SUMMARY OF FUNCTION PROPERTIES

$$
\begin{array}{ccccccc}
\begin{gathered} f' \text{ exists everywhere,} \\ f' \text{ bounded} \end{gathered} & & & & & & \\
\Downarrow & & & & & & \\
\begin{gathered} (D^+f)(x) \text{ bounded,} \\ f \text{ continuous} \end{gathered} & & & & & & \\
\Downarrow & & & & & & \\
f \text{ Lipschitz} & \Rightarrow & \begin{gathered} f' \text{ bounded} \\ \text{where it exists} \end{gathered} & & & & \\
\Downarrow & & & & & & \\
\begin{gathered} f \text{ absolutely} \\ \text{continuous} \end{gathered} & \Rightarrow & \begin{gathered} f \text{ bounded} \\ \text{variation} \end{gathered} & \Leftrightarrow & f = \text{mono} - \text{mono} & \Leftrightarrow & f = \text{inc} - \text{inc} \\
\Downarrow & & \Uparrow & & \Downarrow & & \Downarrow \\
\int_a^b f' = f(b) - f(a) & & f \text{ monotone} & & f' \text{ integrable} & & f' \text{ exists a.e.} \\
\Downarrow & & \Uparrow & & & & \Downarrow \\
\int_a^b f' \leq f(b) - f(a) & \Leftarrow & f \text{ increasing} & & & & f, f' \text{ measurable}
\end{array}
$$

NOTES:

- Throughout the diagram, f is a real-valued function on the compact interval $[a, b]$.
- The notation D^+ refers to one of the four Dini derivates; the summary diagram remains valid when D^+ is replaced by any of the remaining three.
- The summary diagram remains valid when, in the bottom line, the inequality is reversed and "f increasing" is replaced with "f decreasing".

SUGGESTED STUDY QUESTIONS/PROBLEMS:

- Which arrows are trivially true, and which represent deeper results? [The summary diagram includes trivially true arrows so that deep results are broken into chains of easier results.]
- Prove each implication.
- For each one-way implication, show by suitable example that the converse implication fails.