no continuous extension exists. Here is a proof.

Assume that there exists a continuous functional $\bar{\Phi}$ on $\mathcal{C}[0,1]$ such that $\bar{\Phi}P = P(0) + P'(0)$ for any polynomial P. Then $\bar{\Phi}(f) = \mathrm{LIM}_{n\to\infty} \bar{\Phi}(P_n) = \mathrm{LIM}_{n\to\infty} P_n(0) + P_n'(0)$ for any sequence P_n of polynomials converging to f. **I NEED A NON-DIFFERENTIABLE FUNCTION (AT 0) LIKE $x\,\mathrm{SIN}(1/x)$ and a sequence of polynomials converging to it so that the limit is infinity or undefined.**

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8. Let A and B be Lebesgue measurable subsets of $\mathbb{R}$. Let m denote Lebesgue measure. Define a measure μ on measurable sets by
$$\mu(E) = 2m(E \cap A) + m(E \cap B).$$
(a) Show that μ is absolute continuous with respect to m.
(b) Find the Radon-Nikodym derivative of μ with respect to m.

SOLUTION:

(a) Let $m(E) = 0$. Then $m(E \cap A) = m(E \cap B) = 0$ since both $E \cap A$ and $E \cap B$ are subsets of the null set E. Thus $\mu(E) = 0$. By definition, μ is absolutely continuous with respect to m.

(b) I seek a nonnegative m-measurable function $g = (\mathrm{d}\mu/\mathrm{d}m)$ such that $\mu E = \int_E \mathrm{d}\mu = \int_E g\,\mathrm{d}m$ for all measurable sets E (measurable with respect to either measure). Let $g = 2 \cdot 1_A + 1_B$. Then
$$\mu E = 2m(E \cap A) + m(E \cap B) = 2\int_{E\cap A} \mathrm{d}m + \int_{E \cap B} \mathrm{d}m$$
$$= 2\int_E 1_A\,\mathrm{d}m + \int_E 1_B\,\mathrm{d}m = \int_E (2 \cdot 1_A + 1_B)\,\mathrm{d}m = \int_E g\,\mathrm{d}m.$$

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN AUGUST 1993

1. Let λ denote Lebesgue measure. Let $f \in L^1\big([0,1]\big)$. If f has the property that for all t, $0 \le t \le 1$,

$$\int_0^t f(x)\,\mathrm{d}\lambda(x) = 0,$$

then must f be zero almost everywhere? Prove your answer.

SOLUTION:

The function $g(t) = \int_0^t f(x)\,\mathrm{d}\lambda(x)$ is absolutely continuous; thus its derivative exists almost everywhere. In fact, the fundamental theorem says that the derivative is almost everywhere equal to $f(x)$, yet one also sees that the derivative is equal to 0. Thus $f(x)$ is zero almost everywhere.

2. Show that for every linear bounded functional Φ on Hilbert space H there is an element $x_0 \in H$ such that

$$\frac{\big|\Phi(x_0)\big|}{\|x_0\|} = \|\Phi\| = \sup_{\|x\|=1}\big|\Phi(x)\big|;$$

that is, the supremum is attained. Consider the special case of the functional Φ on $L^2\big([-1,1]\big)$ given by the formula

$$\Phi(f) = \int_{-1}^{1} f(t)t\,\mathrm{d}t.$$

Find a function f at which the supremum is attained (that is, $\frac{|\Phi(f)|}{\|f\|} = \|\Phi\|$). Show that when the same formula is used to define a linear functional Φ on $C\big([-1,1]\big)$, there is no function f in $C\big([-1,1]\big)$ such that $\frac{|\Phi(f)|}{\|f\|} = \|\Phi\|$.

3. Let $A \subseteq [0,1]\times[0,1]$ be a measurable subset with positive Lebesgue measure $(\lambda\times\lambda)$ equal to $a^2 < 1$, and let $A_t = \{\, y \in [0,1] : (t,y) \in A \,\}$. Let $E = \{\, t : \lambda(A_t) \ge a \,\}$. Find the best upper bound for the Lebesgue measure of E.

SOLUTION:

I claim that $\lambda E \le a$ and that this is the best upper bound. Note that the indicator function $1_A(t,y)$ is an integrable function on $[0,1]\times[0,1]$; thus the Fubini theorem may be applied. The cross-section function $g_t(y) = 1_A(t,y)$ is precisely $g_t(y) = 1_{A_t}(y)$. Thus $\int g_t\,\mathrm{d}\lambda = \lambda(A_t)$. To establish my claim that $\lambda E \le a$ is the best upper bound, I shall first establish that a is an upper bound, and then show by example that there exist sets such that $\lambda E = a$ occurs. If $\lambda E > a$, then $(\lambda\times\lambda)A = \int 1_A\,\mathrm{d}(\lambda\times\lambda) = \iint g_t(y)\,\mathrm{d}\lambda(y)\,\mathrm{d}\lambda(t) = \int \lambda(A_t)\,\mathrm{d}\lambda \ge \int_E \lambda(A_t)\,\mathrm{d}\lambda \ge \int_E a\,\mathrm{d}\lambda > a^2$, a contradiction. Thus $\lambda E \le a$ is necessary. Let $A = [0,\frac12]\times[0,\frac12]$. Then $a = \frac12$. For each t in $[0,1]$, the set A_t is either $[0,\frac12]$ or the emptyset, according as whether $0 \le t \le \frac12$ or $t > \frac12$. Thus $E = [0,\frac12]$, and $\lambda E = \frac12 = a$.

4. Find the subspace $M^\perp$, when

(a) $M = \{\,$functions in $L^2[-\pi,\pi]$ that vanish on the interval $[-\pi,0]\,\}$;

(b) $M = \{\,$polynomials in $L^2[0,1]$ with constant term equal to zero$\,\}$.

SOLUTION:

(a) I claim that $M^{\perp}$ consists of functions in $L^2[-\pi,\pi]$ that vanish almost everywhere on the interval $[0,\pi]$. Certainly any such function is orthogonal to every function in M. On the other hand, given a function f that does not vanish almost everywhere on $[0,\pi]$, either the positive part f^+ or the negative part f^- does not vanish almost everywhere on $[0,\pi]$; assume, without loss of generality, that f^+ does not vanish almost everywhere on $[0,\pi]$; then the function $f^+ \cdot 1_{[0,\pi]}$ lies in M and is not orthogonal to f. Thus $M^{\perp}$ consists of precisely those functions that vanish almost everywhere on the interval $[0,\pi]$.

(b) I would conjecture that $M^{\perp}$ consists of precisely those functions that are zero almost everywhere in $[0,1]$. I do not know how to make the argument.

[PERHAPS I SHOULD FIGURE IT OUT!]

---

5. Determine whether the function $f(x)$ is continuous on $[0,1]$, where

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n(1-x)}{n}.$$

SOLUTION:

For each x with $0 \le x < 1$,

$$f(x) = (1-x)\sum_{n=1}^{\infty}\frac{x^n}{n} = (1-x)\,\mathrm{LN}(1-x);$$

this function is continuous for such x. To determine whether $f(x)$ is continuous on all of $[0,1]$, I must determine whether $\mathrm{LIM}_{x\to 1} f(x) = 0$. To do so, I enlist the aid of l'Hôpital and calculate

$$\mathop{\mathrm{LIM}}_{x\to 1}(1-x)\,\mathrm{LN}(1-x) = \mathop{\mathrm{LIM}}_{x\to 1}\frac{\mathrm{LN}(1-x)}{(1-x)^{-1}} = \mathop{\mathrm{LIM}}_{x\to 1}\frac{(1-x)^{-1}}{2(1-x)^{-2}} = \mathop{\mathrm{LIM}}_{x\to 1} 2(1-x) = 0.$$

Thus I conclude that $f(x)$ is continuous on $[0,1]$.

