GEOMETRY/TOPOLOGY QUALIFYING EXAMS

Herein are solutions to many of the problems on old qualifying exams. Sometimes multiple solutions are given. Some solutions were written years ago when I knew much less (that may make the solutions better or worse, depending on your point of view—but it also might mean that the solutions are incorrect). Comments to myself appear in brackets, whose presence further indicates that you're peeking at a preliminary draft of something. Although I sometimes obtained solutions from listening to others or reading their work (and I try to give credit wherever this is the case), I typed up the solutions myself using style according to my whim and taste (or lack thereof); hence if anyone is to be blamed for incorrect solutions, typos, style, and such, it is I, me, myself, I, and no one else.

I would appreciate your help with this document; if you have solutions to unsolved problems or have corrections or comments on material that appears here, please let me know. Thanks.

SAMPLE PROBLEMS FROM LOUISIANA STATE UNIVERSITY

1. Let F be a closed set and K a compact set in a Hausdorff space X. Show that $F \cap K$ is both compact and closed.

2. Let C be a connected set and let A be a set such that $C \subseteq A \subseteq \overline{C}$ (closure) in a space X. Show that A is connected.

3. Find an example of a metric space (X, d) containing a point p such that $\overline{N_r(p)} \neq \{\, x : d(x,p) \leq r \,\}$ for some $0 < r$, where $N_r(p) = \{\, x : d(x,p) < r \,\}$.

4. Let $f\colon X \to Y$ be continuous, and let $A \subseteq X$. Show the following:

- If A is compact, then $f(A)$ is compact; and,
- If A is connected, then $f(A)$ is connected.

5. Do the following.

- Give an example of a nest of connected sets whose intersection is not connected.
- Give a sufficient condition for a nest of connected sets to have a connected intersection.

6. Show that the open unit n-cell E^n is homeomorphic to R^n.

7. Do the following.

- Give an example of a T_1-space that is not a T_2-space.
- Give an example of a T_2-space that is not regular.
- Show that every discrete space is metrizable.
- Give an example of a metrizable space that is not second countable.

8. Let $f\colon X \to Y$ be a function from a space X into a space Y. Show that the following are equivalent:

- f is continuous, i.e., $f^{-1}(V)$ is open for each open set $U \subseteq Y$;
- $f^{-1}(V)$ is open for each basic open set $V \subseteq Y$;
- $f(\overline{A}) \subseteq \overline{f(A)}$ for each set A in X;
- $f^{-1}(F)$ is closed for each closed set $F \subseteq Y$.

9. Do the following:

- Prove that each second countable space is separable.
- Give an example of a metric space that is not separable.

10. Let $f\colon X \to Y$ and $g\colon Y \to Z$ be functions on spaces.

- Show that if f and g are each continuous, then $g \circ f$ is also continuous.
- If $g \circ f$ is continuous, are f and g each necessarily continuous? Justify your answer.

11. Show that "homotopy" is an equivalence relation.

12. Show that $S^n = \{\,(x_1, \ldots, x_{n+1}) : \sum_{i=1}^{n+1} x_i^2 = 1\,\}$ is a smooth submanifold of Euclidean $(n+1)$ space.

SAMPLE PROBLEMS FROM UNIVERSITY OF ARIZONA

1. Find a linear fractional transformation $\mathbb{P}^1(\mathbb{R}) \to \mathbb{P}^1(\mathbb{R})$ that maps the set

$$\{\, x \in \mathbb{R} : |x| \geq 1 \,\} \cup \{\infty\}$$

to the set

$$\{\, x \in \mathbb{R} : x \geq 0 \,\} \cup \{\infty\}.$$

SOLUTION:

Thinking of $\mathbb{P}^1(\mathbb{R})$ as a circle, one solution is to find a linear fractional transformation that sends $1 \mapsto 0$ and $-1 \mapsto \infty$, and sends ∞ to some arbitrary positive value. The map

$$x \mapsto \frac{x-1}{x+1}$$

has the necessary effect and sends $\infty \mapsto 1$.

[One may tediously verify that this map does the right thing. First write $(x-1)/(x+1) = \alpha$ and solve for x to obtain $x = (1+\alpha)/(1-\alpha)$. Then one easily sees that $\alpha \geq 0$ corresponds to $|x| \geq 1$.]

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2. Let $p(z)$ be a polynomial of degree n with the property that if $p(\alpha) = 0$ then $|\alpha| < R$. Let C be the contour $|z| = R$ oriented in the counterclockwise sense. Compute

$$\int_C \frac{p'(z)}{p(z)}\, \mathrm{d}z.$$

SOLUTION:

Aside: The expression $p'(z)/p(z)$ is sometimes called the "logarithmic derivative" since it equals $(\mathrm{d}/\mathrm{d}z)\, \textsc{ln}\, p(z)$.

Write $p(z) = a_n \prod_{k=1}^{n} (z - \alpha_k)$, where the roots α_k are not necessarily distinct. Then $p'(z) = a_n \sum_{k=1}^{n} \prod_{\substack{1 \le \ell \le n \\ \ell \neq k}} (z - \alpha_\ell)$, so that

$$\frac{p'(z)}{p(z)} = a_n \sum_{k=1}^{n} \frac{1}{z - \alpha_k}$$

The integral in question picks up residues at each singularity. The expression $p'(z)/p(z)$ has a simple pole at each root α_k, the residue being $m_k a_n$, where m_k is the multiplicity of the root α_k. (In the summation expression for $p'(z)/p(z)$ above, note that some of the terms may coincide.) The integral is $2\pi i$ times the sum of the residues, hence it is $2\pi i n a_n$.

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3. Let $D = \{\, x \in \mathbb{R}^2 : |x| \leq 1 \,\}$, and let $S^1 = \{\, x \in \mathbb{R}^2 : |x| = 1 \,\}$ denote the boundary of D. Show that, if $f\colon D \to D$ is a homeomorphism of D onto D, then f must take the boundary S^1 of D onto S^1. Hint: Consider the fundamental group $\pi_1(D \setminus \{x\})$ in the two different cases $x \in S^1$ and $x \in \textsc{interior}(D)$.

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4. Show that a continuous map of a compact space into $\mathbb{R}$ always assumes its maximum.

SOLUTION:

Two theorems are important here. First, a continuous map sends compact sets to compact sets. Second, the compact sets in $\mathbb{R}^n$ (and in particular, in $\mathbb{R}$) are precisely the closed and bounded sets. (The second theorem is the Heine–Borel theorem.) Combining these, we see that the image of the map under question is closed and bounded in $\mathbb{R}$, hence includes its least upper bound, which is the maximum achieved by the map.

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5. Let Y be a smooth manifold and let $\tilde{Y} \to Y$ be a covering map. Prove that $\tilde{Y}$ is a manifold.

6. Write $S^1 = U_0 \cup U_1$, where

$$U_0 = S^1 \setminus \{(0,1)\} \qquad \text{and} \qquad U_1 = S^1 \setminus \{(0,-1)\}.$$

Consider the vector field X on U_0 given by $X = x\partial_x$ in the local coordinates on U_0 given by stereographic projection on U_0. Write the vector field X in the local coordinates on U_1 and show that this vector field extends continuously to all of S^1.

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7. Prove that any continuous map $f\colon \mathbf{D}^3 \to \mathbf{D}^3$ has a fixed point. (Here $\mathbf{D}^3$ denotes the closed unit ball in $\mathbb{R}^3$.)

SOLUTION:

[There is a correponding result true for all $\mathbf{D}^n$, $n \geq 1$. I don't know of a special proof for $\mathbf{D}^3$ other than the general proof. However, there *is* a much simpler proof for $\mathbf{D}^2$ that uses the fundamental group; I'll give that proof first, since most of its ideas are used in the general proof as well.]

For the sake of contradiction, assume on the contrary that $f\colon \mathbf{D}^2 \to \mathbf{D}^2$ is continuous with $f(x) \neq x$ for all $x \in \mathbf{D}^2$. Define a new map $r\colon \mathbf{D}^2 \to \partial\mathbf{D}^2 = S^1$ as follows: $r(x)$ is defined to be the unique point on the boundary of $\mathbf{D}^2$ lying in the ray emanating from $f(x)$ and passing through x. That r is continuous is intuitively obvious: if x, y lie close together, then so do $f(x), f(y)$, hence the rays $\overrightarrow{f(x)x}$ and $\overrightarrow{f(y)y}$ must be almost one and the same, so that their intersections with the boundary $\partial\mathbf{D}^2$ must lie close together. However, rigorously proving r to be continuous looks rather tedious and I'm not convinced I'd get through it. (The textbooks seem to always skip this part…) Now for $x \in \partial\mathbf{D}^2$, we clearly have $r(x) = x$. Hence r is the identity map when restricted to $\partial\mathbf{D}^2$, showing $\partial\mathbf{D}^2$ to be a (non-deformation) retract of $\mathbf{D}^2$. My final step is to show that no such retract exists. This is intuitively obvious since the center of the ball gets ripped apart in the retraction. The proof is an appeal to the fundamental group: the fundamental group of $\mathbf{D}^2$ is trivial, hence can't surject onto the nontrivial fundamental group of $\partial\mathbf{D}^2$.

The above proof goes through mutatis mudandis [spelling?] to show that the existence of a fixed-point free map $f\colon \mathbf{D}^n \to \mathbf{D}^n$ implies the existence of a retract $r\colon \mathbf{D}^n \to \partial\mathbf{D}^n$. In fact, these statements are equivalent, as I shall now show. Assume that we have a retract $r\colon \mathbf{D}^n \to \partial\mathbf{D}^n$. Then the composition of the retract map followed by the antipodal map on $\partial\mathbf{D}^n$ gives a map $\mathbf{D}^n \to \mathbf{D}^n$ that is fixed-point free. (To be technically correct, I am really composing the retract $\mathbf{D}^n \to \partial\mathbf{D}^n$ with the antipodal $\partial\mathbf{D}^n \to \partial\mathbf{D}^n$ with the inclusion $\partial\mathbf{D}^n \hookrightarrow \mathbf{D}^n$.)

