The purpose of this document is to summarize, from the standpoint of pure mathematics, the contents of the book *Complex Variables and Applications*, fifth edition, by Ruel V. Churchill and James Ward Brown. The intended audience is graduate students who have completed a graduate core curriculum in pure mathematics that included studies in point-set topology, advanced calculus, and differential topology. Presumably such students are preparing to take qualification examinations and will find this summary useful.

This document was started in summer 1995 while studying for my qualifiers. Additions have been made from time to time, mostly after learning something new and relevant while trying to help someone else solve a complex analysis problem. Please send corrections, comments, and suggestions to `aprl@math.arizona.edu`.

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CHAPTER
ONE
COMPLEX NUMBERS

8. REGIONS IN THE COMPLEX PLANE

The use of "connected" in this book is not the same as the usual notion from point-set topology. In point-set topology, a space X is connected if it admits of no disconnection $X = U \cup V$, where U and V are a disjoint pair of nonempty open subsets; equivalently, a space is connected if the only subsets that are both open and closed are the empty subset and the space itself. On the other hand, a space is path-connected if there is a path (continuous map from the interval $[0,1]$ to the space) connecting any pair of points. (Some authors also call this "arc-connected", but other authors use "arc" for something slightly more restrictive than "path".) See the useful reference book *Counterexamples in topology* for details on various notions of connectivity.

In Churchill/Brown, the authors are calling a set connected if any two points can be joined by a finite polygonal path: a path consisting of a finite number of straight† line segments. I'm pretty sure that the notions point-set connected, path-connected, arc-connected, and polygonal-path connected all agree for open subsets of the complex plane, but I have not thought about this too carefully. (It seems these notions should agree on any manifold.)

The book often talks about something taking place in a "domain". Such a generic term is used for many things in mathematics, and even in complex analysis the usage varies with author.

† Of course you know intuitively what it means for a segment to be "straight", but do you know what it means mathematically? Here is one way to prove that a given path is straight. First parameterize the path, say $t\colon [0,1] \to z(t) = x(t) + iy(t)$. For each value of t, the tangent vector to the path is given by $x'(t) + iy'(t)$. The path is straight precisely when all the tangent vectors point in the same direction, i.e., when they are scalar multiples of each other, i.e., when $y'(t)/x'(t)$ is a constant independent of t. (If you know of other ways of defining/checking "straightness" in the complex plane, then please let me know, and I might mention that in a future version of this document. Not able to come up with a better notion off the top of my head, I made up the one stated here.)

So let's get this straight once and for all: a **domain** is an *open* connected set; a **region** is any connected set; both domains and regions are always connected (which means finite-polygonal path connected).

CHAPTER

TWO

ANALYTIC FUNCTIONS

13. LIMITS INVOLVING THE POINT AT INFINITY

There are various ways of "compactifying" a topological space. The most common ones are the one-point compactification (also called the ALEXANDROFF compactification) and the STONE–ČECH compactification. The one-point compactification ties together a space by adding one extra "point at infinity", and this is the one used in complex analysis to manufacture the RIEMANN sphere from the complex plane. Note that in CHURCHILL/BROWN, the complex plane is called the "finite" plane.