---

6. Let f be a measurable function.

(a) Show that if f is in $L^1(\mathbb{R})$, then for each $a > 0$,

$$\text{measure}\left\{\, x : \left|f(x)\right| \ge a \,\right\} \le \frac{\|f\|_1}{a}.$$

(b) How about a statement in the other direction: If for some $M < \infty$ and for each $a > 0$ the measure of the set $\left\{\, x : \left|f(x)\right| \ge a \,\right\}$ is less than or equal to M/a, then is f in $L^1(\mathbb{R})$?

SOLUTION:

(a) Let μ denote the measure, and let $A = \left\{\, x : \left|f(x)\right| \ge a \,\right\}$. Then

$$\mu A = \int_A d\mu \le \int_A \frac{|f|}{a}\,d\mu \le \int \frac{|f|}{a}\,d\mu = \frac{1}{a}\int |f|\,d\mu = \frac{\|f\|_1}{a}.$$

(b) The statement in the other direction is not true. Here is an example. Define the function

$$f(x) = \begin{cases} 0, & \text{for } x \le 0; \\ \frac{1}{x}, & \text{for } x > 0. \end{cases}$$

Let $M = 1$. Notice that for each $a > 0$, the measure of the set $\left\{\, x : \left|f(x)\right| \ge a \,\right\}$ is the same as the measure of the set $\{\, x \in (0,\infty) : x \le (1/a) \,\}$, and this set has measure $1/a = M/a$. Yet $f \notin L^1(\mathbb{R})$.

---

7. Let $f_t(x) = e^{itx^2} f(x)$. Show that if f is in $L^2(\mathbb{R})$, then $f_t \to f$ in $L^2(\mathbb{R})$ norm as $t \to 0$.

SOLUTION:

[use dominated conv?]

---

8. Let f_n be a sequence of real functions defined on a compact interval $[a,b]$. We write $f_n \downarrow 0$ if for all x we have $f_{n+1}(x) \le f_n(x)$ for all n and $f_n(x) \to 0$ as $n \to \infty$. Show that if the f_n are each continuous and $f_n \downarrow 0$, then $f_n \to 0$ uniformly on $[a,b]$. Hint: Fix $\varepsilon > 0$ and consider the family of sets $A_n = \{\, y : f_n(y) < \varepsilon \,\}$ for $n \ge 1$.

SOLUTION:

[use compactness? Consider A_n cover]

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN JANUARY 1994 (FARIS AND WOJTKOWSKI)

1. Show that if a sequence of real numbers $\{ x_n \}$ for $n \geq 1$ converges to a real number a, then the sequence of arithmetic averages

$$y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$$

converges to the same limit a. Give an example where $\{ x_n \}$ is divergent but $\{ y_n \}$ converges.

2. Is $\sum_{n=0}^{\infty} \frac{1}{3^n} \operatorname{SIN}(5^n x)$ continuous? Discuss.

SOLUTION:

The partial sums of the series, being the finite sum of continuous functions, are continuous. The question is whether the limit of the partial sums is continuous. Continuity is preserved under uniform convergence. The Weierstrass M-Test gives a criterion for uniform convergence. Since

$$\left| \frac{1}{3^n} \operatorname{SIN}(5^n x) \right| \leq \left| \frac{1}{3^n} \right| = \frac{1}{3^n},$$

and furthermore since

$$\sum_{n=0}^{\infty} \frac{1}{3^n} < \sum_{n=0}^{\infty} \frac{1}{2^n} = 2,$$

the conditions of the Weierstrass M-Test are satisfied and thus the series given in the problem statement converges uniformly and thus is continuous.

3. Let $u = f(x, y, t)$ be a smooth function defined on an open subset Γ of $\mathbb{R}^3$. Assume that for some $\varepsilon > 0$ the function satisfies

$$\frac{\partial u}{\partial t} = \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) - \varepsilon.$$

Show that there cannot be a point (x_0, y_0, t_0) in Γ at which u assumes its maximum.

SOLUTION:

[It seems that the statement is still true when $\varepsilon = 0$, i.e., one can just remove all the epsilons from this problem. Is ε included just to confuse things, or does it serve a mathematical purpose?]

This first solution immediately reduces the multi-variable calculus problem to one of freshman calculus. If a function $f(x, y, t)$ has a maximum at some point (x_0, y_0, t_0), then the function $g_1(x) = f(x, y_0, t_0)$ has a maximum at x_0, and similarly the function $g_2(y) = f(x_0, y, t_0)$ has a maximum at y_0, and similarly the function $g_3(t) = f(x_0, y_0, t)$ has a maximum at t_0. Thus the first derivative of each of g_1, g_2, and g_3 is zero at the respective points x_0, y_0, and t_0, while the second derivative of each of g_1, g_2, and g_3 is negative at the respective points x_0, y_0, and t_0. Note that $\partial u / \partial t$ at (x_0, y_0, t_0) is just $g_3'(t_0)$. Similarly, $\partial^2 u / \partial x^2$ at (x_0, y_0, t_0) is just $g_1''(x_0)$. Finally, $\partial^2 u / \partial y^2$ at (x_0, y_0, t_0) is just $g_2''(y_0)$. Thus, if the function u had a maximum at (x_0, y_0, t_0), then the left side of the equation in the problem statement would be zero, while the right side would be strictly negative; this is clearly a contradiction.

Here is the most efficient solution. On an open subset, any maximum is also a local maximum. A local maximum can occur only where the gradient matrix is zero; at such points, one examines the symmetric Hessian matrix, in particular its upper-left determinants of all sizes. If none is zero (equivalent to just checking the largest one), then we can definitely decide about the nature of the critical point under consideration. In that case, if all the determinants are positive, then the quadratic form associated to the Hessian is positive definite, and we have a local minimum. If the determinants alternate in sign, starting with the upper-left entry of the Hessian being negative, then the quadratic form is negative definite, and we have a local maximum. Otherwise we have a saddle point.

Say that (x_0, y_0, t_0) is a local maximum. I first need to make sure that the three determinants of the Hessian are nonzero so that I can make use of the test described above to arrive a contradiction. [HOW DO I SHOW THIS?] Then the gradient vanishes, so that $(\partial u/\partial t)(x_0, y_0, t_0) = 0$, and the three determinants of the Hessian must alternate in sign, i.e.,

$$\begin{gathered} f_{xx} < 0, \\ f_{xx} f_{yy} - f_{xy}^2 > 0, \\ 3 \times 3 \det < 0. \end{gathered}$$

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4. Show that the subspace $M \subset C\big([0,1]\big)$ of all piecewise linear continuous functions is dense in $C\big([0,1]\big)$.

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5. Let A_1, A_2, ... be a sequence of measurable subsets of $S = [0,1] \times [0,1]$ such that $\bigcup_{i=1}^{\infty} A_i = S$. Let λ be two-dimensional Lebesgue measure and assume that $\sum_{i=1}^{\infty} \lambda(A_i) < \infty$. Let

$$N(x) = \#\{\, i : x \in A_i \,\},$$

where # denotes the number of points in the set (possibly infinite). Show that N is an integrable function on S.

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6. Give an example of a measurable function on $[0,1]$ that is not equal almost everywhere to a continuous function on $[0,1]$. Provide a detailed proof.

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7. Let $f \geq 0$ be integrable with respect to μ. Let λ denote one-dimensional Lebesgue measure. Show that the $\mu \times \lambda$ measure of the set $S = \big\{\, (x,y) : 0 < y < f(x) \,\big\}$ is $\int f(x)\, \mathrm{d}\mu(x)$.