Finally, I must show that either of these equivalent statements is false. For $n = 2$, it was easy, using the fundamental group, to show that there exists no retract $r\colon \mathbf{D}^n \to \partial\mathbf{D}^n$. Here is a proof that works in general, but assumes knowledge of the notion of degree (or "Brouwer degree") of a map from a sphere to itself, and requires the fact that degree is invariant under homotopy. So assume that $r\colon \mathbf{D}^n \to \partial\mathbf{D}^n = S^{n-1}$ is a retract. Now coordinatize so that S^{n-1} consists of n-tuples such that the sum of the squares of the coordinates is 1. Define $h\colon S^{n-1} \times I \to S^{n-1}$ via $h(x,t) = r(tx)$. In other words, h is the result of "pulling in" the boundary $\partial\mathbf{D}^n$ towards the center, then applying r to retract it back out. But when $t = 0$, we pull the boundary to a single point, so h is constant there. To be precise, $h(x,1)\colon S^{n-1} \to S^{n-1}$ is just the identity map (namely the restriction of r to S^{n-1}), whereas $h(x,0)\colon S^{n-1} \to S^{n-1}$ is a constant map. And h gives a homotopy between these two maps. But the identity map has degree 1, whereas the constant map has degree 0, so that's our contradiction.

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8. Find $H_*(S^2)$. Take S^2 and squeeze (identify) the equator S^1 to a point; denote the resulting quotient space S^2/S^1. Find $H_*(S^2/S^1)$. Justify your answer.

SOLUTION:

Since S^2 has a single component, I have $H_0(S^2) = \mathbb{Z}$. Now S^2 is simply connected, hence $\pi_1(S^2) = 0$, and then abelianizing gives $H_1(S^2) = 0$. Finally, S^2 is a 2-dimensional compact oriented manifold, hence Poincaré duality tells me $H_2(S^2) = H_0(S^2) = \mathbb{Z}$.

The space S^2/S^1 looks like a surface version of the figure eight loop, namely like two spheres meeting at a point. The space is connected and simply connected (let the common point be the base point; any closed loop from there is a product of closed loops on each sphere, but each such loop is trivial). The surface is compact, and I'm pretty sure it's orientable, but I'm not certain. If it were orientable, then I would get the same homology as before, namely

$$\begin{aligned} H_0(S^2/S^1) &= \mathbb{Z}, \\ H_1(S^2/S^1) &= 0, \\ H_2(S^2/S^1) &= \mathbb{Z}. \quad (?) \end{aligned}$$

I do observe that $\pi_2(S^2/S^1)$ appears to be $\mathbb{Z}*\mathbb{Z}$, pretty much for the same reason that π_1 of a figure eight is $\mathbb{Z}*\mathbb{Z}$, but of course I'm not expert enough to justify my intuition on higher loop groups. If there is a relation between H_2 and π_2 similar to the relation that H_1 is the abelianization of π_1, then I would guess that S^2/S^1 must not be orientable and must have $H_2(S^2/S^1) = \mathbb{Z}^2$. [In fact, it's a theorem that the higher loop groups are necessarily abelian, so the only simple relationship would be that the higher loop groups are identical to the higher homology groups, but that cannot be, for surely I would heard of such a remarkable fact.]

I can calculate things precisely by using the special MAYER–VIETORIS sequence in reduced homology. It gives me the exact sequence

$$\to H_3(S^1) \to H_3(S^2) \to H_3(S^2/S^1) \to H_2(S^1) \to H_2(S^2) \to H_2(S^2/S^1) \to H_1(S^1) \to H_1(S^2) \to H_1(S^2/S^1) \to \tilde{H}_0(S^1) \to \tilde{H}_0(S^2) \to \tilde{H}_0(S^2/S^1) \to 0,$$

where $\tilde{H}_0(M) \oplus \mathbb{Z} = H_0(M)$ for all manifolds M. Putting in the dimensions, the exact sequence is

$$\to 0 \to 0 \to ? \to 0 \to 1 \to ? \to 1 \to 0 \to ? \to 0 \to 0 \to ? \to 0,$$

so that I conclude $H_3(S^2/S^1) = 0$ (as it must have been), $H_2(S^2/S^1) = \mathbb{Z}^2$ (ah, quite the surprise!), $H_1(S^2/S^1) = 0$, and $\tilde{H}_0(S^2/S^1) = 0$, so that $H_0(S^2/S^1) = \mathbb{Z}$. Thus I am additionally forced to conclude that S^2/S^1 is not orientable, a fact that is not obvious to me visually, nor in terms of a simplicial cell decomposition. [Something to think about: how does the point connected various orientable 2-cells affect the orientation of the entire simplicial complex?]

COLLECTION OF VECTOR CALCULUS AND ADVANCED CALCULUS QUESTIONS (FROM QUIZ OF ERCOLANI)

1. Derive the formula for the Laplacian in polar coordinates. It is given below:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

2. Is there a C^2-function $g\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\frac{\partial g}{\partial x} = y\,\mathrm{SIN}(x^2y^2),$$
$$\frac{\partial g}{\partial y} = x\,\mathrm{SIN}(x^2y^2)?$$

3. Let $M = \{\,(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1,\ x^2+2y^2-z^2 = 0\,\}$. Is M compact? Is M connected? Explain.

4. Let $x = 2u+v$ and $y = 3u+v$. Suppose we know $\partial f/\partial x$ and $\partial f/\partial y$. Find $\partial f/\partial u$ and $\partial f/\partial v$.

5. Answer the following:

- Find the differential of $f(x,y) = \mathrm{ARCTAN}(y/x)$.
- Is $\mathrm{d}f$ the differential of a function defined on the punctured plane $\mathbb{R}^2 \setminus \{0\}$?
- On what domain is $\mathrm{d}f$ the differential of a function?

6. Find a parametric representation for the line normal to the surface in space given by $x^3+y^3+\mathrm{SIN}(xyz) = 0$ at the point $(1,-1,0)$.

7. Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable homeomorphism given by $\phi(u,v) = (x(u,v), y(u,v))$, and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be an integrable function on a region U in $\mathbb{R}^2$. Rewrite the integral

$$\int_U f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

in terms of the coordinates (u,v).

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN JANUARY 1996 (ERCOLANI AND PICKRELL)

1. Let D denote an open subset of $\mathbb{C}$. A function $f\colon D \to \mathbb{C}$ is said to be complex analytic on D if the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for all $z_0 \in D$. Suppose that f is complex analytic.

- Directly from this definition, derive the Cauchy–Riemann equations, i.e., the relations satisfied by the partial derivatives of the real and imaginary parts of f.
- Suppose in addition that f is C^1 in D. Use Stokes' theorem to prove that

$$\int_{\partial\Omega} f(z)\,\mathrm{d}z = 0,$$

where $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$, and $\Omega \subseteq D$ has a smooth boundary.

SOLUTION:

Isn't C^1 automatic from being complex analytic?

2. Find models for the universal coverings of the following spaces, and indicate how the fundamental group acts in each case:

- the 2-torus

$$\mathbf{T}^2 = \{\, (z_1, z_2) \in \mathbb{C}^2 : |z_1| = |z_2| = 1 \,\};$$

- the punctured complex plane $\mathbb{C} \setminus \{0\}$;
- the "figure eight".

3. For each real number c, define

$$X_c = \{\, (x,y,z) \in \mathbb{R}^3 : xy + z^2 = c \,\}.$$

- Explain why X_c is an embedded submanifold of $\mathbb{R}^3$, provided that $c \neq 0$.
- Find a basis for the tangent space to X_2 at the point $(1,1,1)$.
- Let $i\colon X_2 \hookrightarrow \mathbb{R}^3$ denote the inclusion. Compute

$$i^*(\mathrm{d}x \wedge \mathrm{d}y)|_{(1,1,1)}$$

in terms of the basis dual to the basis you found above.

SOLUTION:

The regular value theorem tells me that X_c is an embedded submanifold (in fact, a smooth manifold) whenever $c \neq 0$; for the details, see the almost identical problem on Fall 1996.

Taking the differential of $xy + z^2 = 2$ gives $y\,\mathrm{d}x + x\,\mathrm{d}y + 2z\,\mathrm{d}z = 0$. Hence the equation of the tangent plane at $(1, 1, 1)$ is $(x-1)+(y-1)+2(z-1) = 0$, and a normal vector there is $(1, 1, 2)$. To produce tangent vectors, I could find three random points in the plane and form pairwise differences, but here it is easy enough to guess some vectors that are orthogonal to the normal vector; for example, take $(2, 0, -1)$ and $(0, 2, -1)$. (These are clearly linearly independent.) So in terms of differential operators, the tangent space at $(1, 1, 1)$ has basis $\{ 2\partial/\partial x - \partial/\partial z, 2\partial/\partial y - \partial/\partial z \}$.

At $(1, 1, 1)$, the alternating tensor $i^*(\mathrm{d}x)$ takes a tangent vector $\alpha(2\partial/\partial x - \partial/\partial z) + \beta(2\partial/\partial y - \partial/\partial z)$ and spits out 2α, whereas $i^*(\mathrm{d}y)$ would spit out 2β. Hence $\frac{1}{2}i^*(\mathrm{d}x)$ and $\frac{1}{2}i^*(\mathrm{d}y)$ is a dual basis. In terms of this basis, $i^*(\mathrm{d}x \wedge \mathrm{d}y) = 4(\frac{1}{2}i^*(\mathrm{d}x)) \wedge (\frac{1}{2}i^*(\mathrm{d}y))$.

The last part felt silly, as if I didn't do any work. Partly that's because I was lucky in my choice of basis earlier. Had I instead used $(1, 1, -1)$ and $(1, -1, 0)$, then I would have $i^*(\mathrm{d}x)$ giving $\alpha(\partial/\partial x + \partial/\partial y - \partial/\partial z) + \beta(\partial/\partial x - \partial/\partial y)$ the value $\alpha + \beta$, whereas $i^*(\mathrm{d}y)$ would assign the value $\alpha - \beta$. Hence my dual basis would be $\{ \frac{1}{2}(i^*(\mathrm{d}x) + i^*(\mathrm{d}y)), \frac{1}{2}(i^*(\mathrm{d}x) - i^*(\mathrm{d}y)) \}$. Wedging them together gives

$$
\begin{aligned}
&(\tfrac{1}{2}(i^*(\mathrm{d}x) + i^*(\mathrm{d}y))) \wedge (\tfrac{1}{2}(i^*(\mathrm{d}x) - i^*(\mathrm{d}y))) \\
=&\tfrac{1}{4}i^*(\mathrm{d}x \wedge \mathrm{d}x) - \tfrac{1}{4}i^*(\mathrm{d}x \wedge \mathrm{d}y) + \tfrac{1}{4}i^*(\mathrm{d}y \wedge \mathrm{d}x) - \tfrac{1}{4}i^*(\mathrm{d}y \wedge \mathrm{d}y) \\
=&-\tfrac{1}{4}i^*(\mathrm{d}x \wedge \mathrm{d}y) - \tfrac{1}{4}i^*(\mathrm{d}x \wedge \mathrm{d}y) \\
=&-\tfrac{1}{2}i^*(\mathrm{d}x \wedge \mathrm{d}y).
\end{aligned}
$$

Hence, in my dual basis, $i^*(\mathrm{d}x \wedge \mathrm{d}y)$ is expressed as negative two times the wedge product of my dual basis. Now that was (slightly) more interesting.