The one-point compactification of $\mathbb{C}$ has many constructions. One of them is purely algebraic: you define $\mathbb{P}^1(\mathbb{C})$, called the "projective line over $\mathbb{C}$", to consist of all pairs $(z_1, z_2) \neq (0, 0)$ modulo the equivalence relation that the pairs (z_1, z_2) and $(\lambda z_1, \lambda z_2)$ are equivalent whenever $\lambda \neq 0$. The equivalence class of (z_1, z_2) is denoted $[z_1 : z_2]$. Hence $[3 : i] = [6 : 2i]$. Whenever the second coordinate is nonzero, you can divide through by it, thus representing all those points in the form $[z : 1]$, where $z \in \mathbb{C}$ is arbitrary. Then there is one extra point, the "point at infinity", represented by $[1 : 0]$. The exponent in the notation "$\mathbb{P}^1(\mathbb{C})$" refers to the dimension (over $\mathbb{C}$) of the resulting object. More generally, $\mathbb{P}^2(\mathbb{C})$, known as the "projective plane over $\mathbb{C}$", consists of all *triples* modulo a similar equivalence relation; and so on. That's not important for complex analysis, but it *does* at least explain the notation. The collection of all pairs $(\lambda z_1, \lambda z_2)$ in the equivalence class $[z_1 : z_2]$, together with the origin $(0, 0)$, is a one-dimensional subspace of the vector space $\mathbb{C}^2$; so you often hear people referring to $\mathbb{P}^1(\mathbb{C})$ as the collection of all lines through the origin in $\mathbb{C}^2$. In general, given any vector space V, the notation $\mathbb{P}(V)$ means the collection of all 1-dimensional subspaces; thus $\mathbb{P}^1(\mathbb{C})$ is also denoted $\mathbb{P}(\mathbb{C}^2)$. The dimension of $\mathbb{P}(V)$ is one less than the dimension of V; hence $\mathbb{P}(\mathbb{C}^2)$ is 1-dimensional, as it should be. Another name for $\mathbb{P}(V)$ is the "first GRASSMANIAN of V"; in general, define $\text{GRASSMANNIAN}(k, V)$ to be the collection of all k-dimensional subspaces of V, so that $\mathbb{P}(V) = \text{GRASSMANIAN}(1, V)$. Finally, there are also more concrete geometric constructions of the RIEMANN sphere, in which one uses "stereographic projection" to think of the compactified complex plane as being a sphere. There are many variants for doing stereographic projection; the most common are to lay a sphere on top of the plane (so that its south pole touches at the origin) and project from the north pole, or to cut the sphere in two via the plane (so that the equator touches along the plane's unit circle) and again project from the north pole (or sometimes from the south pole). The formulas change slightly with these variations, but the net effect is the same. I hope it was clear throughout this paragraph that I kept talking about the same thing, namely the RIEMANN SPHERE, which is variously constructed and denoted as

$$\hat{\mathbb{C}} = \mathbb{C}^\infty = \mathbb{CP}^1 = \mathbb{P}^1(\mathbb{C}) = \mathbb{P}(\mathbb{C}^2) = \text{GRASSMANIAN}(1, \mathbb{C}^2).$$

17. CAUCHY–RIEMANN EQUATIONS

Let $f(z)$ be a complex-valued function of one complex variable. The inputs and outputs of such a function can be broken into two real values. Thus a complex-valued function of one complex variable consists of two real-valued functions each of two real variables:

$$f(z) = u(x,y) + iv(x,y).$$

The following is very important. If f is differentiable at a point z, then the Cauchy-Riemann equations

$$u_x = v_y, \qquad u_y = -v_x$$

hold there; furthermore,

$$f'(z) = u_x(x,y) + iv_x(x,y).$$

A type of converse is the following. If the partial derivatives u_x, u_y, v_x, v_y exist in a neighborhood of z and are continuous at z, and if they satisfy the CAUCHY–RIEMANN equations, then f is differentiable at z.