---

8. Let

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{2n+1} \operatorname{SIN}(nx).$$

(a) Show that $f \in L^1\big([0, 2\pi]\big)$.
(b) Show that the series can be integrated term-by-term, that is, for all a and b with $0 \leq a \leq b \leq 1$ we have

$$\int_a^b f(x)\, \mathrm{d}x = \sum_{n=1}^{\infty} \frac{1}{2n+1} \int_a^b \operatorname{SIN}(nx)\, \mathrm{d}x.$$

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN AUGUST 1994

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1. Let f, g be real-valued functions on $[0,1]$ and let $h = \operatorname{MAX}\{\, f, g \,\}$. Prove the following.
(a) If f, g are continuous at x_0, then h is continuous at x_0.
(b) If f, g are Lebesgue measurable, then h is Lebesgue measurable.
(c) If f, g are Lebesgue integrable, then h is Lebesgue integrable.
(d) If f, g are Riemann integrable, then h is Riemann integrable.

SOLUTION:

(a) Let f, g be continuous at x_0. I shall show that h is continuous at x_0 using the definition. Let $\varepsilon > 0$ be given. The continuity of f, g gives me δ_x, δ_y such that $\left|f(x_0) - f(y)\right| < \varepsilon$ whenever $|x_0 - y| < \delta_x$ and $\left|g(x_0) - g(y)\right| < \varepsilon$ whenever $|x_0 - y| < \delta_y$. If $f(x_0) > g(x_0)$, then the continuous function $f - g$ is positive at x_0 and thus there exists a $\rho > 0$ such that $f - g$ remains positive in a neighborhood of radius ρ about x_0; set $\delta = \text{MIN}(\delta_x, \rho)$; inside the δ-neighborhood of x_0, the function h is just f; thus $\left|h(x_0) - h(y)\right| = \left|f(x_0) - f(y)\right| < \varepsilon$ whenever $|x_0 - y| < \delta$. If $g(x_0) > f(x_0)$, then the same argument with the rôles of f, g reversed shows that h is continuous at x_0. Finally, if $f(x_0) = g(x_0)$, then set $\delta = \text{MIN}(\delta_x, \delta_y)$; for any y, either $h(y) = f(y)$ or $h(y) = g(y)$; thus, whenever $|x_0 - y| < \delta$, either $\left|h(x_0) - h(y)\right| = \left|f(x_0) - f(y)\right| < \varepsilon$ or $\left|h(x_0) - h(y)\right| = \left|g(x_0) - g(y)\right| < \varepsilon$; either way, h is continuous at x_0.

(b) Let f, g be Lebesgue measurable. To show that h is measurable, I must show that the set $A_h = \{\, x : h(x) < a \,\}$ is measurable for any real number a. But if $h = \text{MAX}(f, g)$ is less than a, then both f, g are less than a, and vice versa. Thus $A_h = A_f \cup A_g$, where $A_f = \{\, x : f(x) < a \,\}$ and $A_g = \{\, x : g(x) < a \,\}$. Since the latter two are measurable, also A_h is measurable. Thus h is measurable.

(c) Let f, g be Lebesgue integrable. By definition, each of the nonnegative functions f^+, f^-, g^+, g^- is integrable. Notice that $h^+ \le f^+ + g^+$ and $h^- \le f^- + g^-$. Thus h^+, h^- are integrable (since, for example, $\int h^+ \le \int f^+ + g^+ = \int f^+ + \int g^+ < \infty$); thus h is integrable.

(d) One only discusses (proper) Riemann integrability of bounded functions with compact support. Of the bounded functions with compact support, those that are Riemann integrable are characterized as precisely those functions whose set of discontinuities is a null set. The support of f, g is compact. Let f, g be Riemann integrable. Thus f, g are bounded and their sets of discontinuities are each a null set. I claim that h is bounded; since there exist M, N such that $|f| \le M$ and $|g| \le N$, then $|h| \le |f| + |g| \le M + N$. Furthermore, from part (a), the set of discontinuities of h is at most the union of the sets of discontinuities of f and g; the union of two null sets is a null set. Therefore, h is Riemann integrable.

Here is a much simpler solution. One easily checks that $h = \frac{1}{2}(f + g + |f - g|)$.

(a) The absolute value is continuous. Composing continuous functions gives back a continuous function.

(b) The absolute value is Lebesgue measurable. Composing measurable functions gives back a measurable function. (Since f, g are real-valued, there is no concern about "$\infty - \infty$" occurring anywhere.)

(c) Since f, g are integrable, so is $f - g$, thus also $|f - g|$, and so h is seen to be integrable.

(d) A characterization for a function on a compact set to be Riemann integrable is that the function be bounded and that its set of discontinuities form a set of Lebesgue measure zero. Since f, g have these properties, also h is bounded and furthermore the set of discontinuities of h is at worst the union of the discontinuities of f and g, and thus still has measure zero. So h is Riemann integrable.

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2. Do the following.

(a) What is a Borel set?

(b) Prove that if f is a real-valued function on $(-\infty, \infty)$ such that $f^{-1}\big((a, b)\big)$ is a Lebesgue measurable set for each open interval (a, b), then $f^{-1}(B)$ is a Lebesgue measurable set for every Borel set B.

SOLUTION:

(a) A Borel set is any member of the Borel σ-algebra. The Borel σ-algebra is the smallest σ-algebra of subsets containing the open sets. The Borel σ-algebra can also be characterized as the smallest σ-algebra containing the open intervals.

(b) Define $\mathcal{A} = \{\, S : f^{-1}(S) \text{ is measurable} \,\}$. If I show that $\mathcal{A}$ is a σ-algebra, then the fact that $\mathcal{A}$ contains the open intervals (this is given in the hypotheses) guarantees that it contains all Borel sets (from the characterization given in part (a) above).

To show that $\mathcal{A}$ is a σ-algebra, I must show that $\mathcal{A}$ is closed under complement and countable union. This argument relies on the fact that the operation of taking function preimages behaves nicely with respect to set operations, and the fact that the measurable sets themselves are closed under complement and countable union. Let $S \in \mathcal{A}$. Then $M = f^{-1}(S)$ is a measurable set. Then $f^{-1}(S^{\mathsf{C}}) = M^{\mathsf{C}}$. Since M^{C} is measurable, $S^{\mathsf{C}} \in \mathcal{A}$. Let $S_n \in \mathcal{A}$ for each positive integer n. Then S_n is a countable family of sets. Then $M_n = f^{-1}(S_n)$ is a measurable set for each n. Let $S = \bigcup S_n$. Then $f^{-1}(S) = \bigcup M_n$. Since a countable union of measurable sets is measurable, $S \in \mathcal{A}$.

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3. Let f be a real-valued function on $[0,1]$ such that the set $\{\, x : f(x) = c \,\}$ is Lebesgue measurable for every real c. Does this imply that f is Lebesgue measurable? Give a proof or an example to justify your answer. If you need a non-measurable set you may assume that it exists.

SOLUTION:

The answer is "no"; f is not necessarily Lebesgue measurable. Here is an example. Let E be a nonmeasurable set in $[0,1]$. Define

$$f(x) = \begin{cases} x, & \text{if } x \in E; \\ 1 + x, & \text{if } x \notin E. \end{cases}.$$

For each real number c, the set $\{\, x : f(x) = c \,\}$ is a null set and thus measurable; after all, each such set contains at most one point. However, f is not measurable; after all, the set $\{\, x : f(x) \le 1 \,\}$ is the nonmeasurable set E.

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4. Let S be a set of real numbers of finite Lebesgue measure $m(S)$. Prove that there is a point p such that $m\big(S \cap (-\infty, p)\big) = m\big(S \cap (p, \infty)\big)$.

---

5. The distance between a point and a set is the infimum of the distance between the point and points in the set. Consider the space B of bounded real functions on the line with distance between two such functions f, g defined as the supremum of $|f - g|$. Let $f = 1_{[0,1]}$ be the indicator function of the interval $[0,1]$ on the line.

(a) Find the distance of f to the set BC of all bounded continuous real functions on the line. (Be explicit about where you use continuity.)

(b) Describe the subset of BC at exactly this distance from the indicator function.

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6. The Laplace transform of a real function f on the line is a function g defined for $a > 0$ by

$$g(a) = \int_0^\infty e^{-ax} f(x)\,\mathrm{d}x,$$

provided that the integral converges absolutely.