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4. Consider the map $f\colon S^2 \to \mathbb{R}^6$ defined by

$$f(x_1, x_2, x_3) = (x - 1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3).$$

- Show that f is an immersion, i.e., that the derivative of f is injective at each point.
- Explain why the image of f is an embedded submanifold of $\mathbb{R}^6$.
- Show that f induces a diffeomorphism of $\mathbb{RP}^2$ with the image of f.

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5. For each of the following, give an example or prove that none exists:

- two spaces with the same first homology group over $\mathbb{Z}$ but different fundamental groups;
- a compact surface with a vector field that is nonzero at all points;
- a continuous function
$$f\colon \overline{D} \to S^1$$
such that the restriction $f|_{S^1}$ is the identity—here, D denotes the unit disk in the complex plane;
- a compact oriented manifold X with $\mathrm{H}^n_{\mathrm{DR}}(X) = 0$, where $\mathrm{H}^*_{\mathrm{DR}}$ denotes de Rham cohomology.

SOLUTION:

- The torus and the torus with a point removed each have the free abelian group on two generators as their first homology group; the torus has the same for fundamental group, but the torus with a point removed has the free group on two generators as fundamental group.
- There certainly is no smooth nonvanishing vector field on S^2; I'm not sure about other compact surfaces. [GO BACK AND FIGURE THIS OUT.]
- There is no retract from the disk to its boundary; this is equivalent to the fact that every continuous map $D \to D$ has a fixpoint. [Both assertions are proved elsewhere in this packet. FIND OUT WHERE AND PUT THE APPROPRIATE REFERENCE HERE.]
- For compact oriented manifolds, Poincaré duality asserts, among other things, that $\mathrm{H}^0_{\mathrm{DR}}(X) \cong \mathrm{H}^n_{\mathrm{DR}}(X)$. Since $\mathrm{H}^0_{\mathrm{DR}}$ effectively counts the number of components, we always have $\mathrm{H}^0_{\mathrm{DR}}(X) \neq 0$, thus the same holds for $\mathrm{H}^n_{\mathrm{DR}}(X)$.

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6. Let $M \subseteq \mathbb{R}^3$ denote a compact 3-dimensional submanifold with a smooth boundary. Let n denote the unit normal vector along the boundary of M. Suppose that v is a smooth vector field defined on M. The divergence theorem from vector calculus states that

$$\int_{\partial M} (v \cdot n)\, \mathrm{d}A = \int_M \mathrm{DIV}(V)\, \mathrm{d}V.$$

Formulate this result in terms of differential forms, and explain why it is a special case of the general differential forms version of STOKES' theorem.

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7. Let M denote a compact manifold. The theorem of DE RHAM asserts the existence of a natural isomorphism

$$\mathrm{H}^k_{\mathrm{DR}}(M;\mathbb{R}) \xrightarrow{\sim} \mathrm{H}_k(M;\mathbb{R})^* \qquad \text{for each } k,$$

where $\mathrm{H}^k_{\mathrm{DR}}$ denotes DE RHAM cohomology, H_k denotes singular (= simplicial) homology, and V^* denotes the dual of a vector space V. Explain the statement of this theorem in detail.

SOLUTION:

By definition, $\mathrm{H}^k_{\mathrm{DR}}(M;\mathbb{R})$ is the real vector space of closed k-forms on M modulo the subspace of exact k-forms; $\mathrm{H}_k(M;\mathbb{Z})$ is the abelian group of k-chains that are cycles modulo the subgroup of k-chains that are boundaries; finally, $\mathrm{H}_k(M;\mathbb{R})$ is the corresponding real vector space obtained by extending coefficients to $\mathbb{R}$: $\mathrm{H}_k(M;\mathbb{R}) = \mathrm{H}_k(M;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}$.

We have two real vector spaces, $\mathrm{H}^k_{\mathrm{DR}}$ and H_k, which turn out to each have finite dimension. In general, given two finite-dimensional vector spaces, V and W, over some field k, there is a correspondence

$$\left\{\text{NONDEGENERATE BILINEAR PAIRINGS } V \times W \to k\right\} \longleftrightarrow \left\{\text{ISOMORPHISMS } V \xrightarrow{\sim} W^*\right\}.$$

Thus, in the present situation, the DE RHAM isomorphism is obtained by exhibiting a nondegenerate bilinear pairing

$$\mathrm{H}^k_{\mathrm{DR}}(M;\mathbb{R}) \times \mathrm{H}_k(M;\mathbb{R}) \longrightarrow \mathbb{R}.$$

The pairing of a form with a chain is simply the integral of the form along the chain:

$$\langle \omega, C\rangle \longmapsto \int_C \omega.$$

The proof of the DE RHAM theorem establishes that this pairing is nondegenerate for all k.

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN FALL 1996

1. Suppose that $R(z) = p(z)/q(z)$, where p and q are polynomials with $\text{DEG}(p) < \text{DEG}(q)$. Let $a_1, \ldots, a_n$ denote the distinct roots of $q(z) = 0$, and suppose that $p(a_i) \neq 0$ for each i. Show that

$$R(z) = \sum_{i=1}^{n} r_i(z),$$

where each r_i is a rational function that vanishes at ∞ and has a pole only at $z = a_i$.

SOLUTION:

[Solution One.] If $n = 1$ then we are done, viewing $R(z) = p(z)/q(z)$ as a sum of a single term. Now let n be minimal so that we have a counterexample $R(z)$, namely a rational function $p(z)/q(z)$ for which the desired representation does not exist. Let k denote the multiplicity of the root a_n. Expanding a Laurent series at $z = a_n$, we have:

$$R(z) = \frac{b_{-k}}{(z-a_n)^k} + \cdots + \frac{b_{-1}}{z-a_n} + b_0 + b_1(z-a_n) + \cdots$$

Let $r_n(z)$ denote the negative power portion of this expansion. Then $r_n(z)$ is a rational function that vanishes at ∞ and has a pole only at a_n. Furthermore, $R(z) = r_n(z) + R_0(z)$, where $R_0(z)$ is analytic at a_n and has the same pole behavior at other points (since $r_n(z)$ is holomorphic elsewhere). In fact, we can write

$$r_n(z) = \frac{b_{-k} + b_{-k-1}(z-a_n) + \cdots + b_{-1}(z-a_n)^{k-1}}{(z-a_n)^k} = \frac{p_0(z)}{(z-a_n)^k},$$

where $\text{DEG}\, p_0 < k$. Set q_0 so that $(z-a_n)^k q_0(z) = q(z)$. Then we have

$$R_0(z) = \frac{p(z)}{q(z)} - r_n(z) = \frac{p(z)}{q(z)} - \frac{p_0(z)}{(z-a_n)^k} = \frac{p(z) - p_0(z)q_0(z)}{q(z)}.$$

Since $R_0(z)$ has no pole at $z = a_n$, we must be able to cancel the factor $(z-a_n)^k$ appearing in the denominator against a corresponding factor in the numerator. Hence the degree of the numerator is at most $\text{MAX}(\text{DEG}\, p, \text{DEG}\, p_0 + \text{DEG}\, q_0) - k$, while the degree of the denominator is precisely $(\text{DEG}\, q) - k$. We assumed from the start that $\text{DEG}\, p < \text{DEG}\, q$, and we also know $\text{DEG}\, p_0 < k$ and $\text{DEG}\, q_0 = (\text{DEG}\, q) - k$. Hence we see that both arguments to the MAX are $< \text{DEG}\, q$, so that the degree of the numerator is $< (\text{DEG}\, q) - k = \text{DEG}\, q_0$. Since our example $R(z)$ was a minimal counterexample, $R_0(z)$ cannot be a counterexample and thus has the properties stated in the problem. But then so does $R(z)$, showing that it wasn't a counterexample to begin with. This absurdity contradicts our outgoing assumption that a counterexample exists, and hence the problem has been solved.

[Solution Two.] Basically I am proving a type of partial fraction decomposition. I'll establish the result inductively. If $n = 1$, then $q(z)$ vanishes only at a_1 so that $p(z)/q(z)$ has a pole only at $z = a_1$, and $\text{DEG}(p) < \text{DEG}(q)$ tells me that $p(z)/q(z)$ vanishes at ∞; hence $R(z) = p(z)/q(z)$, viewed as a sum consisting of a single term, already has the required form.

Now let $n > 1$. Let k denote the multiplicity of the root a_n of $q(z) = 0$, and set $q_0(z)$ so that $(z-a_n)^k q_0(z) = q(z)$. Then $(z-a_n)^k$ and $q_0(z)$ are relatively prime (as elements in the polynomial ring $\mathbb{C}[z]$), when there exists polynomials $b(z)$ and $c(z)$ satisfying $(z-a_n)^k b(z) + c(z)q_0(z) = 1$. Note from this equation that $c(a_n) \neq 0$, while $b(z)$ and $q_0(z)$ can likewise not vanish simultaneously. Now write:

$$R(z) = \frac{p(z)}{q(z)} = \frac{p(z)c(z)}{(z-a_n)^k} + \frac{p(z)b(z)}{q_0(z)}.$$

Notice that $p(z)b(z)$ has no roots in common with $q_0(z)$, the latter having one fewer roots than $q(z)$. Hence the induction is satisfied so long as I can show $\text{DEG}\, p + \text{DEG}\, c < k$ and $\text{DEG}\, p + \text{DEG}\, b < \text{DEG}\, q_0$. [So how do I do that?]