Here is a good way for remembering the CAUCHY–RIEMANN equations, which also connects complex analysis with multi-variable real calculus (also called "advanced calculus"). Forgetting some of the structure, you can think of $\mathbb{C}$ as being $\mathbb{R}^2$, and thus you can think of a function $\mathbb{C} \to \mathbb{C}$ as a function $\mathbb{R}^2 \to \mathbb{R}^2$, namely $(x,y) \mapsto (u(x,y), v(x,y))$. In advanced calculus, the differentiability of functions $\mathbb{R}^n \to \mathbb{R}^m$ is studied, the derivative being an $m \times n$ matrix. Question: how does differentiability in real calculus relate to differentiability in complex analysis, and how does the derivative matrix relate to the complex derivative? To understand this, you need to know about the way of thinking of $\mathbb{C}$ in terms of matrices. Associated to each element $a + ib \in \mathbb{C}$ is a multiplication map $\mathbb{C} \to \mathbb{C}$, which is a linear transformation for $\mathbb{C}$ as a $\mathbb{C}$-vectorspace, hence also for $\mathbb{C}$ as a $\mathbb{R}$-vectorspace, and thus each element $a + ib \in \mathbb{C}$ has associated to it a 2×2 *real* matrix. In the standard basis $\{1, i\}$ for $\mathbb{C}$ as a R-vectorspace, applying the multiplication-by-$(a+ib)$ map to each basis element gives the corresponding matrix:

$$a + ib \longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

Notice that the first column is the column vector $[a, b]$ corresponding to the complex number $a + ib$, and the second column is the column vector $[-b, a]$ corresponding to the same complex number multiplied by i. Although it *has* to be true, do yourself a favor and right now verify that usual matrix multiplication of these matrices corresponds to usual multiplication of the corresponding complex numbers. The upshot of all this is that the field of complex numbers may be identified with the subring of 2×2 real matrices of the form shown above. Now keep the following principle in mind: *Whenever you see a 2×2 real matrix, that matrix represents a complex number if and only if the two entries on the diagonal are equal and the two entries on the off-diagonal are each other's negatives.* If a 2×2 matrix is not a complex number, then think of it as a bizarre messed-up higher dimensional real number, or don't think of it at all; but when it looks like a complex number, it *is* a complex number!

Now that we know about these special matrices, what good are they? Read on. If $z \mapsto f(z)$ is differentiable in the complex sense, then also $(x,y) \mapsto (u(x,y), v(x,y))$ is differentiable in the real sense. What about the converse? Well, if $(x, y) \mapsto (u(x, y), v(x, y))$ is differentiable in the real sense, go ahead and form is 2×2 real derivative matrix:

$$\begin{bmatrix} u_x(x, y) & u_y(x, y) \\ v_x(x, y) & v_y(x, y) \end{bmatrix}.$$

The amazing fact is that the function is differentiable in the complex sense precisely when its derivative *is a complex number*! In other words, we need the diagonal entries to be equal, the off-diagonal entries to be each other's negatives. And those are the CAUCHY–RIEMANN equations. Furthermore, in the case that the matrix *is* a complex number, then you can read off that number from the first column. And that's how you remember that the derivative is given by $u_x + iv_x$. Finally, from multivariable real calculus you should know that even when you can compute the partial derivatives to form a matrix that *looks like* the derivative matrix, that matrix will not have the meaning of the derivative unless the derivative exists; one sufficient condition for the derivative to exist is that the first partials each exist and be continuous. That result carries over unchanged to complex analysis to give us the converse statement about when the CAUCHY–RIEMANN equations imply differentiability.

Here follow some comments on the words "holomorphic", "meromorphic", "analytic", "conformal", "smooth", and "entire".

A function is said to be C^0 if it is continuous, C^1 if it has continuous first partial derivatives, C^2 if it has continuous second partial derivatives, and so on. Functions with partial derivatives of all orders are called C^∞ or simply "smooth". If you fix a point at which a function is smooth, you can expand a power series for the function about that point, which is a formal process involving the evaluation of lots of derivatives. This power series defines some other function, which may not equal the function you started with. Call the original function "analytic" when it equals its power series expansion. In real calculus, the function e^{-1/x^2} (defined to be 0 at 0) is smooth but not analytic: all derivatives evaluate to 0 at 0, hence the power series is identically zero, which does not equal e^{-1/x^2} is any neighborhood of 0. Hence e^{-1/x^2} is smooth on $\mathbb{R}$ but not analytic on $\mathbb{R}$. Such quandaries do not occur in complex analysis. The function e^{-1/z^2} has an essential singularity at 0, which means that no sensible value can be assigned to the function when $z = 0$. So e^{-1/z^2} is not even continuous, thus can't be holomorphic. An amazing fact in complex analysis is that a function is holomorphic if and only if it is analytic. To be more precise, if a function is holomorphic, i.e., if a complex function is known to have a first derivative *throughout a neighborhood* of a point, then in fact it necessarily has derivatives of all orders (i.e., is "smooth" or C^∞), and furthermore, it equals its power series (i.e., is "analytic"). Conversely, if a complex function is analytic at a point, then replace it locally with a power series expansion, and produce as many derivatives as you like by repeatedly differentiating the series term-by-term. (One separately establishes that such manipulations are valid for functions represented by power series.)