(a) Let f be in L^1. Show that $g(a)$ is continuous in a for $a > 0$.

(b) Let f be in L^2. Show that $g(a)$ is continuous in a for $a > 0$.

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7. Consider a complete metrix space X and a closed subset B. Assume that there is a sequence B_m of compact subsets of B such that for every $\varepsilon > 0$ there is an m such that every point in B is within ε of a point in B_m. Show that B is compact.

---

8. Let f be a positive measurable function on $[0,h]$; let

$$A = \big\{\, (x,z) : |x| \le f(z), 0 \le z \le h \,\big\},$$

and let

$$V = \big\{\, (x,y,z) : \sqrt{x^2 + y^2} \le f(z), 0 \le z \le h \,\big\}.$$

(a) Show that the planar Lebesgue measure of A is bounded by $\sqrt{4h/\pi}$ times the squre root of the solid Lebesgue measure of V.

(b) For which functions f is this bound an equality?

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN JANUARY 1995 (GRÖMER AND PALMER)

1. Suppose that $f(x)$ is defined by the following infinite series:

$$f(x) = \sum_{n=1}^{\infty} n^3 x^n \operatorname{SIN}(nx).$$

Show that $f(x)$ is a continuous function of x for $-1 < x < 1$.

2. Prove that a subset X of the real numbers is Lebesgue measurable if and only if there exists a G_δ set D such that $X \subseteq D$ and $m^*(D \setminus X) = 0$. Recall that a set is called a G_δ set if it is the intersection of a countable collection of open sets. The notation $m^*(A)$ refers to the outer Lebesgue measure of A.

SOLUTION:

Let X be measurable. By definition, $m^*X = \operatorname{INF}_{X \subseteq \bigcup I_n} \sum \ell(I_n)$, where the infimum is taken over all countable covers of X by intervals, and $\ell(I_n)$ is the length of the interval I_n. Thus, for any $\varepsilon > 0$, there exists a specific cover such that $\sum \ell(I_n) < m^*X + \varepsilon$. Since $X \subseteq \bigcup I_n$, the open set $E = \bigcup I_n$ contains X. Note that all open sets, thus also E, are measurable; then also $E \setminus X$ is measurable. Therefore $m^*(E \setminus X) = m(E \setminus X) = mE - mX = m^*E - m^*X$. Now $m^*E = m^* \bigcup I_n \le \sum m^* I_n = \sum \ell(I_n) < m^*X + \varepsilon$. Combining the pieces, corresponding to each $\varepsilon > 0$ I have constructed an open set E containing X such that $m^*(E \setminus X) < \varepsilon$. In particular, I can find a sequence E_n of open sets containing X such that $m^*(E_n \setminus X) < 1/n$. Define the G_δ-set $D = \bigcap E_n$. Certainly $X \subseteq D$ and furthermore $m^*(D \setminus X) = m^*\big((\bigcap E_n) \setminus X\big) = m^*(\bigcap E_n \setminus X) \le m^*(E_n \setminus X) < 1/n$ for each n; thus $m^*(D \setminus X) = 0$.

Now let the subset X be such that there is a G_δ-set D with $X \subseteq D$ and $m^*(D \setminus X) = 0$. I wish to show that X is measurable. That is, I wish to show for any gauge set S that $m^*S = m^*(S \cap X) + m^*(S \cap X^{\mathsf{C}})$. Subadditivity and some trivialities involving infinity reduce this to showing for any gauge set S of finite outer measure that $m^*S \ge m^*(S \cap X) + m^*(S \cap X^{\mathsf{C}})$. Now D is measurable and thus $m^*S \ge m^*(S \cap D) + m^*(S \cap D^{\mathsf{C}})$. Since $X \subseteq D$, certainly $m^*(S \cap D) \ge m^*(S \cap X)$. Now $S \cap X^{\mathsf{C}} \subseteq (S \cap D^{\mathsf{C}}) \cup (D \setminus X)$. Therefore $m^*(S \cap X^{\mathsf{C}}) \le m^*(S \cap D^{\mathsf{C}}) + m^*(D \setminus X) = m^*(S \cap D^{\mathsf{C}})$. Combining the pieces gives the inequality establishing that X is measurable.

3. Let $f(x)$ denote the function $f(x) = x \operatorname{SIN}(1/x)$ where we agree $f(0) = 0$. Answer the following questions and justify your answers.

(a) Is $f(x)$ Lebesgue integrable on $[0, 1]$?

(b) Is $f(x)$ of bounded variation on $[0, 1]$?

(c) Is $f(x)$ absolutely continuous on $[\varepsilon, 1]$ for every $\varepsilon > 0$?

SOLUTION:

(a) The answer is "yes". Certainly f is measurable on $[0, 1]$. (Since f is product and composition of finite measurable functions on $(0, 1]$, certainly f is measurable there; tossing in an extra point does not affect measurability.) Both f^+ and f^- are bounded by the function $g(x) = x$. Thus $\int_{[0,1]} f^+ \le \int_{[0,1]} g = \frac{1}{2}$; similarly f^- is integrable; thus f is integrable.

(b) The answer is "no". I shall produce a sequence of partitions such that the total variation with respect to each partition forms an unbounded sequence; then the supremum over all partitions will certainly be infinite, and thus the function will be shown not to be of bounded variation. Consider the collection of points $2/\pi$, $2/3\pi$, $2/5\pi$, $\ldots$, $2/(2n+1)\pi$. Call these points $x_0, x_1, \ldots, x_n$ and think of them as a partition P; the total variation with respect to this partition is

$$\sum_{k=1}^{n} \big| f(x_k) - f(x_{k-1}) \big| = \sum_{k=1}^{n} \frac{2}{(2k+1)\pi} + \frac{2}{(2k-1)\pi} = \frac{2}{\pi} \sum_{k=1}^{n} \frac{4k}{4k^2 - 1}.$$

Letting $n \to \infty$ gives a sequence of partitions and a corresponding sequence of total variations; notice that this sequence of total variations tends to infinity.

(c) The answer is "yes". Notice that f has a derivative throughout $(0,1]$, namely

$$f'(x) = \operatorname{SIN}(1/x) - x^2 \operatorname{COS}(1/x).$$

Thus f is Lipschitz on $(0,1]$; in particular, f is Lipschitz on $[\varepsilon, 1]$ for every ε satisfying $0 < \varepsilon < 1$; that in turn implies that f is absolutely continuous on $[\varepsilon, 1]$. Notice further that f being absolutely continuous on $[\varepsilon, 1]$ implies that f is of bounded variation on $[\varepsilon, 1]$; compare this interesting result to part (b).

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4. Do the following.

(a) State Fubini's Theorem.

(b) The double infinite series $\sum_{1 \le i,j < \infty} a_{ij}$ is said to converge to $L < \infty$ if

$$L = \operatorname*{LIM}_{N \to \infty} \sum_{1 \le i,j \le N} a_{ij}.$$

The series is said to converge absolutely if $\sum_{1 \le i,j < \infty} |a_{ij}|$ converges to a finite value. Use Fubini's theorem to show that if $\sum_{1 \le i,j < \infty} a_{ij}$ converges absolutely, then $\sum_{1 \le i < \infty} a_{ij}$ converges for all j and $L = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$.

SOLUTION:

(a) I shall state the two theorems known in Royden as Tonelli and Fubini. I have been told that some authors reverse the two names, while some authors even use the name Fubini to refer to either, or both, or mysterious combinations of the two.

Here is Tonelli according to Royden. Let μ, ν be two σ-finite measures. Let $f(x,y)$ be a measurable function (with respect to the product measure $\mu \times \nu$). Then the cross-section functions $f^y(x)$ and $f_x(y)$ are measurable (respectively with respect to μ, ν), the functions $g(x) = \int f_x(y)\,\mathrm{d}\nu$ and $h(y) = \int f^y(x)\,\mathrm{d}\mu$ are measurable (respectively with respect to μ, ν), and $\int f(x,y)\,\mathrm{d}(\mu \times \nu) = \int g(x)\,\mathrm{d}\mu = \int h(y)\,\mathrm{d}\nu$, where this is interpreted to allow infinity across the board.