[Solution Three. The following solution, by Mike Yarbrough August 1998, assumes the existence of partial fraction decompositions and goes from there.] Let $p(z) = p_m z^m + p_{m-1}z^{m-1} + \cdots + p_1 z + p_0$, and let $q(z) = (z-a_1)^{t_1}(z-a_2)^{t_2} \cdots (z-a_n)^{t_n}$. We know that $p(a_i) \neq 0$. I wish to show that I can start with

$$R(z) = \frac{p(z)}{q(z)} = \frac{p_m z^m + \cdots + p_1 z + p_0}{(z-a_1)^{t_1} \cdots (z-a_n)^{t_n}}$$

and write $R(z) = \sum_{i=1}^{n} r_i(z)$, where r_i is a rational function that vanishes at ∞ and has a pole only at $z = a_i$. I will use partial fractions to accomplish this. Since $\text{DEG}\, p < \text{DEG}\, q$, when we divide, the quotient is 0, leaving only the remainder. Hence we have

$$\begin{aligned}
R(z) = {} & \frac{A_{11}}{(z-a_1)} + \frac{A_{12}}{(z-a_1)^2} + \cdots + \frac{A_{1t_1}}{(z-a_1)^{t_1}} \\
& + \frac{A_{21}}{(z-a_2)} + \frac{A_{22}}{(z-a_2)^2} + \cdots + \frac{A_{1t_2}}{(z-a_2)^{t_2}} \\
& + \cdots \\
& + \frac{A_{n1}}{(z-a_n)} + \frac{A_{n2}}{(z-a_n)^2} + \cdots + \frac{A_{nt_n}}{(z-a_n)^{t_n}}
\end{aligned}$$

Since $q(z)$ splits into linear factors, the numerators A_{ij} in the partial fraction decomposition are constants. Let $r_i(z)$ be the ith row in the above decomposition:

$$\begin{aligned}
r_i(z) &= \frac{A_{i1}}{(z-a_i)} + \frac{A_{i2}}{(z-a_i)^2} + \cdots + \frac{A_{it_i}}{(z-a_i)^{t_i}} \\
&= \frac{A_{i1}(z-a_i)^{\hat{1}+2+\cdots+t_i} + \cdots + A_{ij}(z-a_i)^{1+2+\cdots+\hat{j}+\cdots+t_i} + \cdots + A_{it_i}(z-a_i)^{1+2+\cdots+\hat{t}_i}}{(z-a_i)^{1+2+\cdots+t_i}}
\end{aligned}$$

Thus $r_i(z)$ is a rational function. It vanishes at ∞ since the degree of the numerator is less than the degree of the denominator. The denominator vanishes only when $z = a_i$.

---

2. Consider

$$X_c = \{\, (x, y, z) \in \mathbb{R}^3 : z^2 + xy = c \,\}.$$

- For which values of c is X_c a smooth manifold?
- For the values $c = 0$ and $c = 1$, answer the following questions: Is X_c connected? Is X_c compact? What is $\pi_1(X_c)$? Explain your answers.

SOLUTION:

[Solution by Mike Yarbrough, August 1998.] I shall use the regular value theorem. Consider the map $F\colon \mathbb{R}^3 \to \mathbb{R}^1$ given by $(x, y, z) \mapsto z^2 + xy$. The derivative is $(\partial F/\partial x, \partial F/\partial y, \partial F/\partial z) = (y, x, 2z)$. This matrix will have full rank if any entry is nonzero. Hence the only critical point is $(0, 0, 0)$, at which F has the critical value 0. The regular theorem tells me that $F^{-1}(\text{REGULAR VALUE})$ is an embedded submanifold; furthermore, if F is smooth between smooth manifolds, then $F^{-1}(\text{REGULAR VALUE})$ is smooth. Being polynomial, certainly F is smooth, and so I see that X_c is a smooth manifold whenever $c \neq 0$. The regular value theorem tells me nothing when $c = 0$, so I must consider this case separately. As will be shown in the second part of the problem, X_0 turns out *not* to be a smooth manifold.

X_0 is the locus of points in $\mathbb{R}^3$ satisfying $xy = -z^2$. For each fixed value of z, the level curve is a hyperbola, which is degenerate when $z = 0$. Thus X_0 looks like a cone. Another way to see this is to make the change of variables $x' = x + y$ and $y' = x - y$, so that $(x')^2 - (y')^2 = 4xy$, and our surface is $(x')^2 + 4z^2 = (y')^2$, which is more easily seen to be a cone. (Slicing with fixed values of y gives ellipses whose radii increase as y increases.) At any rate, the vertex of the cone is a singular point; X_0 is locally Euclidean at all other points. We see that X_0 is connected, not compact (it isn't bounded), and since X_0 contracts to a point, we have $\pi_1(X_0) = \{0\}$.

X_1 is the locus of points in $\mathbb{R}^3$ satisfying $xy = 1 - z^2$. The changes of variables $x' = x + y$ and $y' = x - y$ changes the equation to $(x')^2 + 4z^2 = 4 + (y')^2$. Again, for each fixed value of y, there is an ellipse, the smallest one occuring when $y' = 0$. There is no singular point. (We knew that already.) Our manifold is connected but not compact. It has the ellipse $(x')^2 + 4z^2 = 4$ (with $y' = 0$) as a deformation retract. Hence $\pi_1(X_1) = \mathbb{Z}$.

---

3. Recall that the Riemann sphere $\hat{\mathbb{C}}$ is the oriented manifold obtained in the following way. As a topological space, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the one-point compactification of the complex plane. One coordinate chart for $\hat{\mathbb{C}}$ is:

$$\hat{\mathbb{C}} \setminus \{\infty\} \to \mathbb{C}; \qquad z \mapsto z.$$

A second chart is:

$$\hat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}; \qquad \infty \mapsto 0, \quad 0 \neq z \mapsto \tfrac{1}{z}.$$

- Let p be a monic polynomial of degree n, i.e.,

$$p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0.$$

  Prove that p has a unique smooth extention to a map

$$\hat{p}\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}.$$

- Let ω denote the 2-form defined on $\mathbb{C}$ by

$$\omega = \frac{1}{(1+z\overline{z})^2}\,\mathrm{d}z \wedge \mathrm{d}\overline{z}.$$

  Show that ω has a unique extension to a smooth 2-form on $\hat{\mathbb{C}}$ (where $z = x + iy$ and $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$).
- Compute the integrals

$$\int_{\hat{\mathbb{C}}} \omega \qquad \text{and} \qquad \int_{\hat{\mathbb{C}}} \hat{p}^*\omega.$$

  For the second one, you don't need any detailed calculations, so be sure to explain your answer.

SOLUTION:

[Solution by Mike Yarbrough, August 1998.] Since $\operatorname{LIM}_{z\to\infty} p(z) = \infty$, the only candidate for a continuous extension is to define $\hat{p}(\infty) = \infty$; I shall verify that this extension is also smooth. Composing $\hat{p}$ with maps to and from coordinate charts gives the map $\mathbb{C} \to \mathbb{C}$ given by

$$0 \mapsto \infty \overset{\hat{p}}{\longmapsto} \infty \mapsto 0,$$

$$0 \neq z \mapsto \tfrac{1}{z} \overset{\hat{p}}{\longmapsto} p(1/z) \mapsto 1/p(1/z).$$

But $1/p(1/z)$ is

$$\frac{1}{(\frac{1}{z})^n + a_{n-1}(\frac{1}{w})^{n-1} + \cdots + a_0} = \frac{w^n}{1 + a_{n-1}w + \cdots + a_0 w^n}.$$

This latter expression also takes $0 \mapsto 0$, hence it gives the induced map on coordinate charts. When z is close to 0, the denominator is close to 1 (hence can't vanish), so this rational function is smooth near $z = 0$; thus $\hat{p}$ is smooth near $z = \infty$.

Let $w = \frac{1}{z}$ be the coordinate in a neighborhood of $z = \infty$. I will write ω locally in a punctured neighborhood of $z = \infty$ in terms of w, and then see how to extend ω to $w = 0$. I have $\mathrm{d}z = -\frac{1}{w^2}\,\mathrm{d}w$. Similarly, I would guess $\mathrm{d}\overline{z} = -\frac{1}{\overline{w}^2}\,\mathrm{d}\overline{w}$; however, I'm going to check it carefully, because I need practice manipulating conjugate differentials. I know that d is really the Dolbeault [is that right?] operator $\mathrm{d}z\,\partial/\partial z + \mathrm{d}\overline{z}\,\partial/\partial\overline{z}$, where $\partial/\partial z = \partial/\partial x - i\partial/\partial y$ and $\partial/\partial\overline{z} = \partial/\partial x + i\partial/\partial y$. (The way to remember the signs is to demand $(\partial/\partial z)z = 1$ and $(\partial/\partial\overline{z})z = 0$.) Now $\overline{w} = \frac{1}{\overline{z}}$, thus

$$\begin{aligned}
\mathrm{d}\overline{w} &= \frac{\partial}{\partial z}\left(\frac{1}{\overline{z}}\right)\mathrm{d}z + \frac{\partial}{\partial\overline{z}}\left(\frac{1}{\overline{z}}\right)\mathrm{d}\overline{z} \\
&= 0\,\mathrm{d}z - \frac{1}{\overline{z}^2}\,\mathrm{d}\overline{z}.
\end{aligned}$$

At any rate, ω then becomes

$$\begin{aligned}
\omega &= \frac{1}{(1+\frac{1}{w\overline{w}})^2}\,(-\tfrac{1}{w^2})\,\mathrm{d}w \wedge (-\tfrac{1}{\overline{w}^2})\,\mathrm{d}\overline{w} \\
&= \frac{1}{w^2\overline{w}^2}\left(\frac{w\overline{w}}{1+w\overline{w}}\right)^2 \mathrm{d}w \wedge \mathrm{d}\overline{w} \\
&= \frac{1}{(1+w\overline{w})^2}\,\mathrm{d}w \wedge \mathrm{d}\overline{w}
\end{aligned}$$

This wasn't originally defined at $w = 0$, but we see that this expression only has one choice (namely itself) if it were to extend smoothly to $w = 0$, and indeed, it makes sense there.

[For the first integral, not really sure what to do.]

The map $\hat{p}\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has degree n (it wraps the sphere around itself n times in some complicated fashion). Hence $\int_{\hat{\mathbb{C}}} (\hat{p})^* \omega = n \int_{\hat{\mathbb{C}}} \omega$, so we get n times whatever we got for the previous integral.

4.

- Show that a 1-form η on S^1 is exact if and only if

$$\int_{S^1} \eta = 0.$$

- Show that a 2-form ω on S^2 is exact if and only if

$$\int_{S^2} \omega = 0.$$

HINT: For one direction, apply MAYER–VIETORIS to DE RHAM cohomology.