The word "entire" means "analytic everywhere". The domain can make a difference. For example, e^z is entire on $\mathbb{C}$, but it is *not* entire on $\mathbb{CP}^1$, since it has a singularity (in fact, an essential one) at infinity.

The word "conformal" means one-to-one, onto, and analytic. So conformal maps are just analytic bijections. You usually talk about a conformal map between two regions in the complex plane; for example, the map $z \mapsto z^2$ is not a conformal map $\mathbb{C} \to \mathbb{C}$ (it isn't one-to-one), but it *is* a conformal map from the first quadrant to the upper half plane.

The words "holomorphic" and "meromorphic" distinguish the types of singularities that are allowed to occur. Singularities come in three flavors: removable discontinuities, poles of finite order, and essential singularities. Of course, singularities don't have to be isolated: you could have a convergent sequence of points, with singularities at each point. But let's say that a function is analytic everywhere except for some isolated singularities. In that case, functions with only removable discontinuities are "holomorphic", functions possibly with poles are "meromorphic". There is no name for functions with essential singularities. Generally speaking, people use "holomorphic" and "analytic" synonymously, but do keep in mind that holomorphic allows isolated removable discontinuities. (Of course, you can always replace a holomorphic function with an analytic one by removing the discontinuities. That why one calls $z(z-1)/(z-1)$ an analytic function, even though as written it isn't defined and thus not analytic at $z = 1$.)

To be honest, matters aren't as simple as given in the previous paragraph. Matters depend heavily on whether your domain is $\mathbb{C}$ or $\mathbb{CP}^1$ and whether your codomain is $\mathbb{C}$ or $\mathbb{CP}^1$. The function e^z is holomorphic on $\mathbb{C}$ (it has no singularities, so it could also be called analytic or entire), but is not even meromorphic on $\mathbb{CP}^1$ (it has an essential singularity at ∞). On the other hand, $1/z$ is meromorphic on $\mathbb{CP}^1$ (it has a pole of order 1 at the origin) so long as you are thinking of the map as taking values in $\mathbb{C}$. When you start talking about maps from the RIEMANN sphere to itself, then ∞ is "just another point", nothing special about it, and $1/z$ is a perfectly good map taking on perfectly good values everywhere; in that context, you might see the map being called "holomorphic" (as a map from the sphere to itself).

19. POLAR COORDINATES

In the rectangular decomposition,

$$f(z) = u(x, y) + iv(x, y),$$

of a complex-valued function of one complex variable into a pair of real-valued functions of two variables, we let the inputs to the functions u, v be the rectangular coordinates of z. One can also do a rectangular decomposition using polar coordinates, namely

$$f(z) = u(r, \theta) + iv(r, \theta).$$

Corresponding to this decomposition there are versions of the Cauchy-Riemann equations:

$$u_r = \frac{1}{r} v_\theta, \qquad \frac{1}{r} u_\theta = -v_r.$$

Although the authors don't do it, why can't we do a polar decomposition of f using either rectangular or polar coordinates:

$$f(z) = \rho(x, y) \cdot e^{i \cdot \phi(x, y)},$$
$$f(z) = \rho(r, \theta) \cdot e^{i \cdot \phi(r, \theta)}.$$

Presumably one can derive equations similar to Cauchy-Riemann for this situation.