Here is Fubini according to Royden. Let μ, ν be two measures. Let $f(x,y)$ be an integrable function (with respect to the product measure $\mu \times \nu$). Then the corss-section functions $f^y(x)$ and $f_x(y)$ are measurable and almost everywhere integrable (respectively with respect to μ, ν), the functions $g(x) = \int f_x(y)\,\mathrm{d}\nu$ and $h(y) = \int f^y(x)\,\mathrm{d}\mu$ are measurable and integrable (respectively with respect to μ, ν), and $\int f(x,y)\,\mathrm{d}(\mu \times \nu) = \int g(x)\,\mathrm{d}\mu = \int h(y)\,\mathrm{d}\nu$.

(b) This problem has little to do with Fubini. Although Fubini allows certain product integrals to be computed in an interated fashion, the corresponding result for series is obtained more easily using the fact that absolutely convergent series may be rearranged. The real problem to be explored here is that convergence of a double-indexed series is being defined in a particular way, and one would like to establish a connection between this definition and an interated approach to computing the series. Fubini does not establish this connection; Fubini connects the product integral to the iterated integral; Fubini does *not* connect the product integral to the definition of convergence of a double-indexed series used in this problem. This connection seems to be the heart of the problem. I shall first briefly discuss the connection between integration and summation. Then I shall solve the entire problem without Fubini. Then I shall solve the problem using Fubini, which involves two parts: first, I shall state what Fubini guarantees about the product integral; second, I shall connect the product integral to this problem and thereby solve it.

Let $\mu = \nu$ be counting measure on the set $\mathbb{N}$ of positive integers. Notice that the measures are σ-finite, all subsets are measurable, only the empty set has measure zero, and the product of counting measures is again counting measure. Integrals with respect to counting measure are merely summations, although one has to be a little careful about order; summation notation usually specifies an order; integration has no order; thus $\int_{\mathbb{N}} a(n)\,\mathrm{d}\mu = \sum_{n \in \mathbb{N}} a_n$, but one should be careful to avoid writing $\sum_{n=1}^{\infty} a_n$ until one is secure that this order of summation is acceptable; in fact, $\sum_{n \in \mathbb{N}} a_n$ makes no sense unless one knows that the series has the same limit independent of order; that happens if and only if the series is absolutely convergent; else one says that it is conditionally convergent; in fact, $\int_{\mathbb{N}} a(n)\,\mathrm{d}\mu$ is only defined when both the positive and negative parts are integrable; that happens if and only if the series is absolutely convergent; to sum it up, the results about series fit together nicely with the Lebesgue integration theory.

Now I shall solve the problem without Fubini. One merely needs the fact that absolutely convergent series (singly-indexed with the limit defined as the limit of partial sums) may be rearranged. Before I use this important fact, I shall establish a connection between the definition in this problem of convergence of a doubly-indexed series and convergence of a particular single-indexed version of the series. To move between singly-indexed and doubly-indexed series, one needs a bijection $\mathbb{N} \leftrightarrow \mathbb{N} \times \mathbb{N}$. Let $a_1 = a_{11}$, $a_2 = a_{21}$, $a_3 = a_{22}$, $a_4 = a_{12}$, $a_5 = a_{31}$, $a_6 = a_{32}$, $a_7 = a_{33}$, $a_8 = a_{23}$, $a_9 = a_{13}$, and so on; the bijection should be clear. Thus the sequence a_k is a particular arrangement of the collection a_{ij}. Furthermore, the sequence (indexed by N) of sums $\sum_{i,j \le N} a_{ij}$ used in the problem statement is a subsequence of the partial sums of the series $\sum_{k=1}^{\infty} a_k$; namely the sum $\sum_{i,j\le N} a_{ij}$ is the partial sum $\sum_{k=1}^{N^2} a_k$. Recall that a subsequence of a convergent sequence converges to the same limit. Therefore, if $\sum_{k=1}^{\infty} a_k$ converges (according to the traditional definition of the limit of partial sums existing), then $\sum_{i,j} a_{ij}$ converges (in the sense of the definition given in the problem statement). On the other hand, what can we say about the convergence of $\sum_{k=1}^{\infty} a_k$ in terms of $\sum_{i,j} a_{ij}$? Recall that a monotone sequence converges if and only if any subsequence converges. If the collection a_{ij} is nonnegative, then the partial sums of the series $\sum_{k=1}^{\infty} a_k$ are monotone and thus the series converges if and only if $\sum_{i,j} a_{ij}$ converges. Everything said so far relies on trivial facts about sequences. Now, in this problem, I am told that $\sum_{i,j} |a_{ij}|$ converges (presumably in the sense defined in the problem statement); but here I have nonnegative terms, and therefore $\sum_{k=1}^{\infty} |a_k|$ converges (in the usual sense of partial sums). Thus $\sum_{k=1}^{\infty} a_k$ is an absolutely convergent series. Now I shall use, for the first time, the important fact that absolutely convergent series may be rearranged. The series may be computed as $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$.

I will now solve the problem again using Fubini. As stated before, Fubini itself doesn't solve the problem but instead connects the product integral to the iterated integral. The iterated integral is just the iterated summation. In this problem, I wish to connect the iterated summation with the convergence definition given in the problem statement. To complete the picture and solve the problem, I shall connect the product integral with the convergence definition given in the problem statement. The fact that absolutely convergent singly-indexed series may be rearranged will still be necessary in this solution to the problem.

In this problem, we assume that $\sum_{i,j=1}^{\infty} |a_{ij}|$ converges (according to the definition given in the problem). That is, $\text{LIM}_{N\to\infty} \int_{\mathbb{N}\times\mathbb{N}} \big|a(i,j)\big| \cdot 1_{\{\,i,j\le N\,\}}\, \mathrm{d}(\mu \times \nu) < \infty$. Notice that the sequence of nonnegative integrands is a monotone increasing sequence. The monotone convergence theorem requires just this setup (one does need to know that the sequence of integrands converges, for in that case the trivial conclusion just involves lots of infinity); the theorem lets me bring the limit inside the integral; since I already know that the integral is finite, I now know that the limit function (which is the function $\big|a(i,j)\big| = |a_{ij}|$) is integrable with respect to the product measure. In general, if the absolute value of a measurable function is integrable, then the function itself is integrable. Therefore $a(i,j) = a_{ij}$ is integrable with respect to the product measure. Fubini says that the cross-section $a^j(i) = a_{ij}$ is measurable and almost everywhere integrable. For counting measure this just means that the cross-section $a^j(i) = a_{ij}$ is integrable for each j; that is, the series $\sum_{i\in\mathbb{N}} a_{ij}$ is absolutely convergent, and thus $\sum_{i=1}^{\infty} a_{ij}$ converges for each j. (I just used the fact that absolutely convergent series may be rearranged.) Furthermore, Fubini says that $\sum_{i=1}^{\infty} a_{ij}$ is integrable; that is, the series $\sum_{j\in\mathbb{N}} \sum_{i=}^{\infty} a_{ij}$ (in j) is absolutely convergent; thus $\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$ converges. Finally, Fubini says $\sum_{i,j\in\mathbb{N}} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$; here $\sum_{i,j\in\mathbb{N}} a_{ij}$ means the integral with respect to the product measure. To finish the problem, I need to connect this back to $\sum_{i,j=1}^{\infty} a_{ij}$ as defined in the problem statement. This time I need the dominated convergence theorem. Each function $a(i,j) \cdot 1_{\{\,i,j\le N\,\}}$ is dominated by the integrable function $a(i,j)$; thus $\text{LIM}_{N\to\infty} \int_{\mathbb{N}\times\mathbb{N}} a(i,j) \cdot 1_{\{\,i,j\le N\,\}}\, \mathrm{d}(\mu\times\nu) = \int_{\mathbb{N}\to\mathbb{N}} a(i,j)\, \mathrm{d}(\mu \times \nu)$; that is, $\sum_{i,j=1}^{\infty} a_{ij} = \sum_{i,j\in\mathbb{N}} a_{ij}$. I know that the latter converges; therefore the former converges. If we call the limit L, then $L = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$.