SOLUTION:

If η is exact, say $\eta = \mathrm{d}f$, then STOKES tells me $\int_{S^1} \eta = \int_{\partial S^1} f$, but $\partial S^1 = \varnothing$, whence $\int_{S^1} \eta = 0$. Now assume that η is such that $\int_{S^1} \eta = 0$. Define $f(P) = \int_1^P \eta$ for points $P \in S^1$, where $1 \in S^1$ is some fixed point. The difference of any two paths from 1 to P will be a certain number of revolutions around the circle, but by assumption these integrate to 0. Hence f is well-defined and the fundamental theorem of calculus applies (after pulling back from the circle onto the line that covers it, but the details are obvious) to give me $\mathrm{d}f = \eta$.

I almost proved above that the sequence

$$0 \to \text{EXACT 1-FORMS} \to \text{CLOSED 1-FORMS} \to \mathbb{R} \to 0$$

is exact; here the map to numbers takes a closed 1-form and integrates it over all of S^1. That is to say, as a consequence of what was shown above, I almost proved that $\mathrm{H}^1_{\mathrm{DR}}(\S^1) = \mathbb{R}$. For the sake of completeness, since I am almost there, I'll finish the proof that the stated sequence is exact. I already showed that the kernel of the integration map $\Omega^1(S^1) \to \mathbb{R}$ precisely comprises the exact 1-forms; but all top forms are closed, so in effect I proved something about an arrow in the sequence whose exactness is in question. Missing only is the surjectivity of the integration map; but this follows from the existence of forms, for example $\mathrm{d}\theta$, whose integral is nonzero.

Now I would like to establish similar results for S^2. My approach will be different from before. I will first establish $\mathrm{H}^2_{\mathrm{DR}}(S^2) = \mathbb{R}$, then use this to show that any 2-form that integrates to 0 must be exact. Since S^2 is compact oriented, applying POINCARÉ duality to $\mathrm{H}^0_{\mathrm{DR}}(S^2) = \mathbb{R}$ immediately gives me $\mathrm{H}^2_{\mathrm{DR}}(S^2) = \mathbb{R}$. But I'll use MAYER–VIETORIS, as suggested in the hint. Let U denote the sphere minus the south pole, V the sphere minus the north pole. Then $U \cap V$ has S^1 as a deformation retract. Here is an extract of the MAYER–VIETORIS long exact sequence in cohomology:

$$\mathrm{H}^1_{\mathrm{DR}}(U) \oplus \mathrm{H}^1_{\mathrm{DR}}(V) \to \mathrm{H}^1_{\mathrm{DR}}(S^1) \to \mathrm{H}^2_{\mathrm{DR}}(S^2) \to \mathrm{H}^2_{\mathrm{DR}}(U) \oplus \mathrm{H}^2_{\mathrm{DR}}(V)$$

Since U and V are simply connected, their DE RHAM cohomology vanishes above dimension 0, and thus I have an isomorphism $\mathrm{H}^2_{\mathrm{DR}}(S^2) \cong \mathrm{H}^1_{\mathrm{DR}}(S^1)$. I already showed $\mathrm{H}^1_{\mathrm{DR}}(S^1) = \mathbb{R}$.

Since all 2-forms on S^2 are closed, I just established the existence of an exact sequence

$$0 \to \text{EXACT 2-FORMS} \to \text{2-FORMS} \to \mathbb{R} \to 0,$$

but I did *not* show that the map to numbers is given by integration, and it is not clear that the sequence remains exact when I insist that the map $\Omega^2(S^2) \to \mathbb{R}$ be the integration map. But we *are* talking about real vector spaces here, so the exactness above shows that the exact 2-forms have codimension 1 inside $\Omega^2(S^2)$. There exist forms with nonzero integral (for example, take an area form); thus the integration map $\Omega^2(S^2) \to \mathbb{R}$ is surjective, and thus its kernel, whatever it may be, has codimension 1 inside $\Omega^2(S^2)$. But STOKES shows that any exact 2-form integrates to 0. Hence the space of exact 2-forms lies inside the kernel of integration; but both spaces have the same dimension, hence must be equal. The problem is solved: a 2-form integrates to 0 if and only if it is exact.

By the way, instead of my vector space dimension argument, a different (and better!) approach would be to trace the behavior of the boundary map $\mathrm{H}^1_{\mathrm{DR}}(S^1) \to \mathrm{H}^2_{\mathrm{DR}}(S^2)$, and compose its inverse with the integration map $\mathrm{H}^1_{\mathrm{DR}}(S^1) \xrightarrow{\sim} \mathbb{R}$. I'll do that now. Starting with $[\eta] \in \mathrm{H}^1_{\mathrm{DR}}(S^1)$, where $\eta \in \Omega^1(S^1)$, we know that $\eta = \omega_U - \omega_V$, and finally produce $\eta' \in \Omega^2(S^2)$ so that η' restricted to U and V gives $\mathrm{d}\omega_U$ and $\mathrm{d}\omega_V$, respectively. The boundary map sends $[\eta] \mapsto [\eta']$, and we happen to know it to be an isomorphism in the present situation. The inverse map is much easier: Start with $[\eta'] \in \mathrm{H}^2_{\mathrm{DR}}(S^2)$ so that $\eta' \in \Omega^2(S^2)$, then restricting η' to U and V gives the exact forms $\mathrm{d}\omega_U$ and $\mathrm{d}\omega_V$, and finally set $\eta = \omega_U - \omega_V$, so that the inverse map sends $[\eta'] \mapsto [\eta]$. The combined isomorphism $\mathrm{H}^2_{\mathrm{DR}}(S^2) \xrightarrow{\sim} \mathbb{R}$ takes $[\eta'] \mapsto [\eta] \mapsto \int_{S^1} \eta$. For the following, it'll be simpler if U is the closed upper half sphere, V the closed lower half sphere, so that $U \cap V$ is precisely the equator S^1. Everything done so far goes through just fine. Then $\int_{S^1} \eta = \int_{S^1} \omega_U - \int_{S^1} \omega_V$. Then STOKES tells me $\int_{S^1} \omega_U = \int_U \mathrm{d}\omega_U = \int_U \eta'$, and since the orientation on V, induced from the orientation on S^2, induces an orientation on S^1 opposite to that induced from U, I have $-\int_{S^1} \omega_V = \int_V \mathrm{d}\omega_V = \int_V \eta'$. And thus $\int_{S^1} \eta = \int_U \eta' + \int_V \eta' = \int_{S^2} \eta'$. Hence the composed map $\mathrm{H}^2_{\mathrm{DR}}(S^2) \to \mathbb{R}$ is indeed integration, as was to be shown. Phew!

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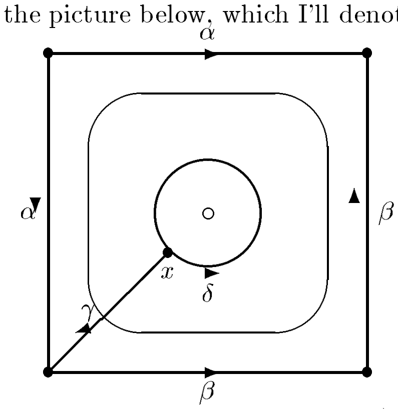
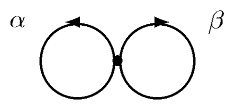
5. Find the fundamental group and the integral homology for the KLEIN bottle.

SOLUTION:

[Solution based on that of Mike Yarbrough, August 1998.] As a (covering) model of the KLEIN bottle, I'll use a square with sides identified as shown in the picture below, which I'll denote X.



Let D denote centered disk (shown as a rounded rectangle in the diagram), and fix a base point $x \in D$. Let E denote the KLEIN bottle with the origin 0 excised. Then $D \cap E$ has a circle as a deformation retract. Now $\pi_1(D, x)$ is trivial, while $\pi_1(D \cap E, x)$ is infinite cyclic, generated by the loop δ that runs once around the origin. Finally, E has the four sides of the square as a deformation retract, and these are identified to give the figure eight loop:



Hence we see that $\pi_1(E)$ would be freely generated by α and β, but in terms of our original basepoint x we have $\pi_1(E, x)$ being freely generated by $\omega = \gamma\alpha\gamma^{-1}$ and $\eta = \gamma\beta\gamma^{-1}$. Now SEIFERT & VAN KAMPEN tells me that $\pi_1(X, x)$ is the freeest possible subgroup of $\pi_1(D, x) * \pi_1(E, x)$ that fits into the following commutative diagram:

$$\begin{tikzcd} & \pi_1(D,x) \arrow[dr] & \\ \pi_1(D\cap E,x) \arrow[ur] \arrow[rr] \arrow[dr] & & \pi_1(X,x) \\ & \pi_1(E,x) \arrow[ur] & \end{tikzcd}$$

Following the top route of the diagram, the loop δ goes to the trivial loop class; following the bottom route, the loop δ is homotopic to $\gamma\beta\beta\alpha\alpha\gamma^{-1}$, and thus goes to $\eta\eta\omega\omega$ in terms of the generators of $\pi_1(E, x)$. Since either route must produce the same result in $\pi_1(X, x)$, I must force the relation $\eta\eta\omega\omega = 1$, and thus

$$\pi_1(X, x) = \frac{\pi_1(D, x) * \pi_1(E, x)}{\langle \eta\eta\omega\omega = 1\rangle},$$

so that $\pi_1(X, x) = \langle \eta, \omega : \eta^2\omega^2 = 1\rangle$.