[DERIVE CR FOR REGULAR SITUATIONS AND FIGURE OUT THESE NEW SITUATIONS.]

21. HARMONIC FUNCTIONS

A real-valued function u of two real variables is **harmonic** if it satisfies the Laplace equation

$$u_{xx}(x, y) + u_{yy}(x, y) = 0.$$

(In particular, for u to be harmonic, it is necessary that u has first-order and second-order derivatives; thus it is also necessary that u and u_x, u_y be continuous.)

Using the fact that, if they are continuous, mixed second-order partial derivatives are equal, one can use the Cauchy-Riemann equations to prove the following:

Theorem. If $f(z) = u(x, y) + iv(x, y)$ is analytic, then the component functions u, v are harmonic.

If two functions u, v are harmonic and also satisfy the Cauchy-Riemann equations, then v is a **harmonic conjugate** of u. Note that the Cauchy-Riemann equations are not symmetric and thus u is not necessarily a harmonic conjugate of v.

Theorem. The function $f(z) = u(x, y) + iv(x, y)$ is analytic if and only if v is a harmonic conjugate of u.

Theorem. The function v is a harmonic conjugate of u if and only if $-u$ is a harmonic conjugate of v.

Theorem. If u, v are each a harmonic conjugate of the other, then each is constant.

Theorem. If u is harmonic on a simply-connected set, then it has a harmonic conjugate. Thus one can find a v such that the real-valued function u becomes the real component of a complex function $f = u + iv$.

CHAPTER

FOUR

INTEGRALS

32. CONTOUR INTEGRALS

To integrate $f(z)$ along some curve C, one must parametrize C as $z(t)$ with $a \leq t \leq b$. Then we define

$$\int_C f(z)\,\mathrm{d}z = \int_a^b f[z(t)]\,z'(t)\,\mathrm{d}t.$$

Note that none of this makes sense unless both $f(z)$ and $z(t)$ are piecewise continuous functions.

34. ANTIDERIVATIVES

Theorem. Let f be continuous on an open connected set. The following statements are equivalent:
(a) f has an antiderivative on the set;
(b) line integrals inside the path are path-independent;
(c) loop integrals inside the set have value zero.

Cauchy-Goursat Theorem. If f is analytic at all points interior to and on a simple closed contour C, then

$$\int_C f(z)\,\mathrm{d}z = 0.$$

There really is no point to separately study these theorems. These results follow immediately from the residue theorem. Anytime you integrate a function around a counterclockwise loop and that function is meromorphic on and inside the loop, the result is simply $2\pi i$ times the sum of the residues at enclosed singular points. If you take a Laurent series (power series with negative-power terms) and differentiate it term-by-term, you see that a power series has an antiderivative if and only if the residue (first negative power term) is zero.

38. SIMPLY AND MULTIPLY CONNECTED DOMAINS

Improved Cauchy-Goursat Theorem. If f is analytic in a simply connected set, then $\int_C f(z)\,\mathrm{d}z = 0$ for every closed contour C lying in that set.

Corollary. A function f that is analytic in a simply connected set has an antiderivative on that set.

Theorem. To integrate f along a closed contour C in an arbitrary set, you integrate f along closed contours bounding each region in which f is not analytic and take the sum.

Again, these theorems follow immediately from the residue theorem.

39. CAUCHY INTEGRAL FORMULA

Elementary Cauchy Integral Formula. Let f be analytic within and on a simple closed contour C oriented in the counterclockwise sense. If z_0 is interior to C, then

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,\mathrm{d}z}{z - z_0}.$$

This is a special case of the general Cauchy integral formula, which appears in a subsequent section.