Let me summarize what happened in this problem. One has to connect the convergence definition given in the problem with the interated summation. The problem says to use Fubini. The hard part of the problem is connecting the convergence definition with some type of known and studied definition. In my first solution, I connected the convergence definition directly to the traditional singly-indexed series definitions and got the connection to the iterated summation practically for free in the last sentence of that solution. In the second solution, I connected the convergence definition to one part of Fubini, namely the iterated integral. This took a lot of work. Fubini then finished the connection to the iterated sum.

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5. Suppose that $f_n(x)$ is a sequence of continuous functions on the interval $[0,1]$ each of which is continuously differentiable for $x \in (0,1)$. Suppose that $f(x)$ is also continuous on $[0,1]$ with a continuous derivative on $(0,1)$. Show that if f_n converges to f in $L^1[0,1]$ and f_n' converges to f' in $L^1[0,1]$, then the sequence f_n converges to f uniformly on $[0,1]$ (that is, with respect to the sup norm).

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN AUGUST 1995 (WHO WROTE IT?)

1. Define the following:
(1) Borel set (in a metric space);
(2) Radon-Nikodym derivative;
(3) Hahn decomposition.

2. Let $\alpha > 0$, $a > 0$, $b > 0$. Prove that in the interval $(0, a)$ the equation

$$x^\alpha + (a - x)^\alpha = b$$

has at most two solutions. [Hint: Consider the function $f(x) = x^\alpha + (a - x)^\alpha$ and its derivative.]

3. Do this:
(a) State the theorem of Fubini.
(b) Let E be a Lebesgue measurable subset of $[0, 1]$, and let χ_E denote the function

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E; \\ 0, & \text{if } x \notin E. \end{cases}$$

Prove that

$$\int_0^1 \int_0^1 \text{SIN}(\pi t \chi_E(x))\, dx\, \mathrm{d}t = \tfrac{2}{\pi} m(E),$$

where m denotes Lebesgue measure.

4. An outer measure μ^* on a space X is said to be *regular* when for every $S \subseteq X$ there is a measruable set E such that $S \subseteq E$ and $\mu^* S = \mu^* E$.
(a) Prove that a regular outer measure has the following property: if $S_k \subseteq S_{k+1} \subseteq \cdots$ is a nested family of subsets of X, then $\text{LIM}_{k\to\infty} \mu^* S_k = \mu^*(\bigcup_{k=1}^\infty S_k)$.
(b) Prove that the Lebesgue outer measure m^* (on the real line) is regular.

5. Suppose that f_n is a sequence of real-valued continuous functions on $[0, 1]$ that converge pointwise to a function f. Suppose also that the functions f_n satisfy

$$|f_n(y) - f_n(x)| \le C|y - x|^\alpha,$$

for some constants $C > 0$ and $\alpha > 0$ that do not depend upon n, x, or y. Prove that the convergence $f_n \to f$ is necessarily uniform.

6. If X is a measure space and H is a Hilbert space, we may call a function $f\colon X \to H$ weakly measurable when each of the real-valued functions $\langle v, f\rangle$ is Lebesgue measurable, where $v \in H$ and $\langle\cdot,\cdot\rangle$ denotes the inner product in H. Suppose that H is separable. Prove that if f is weakly measurable, then the real-valued function $\|f\|$ is Lebesgue measurable, where $\|\cdot\|$ denotes the Hilbert space norm.

7. Give an example illustrating the following situation, or prove that no such example can be found: $f_n \to f$ almost everywhere with respect to Lebesgue measure m on $[0,1]$, but nonetheless it is *not* the case that

$$m\{\, x : |f_n(x) - f(x)| > \alpha \,\} \to 0 \quad \text{for all } \alpha > 0.$$

8. It is well known that the Banach space $\ell^\infty(\mathbb{Z})$ is nonseparable. The dual space $[\ell^\infty(\mathbb{Z})]^*$, which comprises all bounded linear functions on $\ell^\infty(\mathbb{Z})$, is a bit difficult to describe. However, it too is nonseparable. Prove this. [Hint: Try a proof by contradiction, using the Hahn–Banach theorem.]

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN JANUARY 1996 (MAIER AND ?GRÖMER?)

1.

(a) Prove that for all $y > 0$ and all real x the series

$$\sum_{n=1}^{\infty} e^{-yn} \,\mathrm{SIN}\, nx$$

converges absolutely.

(b) Prove that for $y > 0$ and all real x the function

$$f(x,y) = \sum_{n=1}^{\infty} e^{-yn} \,\mathrm{SIN}\, nx$$

has partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$, and that these can be found by term-by-term differentiation of the series.

2. Let f be a real-valued function that is continuous on $[a,b]$ and differentiable on (a,b). Prove that the existence of

$$\operatorname*{LIM}_{h \to 0^+} f'(b-h)$$

implies the existence of

$$\operatorname*{LIM}_{h \to 0^+} \frac{f(b-h) - f(b)}{-h},$$

and that these limits are equal in that case.

SOLUTION:

[Direct approach involves swapping double limits, and there is no easy justification for this. Smart approach uses mean value theorem. In fact, the only justification for swapping the limits is that the result turns out to be true, as proved using the mean value theorem!]

3.

(a) Define absolute continuity.
(b) Define bounded variation.
(c) Prove that absolute continuity implies bounded variation.
(d) Prove that for absolutely continuous functions f the condition $f' = 0$ almost everywhere implies that f is constant.

---

4. Let f be a Lebesgue integrable function on $(-\infty, \infty)$. Prove that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\Big|\int_N f\Big| < \varepsilon \quad \text{whenever } mN < \delta.$$

Here m denotes the Lebesgue measure.

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5. Prove that a set $A \subseteq \mathbb{R}$ is a Borel set of finite Lebesgue measure if and only if it is the disjoint union of an F_σ set of finite Lebesgue measure and a Borel set of Lebesgue measure zero.

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6. Let f_n be a sequence of real-valued Lebesgue measurable functions on $[0,1]$, and let f be another such function. Recall that the sequence f_n is said to converge to f in measure when

$$m\big\{x : |f_n(x) - f(x)| > \alpha\big\} \to 0 \quad \text{for all } \alpha > 0.$$

First, give examples of sequences f_n that converge in measure and furthermore
(a) converge almost everywhere and in mean,
(b) converge almost everywhere but not in mean,
(c) converge in mean but not almost everywhere,
(d) converge neither in mean nor almost everywhere.

Second, prove that if the sequence of functions f_n is restricted to satisfy $|f_n| \le M < \infty$ for all n, precisely two of the four preceding possibilities become impossible. Which two?

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7. Consider the set of all Lebesgue measurable functions $f: \mathbb{R} \to \mathbb{R}$ for which f^3 is integrable and furthermore $\int f^3 > 1$.
(a) Prove that this set is an open subset of $L^3(\mathbb{R})$, in the usual sense.
(b) Prove that this set is a subset of $L^2(\mathbb{R})$, but is not open.

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8. Suppose that $F_n: L^2[0,1] \to \mathbb{R}$ is a sequence of continuous linear functions satisfying

$$\big|F_n(f)\big| \le C\|f\|_2 \quad \text{for all } n, \quad \text{with } C < \infty \text{ constant.}$$

(a) Prove that some subsequence of this sequence converges on a dense subset of $L^2[0,1]$.
(b) Prove that some subsequence converges on all of $L^2[0,1]$.