The next task is to find the singular homology with integral coefficients. Using the same decomposition as above, most of the reduced homology vanishes; the interesting part of the MAYER–VIETORIS sequence is:

$$0 \to H_2(X) \to \underbrace{H_1(D \cap E)}_{\mathbb{Z}} \to \underbrace{H_1(D)}_{0} \oplus \underbrace{H_1(E)}_{\mathbb{Z}\oplus\mathbb{Z}} \to H_1(X) \to 0$$

The homomorphism $H_1(D \cap E) \to H_1(E)$ is completely determined by the image of a generator in $H^1(D \cap E)$; since there is no torsion in $H_1(E)$, the map is either the zero map or it is injective. In fact, we saw in the first part of this problem that the map sends the generator δ of $H^1(D \cap E)$ to twice the sum of the generators of $H_1(E)$. (Well, we actually had the product of the squares of the generators, but now we are thinking additively.) Thus the map is injective, and we conclude $H_2(X) = 0$. To get $H_1(X)$, I could simply abstractly abelianize $\pi_1(X)$, but I'm familiar enough with the maps at this point to figure out $H_1(X)$ directly. From the exact sequence, I have $H_1(X) \cong (\langle\alpha\rangle \oplus \langle\beta\rangle)/\langle 2(\alpha, \beta)\rangle$. This, of course, agrees with the result of abelianizing $\pi_1(X)$. One might ask, what is the structure of this finitely generated abelian group? I basically need to apply TIETZE transformations to clean up my presentation; but I only need to make one such transformation, so I'll just describe it in words. Since $(0, \beta)$ can be expressed in terms of $(\alpha, 0)$ and $\gamma = (\alpha, \beta)$, the group is generated by α and γ, subject only to the relation $2\gamma = 0$; thus the structure is $\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})$. The homology of the KLEIN bottle is:

$$H_k(X; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0; \\ \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}, & \text{if } k = 1; \\ 0, & \text{if } k \geq 2. \end{cases}$$

Since the KLEIN bottle is $\mathbb{RP}^2 \# \mathbb{RP}^2$, it is reassuring to observe that my calculation above agrees with a general fact that I have memorized, namely:

$$H_k(\underbrace{\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2}_{n \text{ copies}}; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } k = 0; \\ \frac{\mathbb{Z}}{2\mathbb{Z}} \oplus \underbrace{\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{n-1 \text{ copies}}, & \text{if } k = 1; \\ 0, & \text{if } k \geq 2. \end{cases}$$

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6. Give examples of the following, or prove that none exists:

- a compact 2-manifold X with $H_2(X; \mathbb{Z}) = 0$;
- a 1-form η on $\mathbb{R}^2 \setminus \{0\}$ such that
$$\int_{S^1} \eta \neq 0;$$
- a non-normal covering space of the figure eight;
- a space that has the direct product $F_2 \times (\mathbb{Z}/2\mathbb{Z})$ as fundamental group, where F_2 denotes the free group on two generators.

SOLUTION:

- Any compact non-orientable 2-manifold, say $\mathbb{RP}^2$, serves as an example of a compact 2-manifold whose homology vanishes in dimension 2.
- The angle form $\mathrm{d}\theta \in \Omega^1(S^1)$ pulled back to $\mathbb{R}^2 \setminus \{0\}$ is the 1-form
$$\eta = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2},$$
which integrates to 2π on S^1.
-
- Since the fundamental group of a product space is the product of the fundamental groups, I simply form the product of the figure eight with a real projective plane; then the fundamental group is as desired.

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN FALL 1995 (ERCOLANI AND FRIEDLANDER)

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1. Find the images of the following curves under the transformation $\zeta = 1/z$:

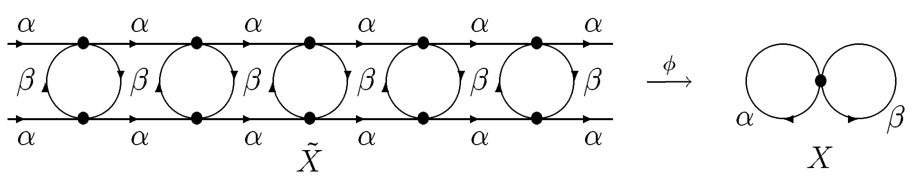
- The family of circles $x^2 + y^2 = ax$;
- The family of parallel lines $y = x + b$.

2. Let X denote a connected topological space that has a universal covering space.

- Describe the basic correspondence between the following three things:

  — subgroups of $\pi_1(X, x_0)$;
  — covering spaces $\tilde{X}$ of X;
  — the group of covering space transformations (Deckbewegungen) of $\tilde{X}$.

  In terms of this correspondence, to what does normality of a subgroup correspond? (You do not need to supply proofs, just statements of the basic facts.)
- Consider the particular example of the covering space $\phi: \tilde{X} \to X$ pictured below, where X is the figure eight, and the basepoint is the vertex of the figure eight. The fundamental group of X can be identified with the free group on the two generators α and β, as shown in the picture. Illustrate your answer to the first part of this problem using the covering space ϕ. In particular, determine the corresponding subgroup of $\pi_1(X, x_0)$, and say whether it is normal.



3. Let $f: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (3x^2 + y, 5xy)$, and let $\omega = \mathrm{d}x \wedge \mathrm{d}y$. Evaluate $f^*\omega(V, \cdot)$, where $V = ye_1 + xe_2$ is a vector field on $\mathbb{R}^2$. (The result should be a 1-form on $\mathbb{R}^2$.)

4. Let V be the 2-dimensional subspace of $\mathbb{R}^3$ defined by $x + y + z = 0$. For what values of $a \in \mathbb{R}$ is the intersection of V with the surface

$$S_a = \{ (x, y, z) \in \mathbb{R}^3 : z = x^2 + y^2 - a \}$$

a 1-dimensional submanifold of V?

5. Suppose that M is a compact, connected, orientable n-dimensional manifold (without boundary), and that η is an $(n-1)$-form on M. Show that $\mathrm{d}\eta$ must be zero at some point on M.

SOLUTION:

If $\mathrm{d}\eta$ were nowhere zero, then it would be (by definition) a volume form, and volume forms have the property that they integrate to a nonzero value. But Stokes tells us that the exact form $\mathrm{d}\eta$ would integrate to zero.

6. For each of the following, either give an example or show that none exists.

- A bounded, connected open subset U of $\mathbb{R}^n$ such that $\mathbb{R}^n - U$ is connected but U is not contractible to a point.
- Two simply connected open subsets U and V of $\mathbb{R}^n$ such that $U \cap V$ is connected but $U \cup V$ is not simply connected.
- A simply connected non-orientable surface.

7. Calculate π_1 and H_1 of the 2-dimensional sphere with four points removed: $S^2 - \{ p_1, p_2, p_3, p_4 \}$. Give an example of a nontrivial class in π_1 that is trivial in H_1. Calculate π_1 and H_1 of the projective plane with two points removed: $\mathbb{RP}^2 - \{ q_1, q_2 \}$.

8. Consider the following 2-form on $\mathbb{R}^3$:

$$\sigma = x\,\mathrm{d}y \wedge \mathrm{d}z - y\,\mathrm{d}x \wedge \mathrm{d}z + z\,\mathrm{d}x \wedge \mathrm{d}y.$$

Let σ' denote the restriction of σ to S^2, i.e., $\sigma' = i^*\sigma$, where $i\colon S^2 \hookrightarrow \mathbb{R}^3$ is the standard inclusion. Observe that σ' is the area form on S^2.

For all $(x, y, z) \in \mathbb{R}^3$ except those with $x = 0, y = 0$, and $z \in (-\infty, 0]$, define $\phi(x, y, z)$ to be the angle between the positive z-axis and the ray from 0 through (x, y, z).

[picture of xyz axes, point (x,y) in xy-plane, point (x,y,z) above it, and ϕ marking the angle.]

With appropriate conventions, $\phi(x, y, z) = \mathrm{ARCTAN}\bigl(\sqrt{x^2 + y^2}/z\bigr)$. If $\nu(p) = |p|$ and θ is considered as a function on $\mathbb{R}^3$, $\theta = \mathrm{ARCTAN}(y/x)$, then (ν, θ, ϕ) is a coordinate system on all points $p = (x, y, z) \in \mathbb{R}^3$ except for those with $y = 0$, $x \in [0, \infty)$, or $x = 0$, $y = 0$, $z \in (-\infty, 0]$.

The (polar) coordinate chart is given by

$$\begin{aligned} x &= \nu \,\mathrm{SIN}\,\phi\,\mathrm{COS}\,\theta \\ y &= \nu\,\mathrm{SIN}\,\phi\,\mathrm{SIN}\,\theta \\ z &= \nu\,\mathrm{COS}\,\phi \end{aligned}$$

A "longitude" on the sphere is a curve defined by $\theta =$ constant. A "meridian" on the sphere is a curve defined by $\phi =$ constant.

- Show that if v is a longitudinal unit tangent vector on the sphere $S^2(r)$ of radius r, then $\mathrm{d}\phi(v) = 1/r$, and that if w is a unit tangent vector that points along a meridian through $p = (x, y, z) \in S^2(r)$, then $\mathrm{d}\phi(w_p) = \frac{1}{\sqrt{x^2+y^2}}$.

  [picture of xyz axes, upper half of sphere, a meridian and longitude, and tangent vectors v and w emanating from point where meridian and longitude cross.]
- Show that if θ and ϕ are taken to mean their restrictions to (certain regions of) S^2, then $\sigma' = h\,\mathrm{d}\theta \wedge \mathrm{d}\phi$, where $h\colon S^2 \to \mathbb{R}$ is given by $h(x, y, z) = -\sqrt{x^2 + y^2}$. (The minus sign comes from the orientation.)
- Show that $\sigma' = \mathrm{d}(-\mathrm{COS}\,\phi\,\mathrm{d}\theta)$.
- Calculate the area of the spherical strip between the equator and the meridian at $\phi = \pi/3$.
- Show that if C is a closed singular 1-chain on S^2 that bounds an open region Ω, then

$$\mathrm{AREA}(\Omega) = \left| \int_C \mathrm{COS}\,\phi\,\mathrm{d}\theta \right|.$$

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN JANUARY 1995 (FRIEDLANDER AND ERCOLANI)

1. Find a conformal transformation that maps the strip $\{\, z : 0 < \mathrm{IMAGINARYPART}(z) < 1 \,\}$ onto the wedge $\{\, z : 0 < \mathrm{ARGUMENT}(z) < \frac{\pi}{3} \,\}$.

SOLUTION:

The map $z \mapsto e^{(\pi/3)z} = e^{(\pi/3)x} e^{i(\pi/3)y}$ does the right thing. After all, writing $z = x + iy$, as x ranges over $-\infty < x < \infty$, the length $e^{(\pi/3)x}$ ranges over $0 < e^{(\pi/3)x} < \infty$; as y ranges over $0 < y < 1$, the angle $(\pi/3)y$ ranges over $0 < (\pi/3)y < 1$.