40. DERIVATIVES OF ANALYTIC FUNCTIONS

Theorem. If f is analytic at a point, then its derivatives of all orders exist and are also analytic at that point.

Corollary. If $f(z) = u(x,y) + iv(x,y)$ is analytic at a point, then the component functions u, v are smooth at that point.

Generalized Cauchy Integral Formula. Let f be analytic within and on a simple closed contour C oriented in the counterclockwise sense. If z_0 is interior to C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_C \frac{f(z)\,\mathrm{d}z}{(z - z_0)^{n+1}}.$$

Note that this agrees with the Cauchy Integral Formula by taking $n = 0$.

41. MORERA'S THEOREM

Theorem. If f is continuous throughout the open connected set and if $\int_C f(z)\,\mathrm{d}z = 0$ for every closed contour C lying in that set, then f is analytic in that set.

Combinining this with the Cauchy-Goursat theorem gives us the following corollary.

Corollary. The function f is analytic throughout a simply connected set if and only if f is continuous there and $\int_C f(z)\,\mathrm{d}z = 0$ for every closed contour C in the set.

42. MAXIMUM MODULI OF FUNCTIONS

Lemma. Let f be analytic throughout an ε-neighborhood of z_0. If $\left|f(z)\right| \leq \left|f(z_0)\right|$ for each point z in the neighborhood, then $f(z) = f(z_0)$ throughout the neighborhood.

The proof involves Gauss's mean value theorem, which says that the value of an analytic function at the center of a circle is the arithmetic mean of the values on the perimeter of the circle.

Maximum Modulus Principle. If f is analytic and non-constant in an open connected set, then $|f|$ has no maximum value in that set.

Corollary. Let f be continuous in a compact set R and let f be analytic and non-constant in the interior of R. Then the maximum value of $|f|$ in R, which is always reached, occurs somewhere on the boundary of R and never in the interior.

43. LIOUVILLE'S THEOREM AND THE FUNDAMENTAL THEOREM OF ALGEBRA

The following is a corollary of the previous theorems.

Liouville's Theorem. If f is entire (analytic throughout $\mathbb{C}$) and bounded, then f is constant.

Fundamental Theorem of Algebra. Any polynomial (one variable, complex coefficients) of degree at least 1 has a zero.

It is easy to prove the fundamental theorem of algebra as a corollary to Liouville's theorem. You'll want to be sure to know how!

CHAPTER
FIVE
SERIES

44. CONVERGENCE OF SEQUENCES AND SERIES

A sequence of complex numbers converges if and only if the two component sequences of real and imaginary parts converge.

If a sequence is absolutely convergent, then the two component sequences are absolutely convergent, thus the two component sequences converge, and thus the original complex sequence converges.

45. TAYLOR SERIES

Taylor's Theorem. Let f be analytic throughout an open disk of radius R_0 centered at z_0. Then at each point in that disk,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \qquad a_n = \frac{f^{(n)}(z_0)}{n!}.$$

This is saying a lot. Notice that one can expand a Taylor series about a point for any function that is smooth at that point. The Taylor series is a particular power series, and power series in general have the property that they have a certain radius of convergence inside of which they convergence, outside of which they diverge, and on the boundary of which anything can happen. If f is analytic, then f is smooth. Thus we can expand a Taylor series. This theorem says that the radius of convergence is at least as large (it turns out not to be any larger) as the radius of the largest disk in which f is analytic; furthermore, not only does the series converge in this radius, it converges to the value of the function. Therefore, given an entire function (one which is analytic in the entire plane), the radius is infinitely large and thus the entire function is identical to its Taylor series (expanded about any point).