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN AUGUST 1996 (GRÖMER AND MAIER)

1. Do the following:

(a) Let a_1, a_2, ... be a sequence such that the limit $A = \mathrm{LIM}_{n\to\infty} a_n$ exists and is finite. Prove that

$$\mathrm{LIM}_{n\to\infty} \frac{1}{n}(a_1 + a_2 + \cdots + a_n) = A.$$

(b) Give an example to show that there are sequences b_1, b_2, ... that do not converge but are such that $\mathrm{LIM}_{n\to\infty} \frac{1}{n}(b_1 + b_2 + \cdots + b_n)$ exists and is finite.

2. A real-valued function f on $[a, b]$ is said to be *Lipshitz continuous* when there is a constant M such that

$$\left|f(x_1) - f(x_2)\right| \leq M\left|x_1 - x_2\right| \quad \text{for all } x_1, x_2 \in [a, b].$$

(a) Define absolute continuity of a real-valued function, and prove that Lipshitz continuous functions are absolutely continuous.

(b) Show that Lipshitz continuous functions are not necessarily differentiable at all points of $[a, b]$.

(c) Show that if a function is differentiable everywhere on $[a, b]$, and if its derivative is bounded, then it is Lipschitz continuous.

Solution to Lindelöf problem

(from August 1996 Analysis Qualifying Exam)

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3. Does the set of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$, as a topological space with topology induced by the usual topology on $\mathbb{R}$, have the Lindelöf property? (That is, does each open cover of $\mathbb{R} \setminus \mathbb{Q}$ have a *countable* subcover?) Recall, in the induced topology, that $A \subseteq \mathbb{R} \setminus \mathbb{Q}$ is open precisely when $A = O \cap (\mathbb{R} \setminus \mathbb{Q})$, where O is open in $\mathbb{R}$.

SOLUTION:

SOLUTION:

Summary of solution: $\mathbb{R}$ is second countable, which is a hereditary property; hence $\mathbb{R} \setminus \mathbb{Q}$ is second countable. Every second countable space is Lindelöf; hence $\mathbb{R} \setminus \mathbb{Q}$ is Lindelöf.

Now I shall establish the various claims. To be second countable means to have a countable basis. The open intervals with rational center and rational radius compose a countable basis for $\mathbb{R}$. Thus $\mathbb{R}$ is second countable. Given a subspace of a space, one way of obtaining a basis for the subspace is to start with a basis for the big space and intersect each member with the subspace. Taking a countable basis for $\mathbb{R}$ and pushing it down to $\mathbb{R} \setminus \mathbb{Q}$ in this manner thus gives a countable basis for $\mathbb{R} \setminus \mathbb{Q}$. Finally I shall show that that second countable implies Lindelöf.

Start with an arbitrary open cover $\{O_\alpha\}$. Now construct a countable subcollection: for each member of a countable basis, if that basis set is contained in some O_α, then toss precisely one such O_α into the collection being constructed. Call the resulting collection $\{O_n\}$; I claim that it covers the space. Indeed, any chosen point p lies in some member O_α of the cover $\{O_\alpha\}$; being open, O_α is the union of basis sets; hence p lies in some basis set, which in turn lies in O_α; thus one of the O_n contains this basis set, so that $p \in O_n$.

[HERE FOLLOWS SOME RANDOM RAMBLING ON TOPOLOGICAL STUFF WHICH I SHOULD MOVE TO SOME OTHER LOCATION AT SOME POINT.]

A property of a topological space is called **hereditary** when each subspace (in the induced topology) also has the property. A property is **weakly hereditary** when each *closed* subspace inherits the property.

> [Compactness is a familiar example of a weakly hereditary property: "a closed subset of a compact space is compact". Actually, this statement has two interpretations, each correct:
>
> - Any cover of the subset by open sets (in the sense of the big space's topology) has a finite subcover.
> - Any cover of the subset by open sets (in the sense of the induced topology) has a finite subcover.
>
> The second interpretation is the statement that compactness is weakly hereditary. The proof of either interpretation may be gleaned from what appears below on the Lindelöf property.]

I shall now prove that the Lindelöf property is weakly hereditary. Let $Y \subseteq X$, where X is a Lindelöf topological space, and Y is closed in X. Consider any cover $\{Y_\alpha\}$ of Y, where each Y_α is an open set in the induced topology of Y. (Here the indices α come from some index set.) Each Y_α is of the form $Y \cap X_\alpha$, where X_α is an open set in X. The collection $\{X_\alpha, X \setminus Y\}$ is a cover of X by open sets in X (here I have used the fact that Y is closed in X), and since X is Lindelöf, in fact X must be covered by a countable collection of the form $\{X_n, X \setminus Y\}$, where the X_n come from among the X_α. (The set $X \setminus Y$ may be unnecessary for this subcover, but my statement is nonetheless true.) Then Y is covered by the countable collection $\{X_n\}$, and thus Y is also covered by the countable collection $\{X_n \cap Y\}$. But this is a subcover of the original cover $\{Y_\alpha\}$, thus showing that Y is Lindelöf in its own topology.

Robert Maier suggests that interesting problems about relationship between lindeloefness and countability is in Kelley, General Topology, problems 1.J through 1.L.

4.

(a) Suppose that $f_n: \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable for each n, and that the f_n converge pointwise to f, i.e., $f_n(x) \to f(x)$ for each x. Prove that f is Lebesgue measurable.

(b) Suppose that the f_n, in addition to being Lebesgue measurable, are furthermore continuous. Give an example to show that f need not be continuous. However, the sets $f^{-1}(O)$, for $O \subseteq \mathbb{R}$ open, cannot be too far up the Borel hierarchy; show how far up they can be.

SOLUTION:

Let $S = \{ x : x > \alpha \}$. Then $f^{-1}(S) = \{ x : f(x) > \alpha \}$, which is same as $\{ x : f_n(x) > \alpha \text{for all } n \geq N_x \}$, which is same as $\bigcup_{n=1}^{\infty} \bigcap_{i=n}^{\infty} f_i^{-1}(S)$, and so our set is a $G_{\delta\sigma}$, certainly measurable.

I forgot second part. Example should be easy, and I just showed that we can't be too far up Borel hierarchy.

---

5. If $f: \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, explain what it means for f to be Lebesgue integrable, and explain how, if f is integrable, the integral $\int_{\mathbb{R}} f$ is computed. Your explanation should be a careful one, distinguishing among several cases (e.g., f bounded versus f unbounded, f vanishing outside a set of finite measure versus f not vanishing outside any set of finite measure, et cetera).

6.

(a) State the Hahn–Banach theorem.

(b) Define the dual X^* of a Banach space X, and explain why, irrespective of the choice of X, X^* is nonempty.

(c) Explain why the dual of $L^\infty[0,1]$ contains an uncountable set of linearly independent elements that are *not* of the form $f \mapsto \int_{[0,1]} fg$ for some $g \in L^1[0,1]$. (This is essentially the reason why we say that the dual of $L^\infty[0,1]$ is not $L^1[0,1]$.)

SOLUTION:

[for c, use evaluation-at-a-point maps on the unique continuous representative of each L^p equivalence class. Since the Dirac delta functions are not really functions, these evaluation maps (infinitely many, independent of each other) are not of specified form.]

7.

(a) If X is a set, and $\mathcal{F}$ is a collection of functions from X to $\mathbb{R}$, explain what is meant by the weak topology on X generated by $\mathcal{F}$.

(b) Suppose that X is $C(\mathbb{R})$, the set of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and that $\mathcal{F}$ is the collection of "evaluation maps" of the form $f \mapsto f(y)$, indexed by $y \in \mathbb{Q}$. What topology on $C(\mathbb{R})$ does this countable set $\mathcal{F}$ generate? In particular, what does it mean for a sequence of elements of $C(\mathbb{R})$ to converge with respect to this topology? Is this topology the same as the topology of pointwise convergence?

SOLUTION:

[I say yes, top is same as that of ptwise conv., but would like to check with an authority on this.]

8.

(a) Explain what it means for a measure on some measurable space to be absolutely continuous with respect to another measure on that space, and state the Radon–Nikodym theorem.