2. Let $\mathbb{Z}$ be the additive group of integers. Define an action of $\mathbb{Z}$ on the positive reals $\mathbb{R}_{>0}$ as follows:

$$\rho(n)x := 2^n x \qquad \text{for } n \in \mathbb{Z} \text{ and } x \in \mathbb{R}_{>0}.$$

- An action ρ of $\mathbb{Z}$ is said to be discontinuous if each point x has a neighborhood U such that $\rho(n)U$ and $\rho(m)U$ are disjoint whenever $n \neq m$. Is the action above discontinuous?
- Declare two points x and y equivalent in $\mathbb{R}_{>0}$ if there is an integer n such that $x = 2^n y$. Let E denote this equivalence relation. Is the set $M := \mathbb{R}_{>0}/E$ of equivalence classes a Hausdorff space in the quotient topology? Is M homeomorphic to the half open interval $(\frac{1}{2}, 1]$ in the relative topology it acquires as a subset of $\mathbb{R}$?

SOLUTION:

[REREAD SOLUTION]

(a) I claim that the action is discontinuous. Given any $x \in \mathbb{R}^+$, I claim that the neighborhood $(x-\varepsilon, x+\varepsilon)$ with $\varepsilon = x/3$ has disjoint images under the action. One easily sees that the action is monotone (that is, $\rho(n)x < \rho(n+1)x$) and thus it is sufficient to verify that the images of $(x-\varepsilon, x+\varepsilon)$ under $\rho(n)$ and $\rho(n+1)$ are disjoint; furthermore, this is accomplished by verifying that $\rho(n)(x+\varepsilon) \leq \rho(n+1)(x-\varepsilon)$; that is, I must verify that the right endpoint of one image lands to the left of the left endpoint of the other image. But this is true since $2^n(x+\varepsilon) \leq 2^{n+1}(x-\varepsilon)$ is equivalent to $x+\varepsilon \leq 2x - 2\varepsilon$ and that is equivalent to $\varepsilon \leq x/3$.

(b) **[Learn more about the quotient topology.]**

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3. Let X be a path connected space. Show that the following are equivalent:

- X is simply connected.
- Every map of the unit circle S^1 into X extends to a continuous map of the closed unit disk B^2 into X. HINT: Represent B^2 as a quotient of $I \times I$ by sending $(s,t) \in I \times I$ to $te^{2\pi i s}$.

SOLUTION:

Let X be simply connected, and let $f: S^1 \hookrightarrow X$ be an injection of the unit circle; clearly I may instead think of f as a map $f: I \to X$ with $f(x) = f(y)$ precisely when $\{x,y\} = \{0,1\}$. By definition of being simply connected, each loop in X may be deform retracted to a point; that is to say, there exists a continuous map $g: I \times I \to X$ so that we always have $g(t,0) = g(t,1)$, also $g(1,x) = f(x)$, and finally, $g(0,x) = \text{constant}$. (I'm letting time run backwards, with the original loop at time 1 and the final point at time 0.) These identifications are precisely such so that g is in effect defined on B^2, or more precisely, that g factors through the map $I \times I \to B^2$. In general, when factoring a continuous map through another continuous map, there is no guarantee that the remaining map will be continuous. [For example, the identity map $[0,1) \to [0,1)$ factors through S^1, but not continuously at both stages.] However, in the present situation, it is clear that we are merely identifying points on which g assumes the same value, hence the remaining map (really another copy of g) is continuous.

Let X have the property that each continuous map $S^1 \hookrightarrow X$ extends to a map $B^2 \to X$. I claim that X is simply connected. Indeed, let $f: I \to X$ be a simple closed loop; then f induces a continuous map $S^1 \hookrightarrow X$, hence there is an extension $g: B^2 \to X$. But our identification map $I \times I \to B^2$ is injective everywhere except three edges, so we me lift g to a map $I \times I \to B^2$, realizing that we will necessarily have g satisfying the properties of being a homotopy from f to a point.

[Old solution of part (b), perhaps more clearly stated:] Let $f: S^1 \to X$ be a closed path. By (2), there is an extension, namely a map $\hat{f}: B^2 \to X$ with $\hat{f}|S^1 = f$. Define the continuous map $g: I \times S^1 \to B^2$ via $g: (t,s) \mapsto ts$. I claim that $f \circ g: I \times S^1 \to X$ is a homotopy between the closed path f and the constant map $S^1 \to X$ given by $s \mapsto \hat{f}(0)$. After all, $(f \circ g)(1,s) = (\hat{f}|S^1)(s) = f(s)$, and $(f \circ g)(0,s) = \hat{f}(0)$. Since any closed path f is homotopic to a constant map, the space X is simply connected.

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4. Consider the set

$$M = \left\{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \quad \text{and} \quad x^2 + 2y^2 - z^2 = 0 \right\}.$$

- Is M compact? Is M connected? Is M a manifold? Give reasons for your answers.
- Let M_0 be a connected component of M and let $i: M_0 \hookrightarrow \mathbb{R}^3$ be the inclusion map. Consider the differential 1-form $i^*\mathrm{d}y$ on M_0. Choose an open cover of M_0 by coordinate charts and give a formula for $i^*\mathrm{d}y$ in each chart.

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5. A map $f: M \to N$ between two manifolds M and N is said to be a submersion if at each point $m \in M$ the derivative

$$\mathrm{d}f_m: T_m M \to T_{f(m)} N$$

is surjective.

- Let M and N be surfaces (that is, $\mathrm{DIM}\, M = \mathrm{DIM}\, N = 2$), and let $f: M \to N$ be a submersion. Show that for each point $m \in M$ there exist local coordinates about m and local coordinates about $f(m)$ so that in these local coordinates $f(x_1, x_2) = (x_1, x_2)$.
- Show that if M is compact and N is connected, then any submersion $f: M \to N$ is necessarily surjective.
- Show that there does not exist a submersion of a compact surface into $\mathbb{R}^2$.

6.

- Suppose that M and N are smooth, orientable manifolds, and $M = \partial X$, where X is a compact orientable manifold with boundary. Suppose $f: M \to N$ is a smooth map. Let k denote the dimension of M and suppose that ω is a closed k-form on N. Prove that if f can be smoothly extended to all of X, then
$$\int_M f^*\omega = 0.$$
- Suppose that $f, g: M \to N$ are smooth maps that are smoothly homotopic via
$$F: M \times I \to N.$$
Suppose that M is a compact orientable manifold of dimension k. Use the first part of the problem to show that for each closed k-form ω on N,
$$\int_M f^*\omega = \int_M g^*\omega.$$
HINT: $\partial(M \times I)$ consists of two copies of M with opposite orientations.

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7. Let ω be a closed 1-form on a 2-dimensional torus

$$T^2 = S^1 \times S^1.$$

Let ϕ and ψ denote angular coordinates on T^2 (which are only defined up to the addition of $2\pi k$ for $k \in \mathbb{Z}$).

- Show that for the appropriate choice of constants a and b, the 1-form ω differs from $a\,\mathrm{d}\phi + b\,\mathrm{d}\psi$ by an exact 1-form.
- Let
$$\eta_1 = a_1\,\mathrm{d}\phi + b_1\,\mathrm{d}\psi \qquad \text{and} \qquad \eta_2 = a_2\,\mathrm{d}\phi + b_2\,\mathrm{d}\psi.$$
Show that $\eta_1 - \eta_2$ is exact if and only if $a_1 = a_2$ and $b_1 = b_2$.
- Use the previous results to determine the DE RHAM cohomology of T^2.

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN AUGUST 1994 (FRIEDLANDER AND PALMER)

1. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^4}\,\mathrm{d}x.$$

2. Find a conformal transformation from the first quadrant $\{x>0, y>0\}$ of the complex plane $z = x+iy$ onto the strip $0<y<1$.

SOLUTION:

Going in the other direction, the map $z \mapsto e^{(\pi/2)z}$ clearly works. Hence I seek the appropriate branch of the inverse map, namely $z \mapsto (2/\pi)\,\textsc{log}\, z$ with $0 < \textsc{argument}(z) < 2\pi$.

3. Characterize exact 2-forms on a 2-dimensional sphere. Explain your answer.

4. Let T^5 be a 5-dimensional torus. What is the dimension of $\mathrm{H}^3_{\mathrm{DR}}(T^5)$?

5. Let M be a simply connected manifold. Prove that $\mathrm{H}^1_{\mathrm{DR}}(M) = 0$.

6. Can one manufacture a map of Arizona in such a way that the distance between any two points on the map is exactly 100 000 times smaller than the distance between corresponding points on the ground? Give a proof for your answer.

7. Give an example of a surface of constant Gaussian curvature -1.

8. A manifold M is obtained by making a circular hole in a 2-dimensional torus, and then by attaching a Möbius strip to it. The boundary of the Möbius strip (that is, a circle) is attached to the boundary of the circular hole.

- Compute the de Rham cohomology $\mathrm{H}^1_{\mathrm{DR}}(M)$.
- Compute simplicial (= cell) homology $\mathrm{H}_1(M,\mathbb{Z})$ with integer coefficients.
- Compute simplicial (= cell) cohomology $\mathrm{H}^1(M,\mathbb{Z})$.

9. Find all connected covering spaces for $\mathrm{SO}(3)$, the space of all orthogonal 3×3 matrices of determinant 1.

SOLUTION:

[Note: $\mathrm{SO}(3) = \mathbb{RP}^3$]

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN JANUARY 1994 (PALMER AND FRIEDLANDER)

1. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+1}\,\mathrm{d}x.$$

2. Find a conformal transformation from the first quadrant $\{\, x > 0, y > 0 \,\}$ of the complex plane $z = x + iy$ onto the disk $\{\, z : |z-1| < 1 \,\}$.

SOLUTION:

I know that Möbius transformations send circles on $\mathbb{CP}^1$ to circles on $\mathbb{CP}^1$. (At the level of the plane $\mathbb{C}$, lines go to lines or circles, and circles go to lines or circles.) Since boundaries go to boundaries, I would first like to transform the original region so that its boundary is a straight line. The map $z \mapsto z^2$, although not injective on all of $\mathbb{C}$, conformally maps the first quadrant onto the upper half space. Then I rotate upper half space to the right half space via $z \mapsto -iz$, and shift it over via $z \mapsto z + \frac{1}{2}$. Then the map $z \mapsto 1/z$ sends the vertical line $x = \frac{1}{2}$ to the circle of diameter 2 centered at $z = 1$, sending the right half space to the circle's interior. Finally, I shift the circle via $z \mapsto z - 1$. Putting it all together, the desired conformation transformation is:

$$z \mapsto \frac{1}{\frac{1}{2} - iz^2} - 1 = \frac{\frac{1}{2} + iz^2}{\frac{1}{2} - iz^2}.$$

3. Suppose the $C = [a, b]$ is a closed interval on the real line $\mathbb{R}$. Suppose further that $f\colon C \to \mathbb{R}$ is a continuous injective map. Show that f is necessarily monotone and that $f^{-1}\colon f(C) \to C$ is continuous.