Important Series. The following function is entire and equals its Taylor series everywhere:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

The following function is analytic everywhere except $z = 1$, hence the closest singularity to the origin is at distance 1, thus this function equals its power series expanded about the origin everywhere inside the unit circle:

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \qquad \big(|z| < 1\big).$$

Another useful series is

$$-\operatorname{LN}(1 - z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

Of course, you should also know $\operatorname{COS} z$ and $\operatorname{SIN} z$, but these are easy to figure out from e^z if you start with $e^{iz} = \operatorname{COS} z + i \operatorname{SIN} z$ and then obtain $\operatorname{COS} z = \frac{1}{2}(e^{iz} + e^{-iz})$, then plug iz and $-iz$ formally into the expansion of e^z.

47. LAURENT SERIES

The Taylor Theorem of the previous section told us the behavior of the Taylor series expanded about the center of a disk in which f is analytic. You can also expand series about points where f is not analytic; the resulting series will have both positive and negative powers of the variable.

Laurent's Theorem. Let f be analytic throughout an annular domain centered at z_0 with inside radius R_1 and outside radius R_2; that is, f is not necessarily analytic inside the circle of radius R_1, but it is analytic in the rest of the larger circle of raidus R_2. Let C be any counterclockwise oriented simple closed contour around z_0 lying in the annular domain; that is, let C be a "curvy circle" running counterclockwise between the two circles. Then at each point in the annular domain,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n},$$

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,\mathrm{d}z}{(z - z_0)^{n+1}}, \qquad b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,\mathrm{d}z}{(z - z_0)^{-n+1}}.$$

Equivalently,

$$f(z) = \sum_{-\infty}^{\infty} c_n (z - z_0)^n, \qquad c_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,\mathrm{d}z}{(z - z_0)^{n+1}}.$$

One should reduce this theorem to Taylor's Theorem in the case that f is analytic inside the smaller circle. [**DO THIS**]

48. EXAMPLES

It turns out that the Laurent series (and the Taylor series, since it is a special case of the Laurent series) is unique. Thus, if one can find such a series using techniques other than integration, one in fact has found the Laurent series. That is, if you can find the coefficients of a Laurent-like series using some neato trick, then you have found the actual Laurent series, and thus your coefficients are in fact certain integrals, and thus you know the value of those integrals. This type of trickery is used a lot to evaluate nasty integrals.

49. ABSOLUTE AND UNIFORM CONVERGENCE OF POWER SERIES

A Taylor series is a special example of a power series of the form $\sum_{n=0}^{\infty} a_n (z - z_0)^n$. For power series of this form (this discussion therefore does not include Laurent series), the series has a radius of convergence inside of which it is absolutely convergent and uniformly convergent. (Absolute convergence holds at each point inside the radius of convergence; uniform convergence is a property of the entire inside of radius of convergence.) A power series always represents a continuous function inside its radius of convergence; in fact, we can say more: a power series always represents an analytic function inside its radius of convergence; that is, if you were to start with a power series, thinking of it as a function, and expand a Taylor series for that power series, you get back the same series.

50. INTEGRATION AND DIFFERENTIATION OF POWER SERIES

The gist of this section is that you can integrate and differentiate a power series term-by-term inside its radius of convergence. Furthermore, you can multiply a power series by a continuous function and still integrate term-by-term.

51. UNIQUENESS OF SERIES REPRESENTATIONS

Theorem. If $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges to $f(z)$ at all points interior to some circle centered at z_0, then that series is the Taylor series for f.

Theorem. If $\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$ converges to $f(z)$ at all points inside some annular domain centered at z_0, then that series is the Laurent series for f.

52. MULTIPLICATION AND DIVISION OF POWER SERIES

The gist is that you can multiply and divide power series and doing so formally gives the correct results.

CHAPTER
SIX
RESIDUES AND POLES

53. RESIDUES

A point z_0 is a **singular point** of the function f if f fails to be analytic at z_0 but is analytic at some point in every neighborhood of z_0; if f is in fact analytic in some deleted neighborhood of z_0, then z_0 is an **isolated singular point**.