(b) Give an example of two distinct Borel measures on $\mathbb{R}$ that are mutually absolutely continuous, and work out the corresponding Radon–Nikodym derivatives.

(c) Give an example of three Borel measures on $\mathbb{R}$, none of which is absolutely continuous with respect to either of the other two.

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN JANUARY 1997 (WHO?)

1. Suppose that f is differentiable at each point $x \in (a,b)$ except perhaps at some specified point $x_0 \in (a,b)$. Show that if $\text{LIM}_{x\to x_0} f'(x)$ exists and equals some number $A \in \mathbb{R}$, then f is differentiable at x_0 and $f'(x_0) = A$.

SOLUTION:

Here is a counterexample from Matt: just consider a step function, say equal to 0 for $x < 0$ and 1 for $x > 0$, undefined at 0.

2. Suppose that the continuous function f is differentiable at each point $x \in (a,b)$ except perhaps at some specified point $x_0 \in (a,b)$. Show that if $\text{LIM}_{x\to x_0} f'(x)$ exists and equals some number $A \in \mathbb{R}$, then f is differentiable at x_0 and $f'(x_0) = A$.

SOLUTION:

This is a two-sided version of a similar problem on another exam. The solution involves the mean value theorem. [Which exam?]

3. Give an example of a function $f\colon \mathbb{R} \to (0,\infty)$ that is continuous on $\mathbb{R}\setminus\mathbb{Q}$, but discontinuous at each point in $\mathbb{Q}$. The function f is required to be integrable on $\mathbb{R}$.

SOLUTION:

Here is an example due to Jeff. Let $n\colon \mathbb{Q} \to \mathbb{Z}_{\geq 1}$ be some ordering of the rationals. Define

$$f(x) = \begin{cases} 1/n(x), & \text{if } x \in \mathbb{Q}; \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

At each point of $\mathbb{Q}$, the function value is $1/n$ for some n, yet any neighborhood of that point contains irrational points where the function takes on the value 0, so there is always a jump of size at least $1/n$. At each point not in $\mathbb{Q}$, the function value is 0, and we seek to find an ε-neighborhood so that the function value doesn't jump by more than some prescribed $\delta > 0$. Given such a $\delta > 0$, there are only finitely many denominators n such that $1/n \geq \delta$, and thus choosing ε small enough so that the neighborhood misses those finitely many rational points where the function value is large, indeed the maximum jump is less than δ. Finally, since the rational points have measure zero, this function is zero almost everywhere, and hence is measurable and furthermore integrable.

4. Let I denote the closed unit interval.

(a) Show that a sequence of continuously differentiable real-valued functions on I that converges uniformly need not converge to a continuously differentiable function.

(b) Show that, in fact, any continuous real-valued function on I may be obtained as the uniform limit of a sequence of continuously differentiable functions.

SOLUTION:

Example due to Jeff: Let

$$f_n(x) = \sqrt{(x-\tfrac{1}{2})^2 + \tfrac{1}{n}},$$

so that

$$f_n'(x) = (x-\tfrac{1}{2})/f_n(x).$$

Then $f_n(x) \to f(x)$ where $f(x) = \sqrt{(x-\frac{1}{2})^2} = |x-\frac{1}{2}|$, with uniform convergence. [ONE MUST STILL SHOW THIS.]

AND STILL MUST DO PART B.

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5. Let $\mathcal{M}$ denote the σ-algebra of sets S that are contained in I and have the property that either S or $I \setminus S$ is countable.

(a) Show that the extended real-valued function μ that assigns to each set $S \in \mathcal{M}$ the number of points in S is a measure on $\mathcal{M}$. In particular, show that it is countably additive.

(b) Show that a function $f\colon I \to \mathbb{R}$ is measurable with respect to $\mathcal{M}$ if and only if there exists a real number α such that $I \setminus f^{-1}(\alpha)$ is countable. Another way of saying this is that f must be constant except on a countable set. [Hint: For each real β, both $f^{-1}(-\infty,\beta)$ and $f^{-1}[\beta,\infty)$ are measurable, and $f^{-1}(-\infty,\beta) \cup f^{-1}[\beta,\infty) \ldots$]

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6. Suppose that $f\colon I \to [0,\infty)$ is continuous. Let m denote Lebesgue measure on I, and define $\mu_f(E) = m(f^{-1}(E))$. Show that f is a Borel measure on $[0,\infty)$ and that

$$\int_I f^p(x)\,\mathrm{d}m(x) = \int_{[0,\infty)} t^p\,\mathrm{d}\mu_f(t) \quad \text{for all } p \geq 0.$$

Hint: Show that something similar is true for simple functions and then use a "density argument". (A density argument is one that uses the fact that simple functions are dense in...)

SOLUTION:

[Prof Palmer has a partial writeup on how to solve this problem. I suspect the author of this problem had a different solution in mind. Both Palmer's and the other solution should be given here.]

7.

(a) Show that there is at least one continuous linear map $T\colon L^\infty[I] \to \mathbb{R}$ such that $Tf = 0$ for all continuous f and furthermore $Tf \neq 0$ for at least one noncontinuous f.

(b) Show that if "L^∞" above is replaced by "L^1", no such map T exists.

SOLUTION:

[Use Hahn-Banach, something involving a step function for part a. For part b, use fact that cont functions are dense in L^1.]

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8. Let $\mathbb{R}^\infty$ denote the space of all real sequences of finite length. In an obvious sense, $\mathbb{R} \subset \mathbb{R}^2 \subset \mathbb{R}^3 \subset \cdots \subset \mathbb{R}^\infty$; in fact, $\mathbb{R}^\infty = \bigcup_{n=1}^\infty \mathbb{R}^n$. A standard way of topologizing $\mathbb{R}^\infty$ is to equip it with the strongest topology (most number of open sets) with the property that the inclusion maps $i_n\colon \mathbb{R}^n \to \mathbb{R}^\infty$ are continuous.

Suppose that $x^{(m)}$ is a sequence of points in $\mathbb{R}^\infty$ that converges to a point $x \in \mathbb{R}^\infty$ in this topology, i.e., has the property that for any open set $O \subseteq \mathbb{R}^\infty$ containing x, $x^{(m)} \in O$ for all sufficiently large m. Explain what this sort of convergence means at the level of the elements of the sequence $x^{(m)}$. For every i, does the sequence of real numbers $x_i^{(m)}$ need to converge to x_i? Does this alone ensure convergence?

MISCELLANEOUS STUFF:

A function that is differentiable but has a discontinuous derivative:

$$f(x) = \begin{cases} x^2 \,\mathrm{SIN}(1/x), & \text{if } x \neq 0; \\ 0, & \text{if } x = 0. \end{cases}$$

9. Extra problem: Let f be a real-valued function on $\mathbb{R}$. Let $E = \{\, x : f \text{ is continuous at } x \,\}$ be the points of continuity. Prove that E is a G_δ, i.e., a countable intersection of open sets.

SOLUTION:

By definition, f is continuous at x if, for each $\varepsilon > 0$, there exists a $\delta > 0$, such that whenever $|h| < \delta$, we have $|f(x) - f(x+h)| < \varepsilon$.

For each n, letting $\varepsilon = 1/n$, and for each $x \in E$, there is a δ, hence an open interval $(x-\delta, x+\delta)$, such that for $y \in (x-\delta, x+\delta)$, we have $|f(x) - f(y)| < 1/n$. Let G_n be the union of these intervals. Note that G_n is open. (If there are multiple choices for δ for a particular x and n, any choice $\delta > 0$ is fine.)

Since $E \subseteq G_n$, also $E \subseteq \bigcap_{n=1}^{\infty} G_n$. Now fix $y \in \bigcap_{n=1}^{\infty} G_n$. Then for each n, there exists a point $x_n \in E$ so that $|f(x_n) - f(y)| < 1/n$.

[IS THIS IT? WHAT?]