The coefficient of the first negative power term of the Laurent series at an isolated singularity is the **residue** of f at that isolated singular point. By "first negative power" I don't mean the lowest negative power; I merely mean the coefficient of the $1/(z-z_0)$ term. Looking back at Laurent's Theorem, this coefficient, the residue, is

$$\operatorname{Res}_{z=z_0} f(z) = b_1 = \frac{1}{2\pi i}\int_C f(z)\,\mathrm{d}z.$$

Here C is some counterclockwise curve about z_0 inside a circle in which f is analytic except at z_0. Note that the Laurent series can be discussed anytime f is analytic in an annular domain; we discuss residues only at isolated singularities.

54. RESIDUE THEOREM

If f has only a finite number of singularities, then each must be an isolated singularity. We already know that integrating a counterclockwise loop around circles of non-analyticity amounts to summing the integrals around each such circle; in particular, given a finite number of singularities, integrating a counterclockwise loop about some of those singularities amounts to summing integrals around each singularity; but the latter are just (up to a constant multiple) the residues of those singularities.

Residue Theorem. If C is a counterclockwise loop within and on which f is analytic except for a finite number of singular points $z_1, \ldots, z_n$, then

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{R\scriptstyle{ES}}_{z=z_k} f(z).$$

Stereographic Trick. If we have the hypotheses in the previous theorem and furthermore know that f has no singularities exterior to C, then we can evaluate the integral around C by inverting (thereby making the point at infinity our only singularity) and we get

$$\int_C f(z)\,\mathrm{d}z = 2\pi i \mathrm{R\scriptstyle{ES}}_{z=0} \frac{1}{z^2} f(1/z).$$

55. PRINCIPAL PART OF A FUNCTION

About an isolated singular point one expands a Laurent series. The portion of the series involving negative powers is the principal part. We distinguish between three types of isolated singularities.

If there are a finite number of negative powers, say m, then the isolated singularity is a **pole of order** m. Actually, some of the coefficients might be zero, so we need to be careful. If $-m$ is the lowest negative power term, then we have a pole of order m; if $m = 1$, that is, if the residue term is the only negative power term, then the isolated singularity is a pole of order 1 or a **simple pole**. If the number of negative power terms is infinite, then the isolated singularity is an **essential singular point**. If there are no negative power terms, then the isolated singularity is a **removable singular point**; that is, by replacing $f(z_0)$ with the constant term of the power series (the Laurent series is a power series if there are no negative terms), then f becomes analytic at z_0.

Picard's Theorem. In each neighborhood of an essential singular point, a function assumes every finite value, with one possible exception, an infinite number of times. That is, except possibly for one value, all others values are assumed, and they are assumed infinitely often.

56. RESIDUES AT POLES

To find the residue of a function at an isolated singular point z_0, you must find the coefficient of $1/(z - z_0)$ in the Laurent series. For an essential singular point (when there are infinitely many negative power terms in the series), there is no better method. Otherwise, however, the residue can be found in other ways.

If $f(z)$ can be written $f(z) = \phi(z)/(z - z_0)$ with ϕ analytic and nonzero at z_0, then f has a simple pole at z_0 and the residue of f at z_0 is $\phi(z_0)$.

More generally, if $f(z)$ can be written $f(z) = \phi(z)/(z - z_0)^m$ with ϕ analytic and nonzero at z_0, then f has a pole of order m at z_0 and the residue of f at z_0 is $\phi^{(m-1)}(z_0)/(m-1)!$.

57. ZEROS AND POLES OF ORDER M

If f is analytic at z_0 and $f(z) = (z - z_0)^m g(z)$ where $g(z)$ is analytic and $g(z_0) \neq 0$, then f is said to have a **zero of order** m at z_0. This is equivalent to the first m terms of the Taylor series of f (expanded about z_0) being zero.

Zeros and poles are related. If two functions f, g are analytic at z_0 and $f(z_0) \neq 0$, then f/g has a pole of order m at z_0 if and only if g has a zero of order m at z_0.