4. Suppose that X is a Hausdorff space, that $p \in X$ is a point, and that $K \subset X$ is a *compact* subset that does *not* contain p. Show that there exist *disjoint* open sets U and V so that $p \in U$ and $K \subset V$.

SOLUTION:

Since X is Hausdorff, associated to each point $q \in K$ is a pair of disjoint open sets U_q, V_q that separate p from q: $p \in U_q$ and $q \in V_q$. The collection of all V_q clearly cover K, hence some finite subcollection does the same; let V denote their union. Let U denote the intersection of the corresponding finitely many sets U_q. Then U and V are open, and $p \in U$ and $K \subset V$. Finally, I claim $U \cap V = \varnothing$, for any point in V lies in some V_q, hence does not lie in the corresponding U_q, hence does not lie in the intersection U.

5. Consider the usual exponential function $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$, defined by $\exp(z) = e^z$.

- Show that $\exp$ is a covering space map, and that the local homeomorphisms that appear in the definition of a covering space are actually analytic with analytic inverses in this case.
- Use the above to show that if f is an entire function *without zeros*, then there exists an entire function g so that

$$f(z) = \exp(g(z)).$$

NOTE: An entire function is simply an analytic function defined on $\mathbb{C}$.

6. Let $E_r = \mathbb{R}^2/\Lambda(\tau)$, where $\Lambda(\tau) = \{\,(m, 0) + nr : m, n \in \mathbb{Z}\,\}$ and $\tau = (\tau_1, \tau_2)$ with $\tau_2 \neq 0$. The set $\Lambda(\tau)$ is a *lattice* in $\mathbb{R}^2$, and the quotient E_r is a torus.

- Show that the map $\pi_\tau: \mathbb{R}^2 \to E_\tau$ given by $\pi_\tau(p) = p + \Lambda(\tau)$ is a covering space map.
- Suppose that τ and τ' are points in $\mathbb{R}^2$ with $\tau_2 \neq 0$ and $\tau_2' \neq 0$. If f is a continuous map $f: E_\tau \to E_{\tau'}$, does there exist a continuous map $\tilde{f}: \mathbb{R}^2 \to \mathbb{R}^2$ that covers the map f? That is, can one find $\tilde{f}$ so that $f \circ \pi_\tau = \pi_{\tau'} \circ \tilde{f}$? Give a proof or a counter-example.

7. The orthogonal group $\mathbf{O}(2)$ consists of all 2×2 real matrices of the form

$$\begin{bmatrix} u_1 & -\lambda u_2 \\ u_2 & \lambda u_1 \end{bmatrix},$$

for which $u_1^2 + u_2^2 = 1$ and $\lambda = \pm 1$.

- Find all the homology groups $H_q(O(2))$, for $q = 0, 1, 2, \ldots$
- Let $\mathrm{GL}(2, \mathbb{R})$ denote the group of 2×2 real invertible matrices and let $\mathbf{U}_+$ denote the upper triangular matrices with positive entries on the diagonal. The Gram–Schmidt [SPELLING?] process shows that there exist *continuous* maps

$$\mathcal{O}: \mathrm{GL}(2, \mathbb{R}) \to \mathbf{O}(2),$$
$$\mathcal{U}: \mathrm{GL}(2, \mathbb{R}) \to \mathbf{U}_+,$$

so that for any $g \in \mathrm{GL}(2, \mathbb{R})$ one has

$$g = \mathcal{O}(g)\mathcal{U}(g),$$

and if $g \in \mathbf{O}(2)$, then $g = \mathcal{O}(g)$. Use this to show that the homotopy type of $\mathrm{GL}(2, \mathbb{R})$ is the same as that of $\mathbf{O}(2)$. Use this to calculate $H_q(\mathrm{GL}(2, \mathbb{R}))$ for $q = 0, 1, 2, \ldots$

8. Let $p_0 = (0, 0)$ and $p_1 = (2, 0)$. Let S^1 denote the circle of radius 1 about p_0, and let $i: S^1 \to \mathbb{R}^2 \setminus \{p_0, p_1\}$ denote the inclusion map. Find an example of a closed 1-form ω defined on $\mathbb{R}^2 \setminus \{p_0, p_1\}$ that is not exact on $\mathbb{R}^2 \setminus \{p_0, p_1\}$, but such that $i^*\omega$ is exact on S^1.

9. Give an example of a vector field, on the 3-dimensional sphere S^3, that never vanishes. HINT: S^3 can be identified with the group $\mathrm{SU}(2)$ of 2×2 unitary matrices of determinant 1. How does multiplication by a group element act on the tangent space $T_e\mathrm{SU}(2)$ to $\mathrm{SU}(2)$ at the identity?

Prove that a smooth nowhere vanishing vector field on S^2 does not exist. HINT: Show that the existence of such a vector field implies that the identity on S^2 is homotopic to the antipodal map. What is the degree of the antipodal map on S^2?

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN AUGUST 1993 (FRIEDLANDER AND PALMER)

1. Suppose that $f(z)$ is analytic on an open set $U \subset \mathbb{C}$.

(a) Write $\mathrm{d}z = \mathrm{d}x + i\,\mathrm{d}y$. Show that $f(z)\,\mathrm{d}z$ is a closed one-form on U.

(b) Suppose that U is the annulus $1 < |z| < 2$. Find an example of such a one-form $f(z)\,\mathrm{d}z$ that is *not exact* on U.

SOLUTION:

(a) Let $f(z) = u(x,y) + iv(x,y)$, where $z = x + iy$. Then

$$\begin{aligned} f(z)\,\mathrm{d}z &= \big(u(x,y) + iv(x,y)\big) \cdot \big(\mathrm{d}x + i\,\mathrm{d}y\big) \\ &= \big(u(x,y)\,\mathrm{d}x - v(x,y)\,\mathrm{d}y\big) + i\big(v(x,y)\,\mathrm{d}x + u(x,y)\,\mathrm{d}y\big). \end{aligned}$$

The differential operator is linear. By definition, $\mathrm{d}\big(f(z)\,\mathrm{d}z\big) = \mathrm{d}f(z) \wedge \mathrm{d}z$; thus

$$\mathrm{d}\big(f(z)\,\mathrm{d}z\big) = \big(\mathrm{d}u(x,y) \wedge \mathrm{d}x - \mathrm{d}v(x,y) \wedge \mathrm{d}y\big) + i\big(\mathrm{d}v(x,y) \wedge \mathrm{d}x + \mathrm{d}u(x,y) \wedge \mathrm{d}y\big).$$

Now $\mathrm{d}\big(u(x,y)\big) = \mathrm{D}_1\,u(x,y)\,\mathrm{d}x + \mathrm{D}_2\,u(x,y)\,\mathrm{d}y$; and, by definition, $\mathrm{d}\big(u(x,y)\,\mathrm{d}x\big) = \mathrm{d}\big(u(x,y)\big) \wedge \mathrm{d}x$. Since $\mathrm{d}x \wedge \mathrm{d}x = 0$, we have $\mathrm{d}u(x,y) \wedge \mathrm{d}x = \mathrm{D}_2\,u(x,y)\,\mathrm{d}y \wedge \mathrm{d}x = -\,\mathrm{D}_2\,u(x,y)\,\mathrm{d}x \wedge \mathrm{d}y$. Following similar procedures for the remaining terms gives

$$\begin{aligned} \mathrm{d}\big(f(z)\,\mathrm{d}z\big) &= \big(-\,\mathrm{D}_2\,u(x,y)\,\mathrm{d}x \wedge \mathrm{d}y - \mathrm{D}_1\,v(x,y)\,\mathrm{d}x \wedge \mathrm{d}y\big) + i\big(-\,\mathrm{D}_2\,v(x,y)\,\mathrm{d}x \wedge \mathrm{d}y + \mathrm{D}_1\,u(x,y)\,\mathrm{d}x \wedge \mathrm{d}y\big) \\ &= -\big(D_2 u(x,y) + \mathrm{D}_1\,v(x,y)\big)\,\mathrm{d}x \wedge \mathrm{d}y + i\big(\mathrm{D}_1\,u(x,y) - \mathrm{D}_2\,v(x,y)\big)\,\mathrm{d}x \wedge \mathrm{d}y. \end{aligned}$$

Since $f(z)$ is analytic on U, the CAUCHY–RIEMANN equations hold on U; they say $\mathrm{D}_1\,u(x,y) = \mathrm{D}_2\,v(x,y)$ and $\mathrm{D}_2\,u(x,y) = -\,\mathrm{D}_1\,v(x,y)$. Thus in fact $\mathrm{d}\big(f(z)\,\mathrm{d}z\big) = 0$ on U; that is, the one-form $f(z)\,\mathrm{d}z$ is closed on U.

(b) The "calculus-on-manifolds" integral of a 1-form along a 1-chain agrees in this context with the complex path integral. If a form is exact, then STOKES tells us that it integrates to zero on any closed path. But $(1/z)\,\mathrm{d}z$ integrates to $2\pi i$ around the unit circle, hence cannot be exact.

[One can also see the equivalence between being exact in the complex sense and being exact in the smooth real sense by using the Dolbealt operator. I should write more on this.]

2. Let $p_1 \colon \mathbb{R}^2 \to \mathbb{R}$ denote the projection $p_1(x,y) = x$.

(a) The mapping p_1 is continuous. Is p_1 an open mapping?

(b) Find a closed set F so that $p_1(F)$ is open.

(c) Can one find a closed and bounded set F so that $p_1(F)$ is open?

SOLUTION:

(a) The mapping p_1 is open; to show this, I must show that any open set gets mapped to an open set. Let U be the image of any open set V. If U is empty, then it is open; else, let $x \in U$, and pick any y such that (x,y) is in the preimage of x. Since (x,y) lies in the open set V, there exists an $\varepsilon > 0$ such than an ε-ball about (x,y) lies entirely in V; the image of this ball is an ε-ball (an interval) about $x \in U$. I have shown that any point $x \in U$ has an open neighborhood contained in U; thus U is open. I have now shown that the image of any open set is open; thus the mapping is an open mapping. [Perhaps it is quicker to observe that one may verify that a mapping is open by checking this on basic open sets; but open balls in $\mathbb{R}^2$ map to open intervals, and so we are done.]

(b) The set $\mathbb{R}^2$ is closed; its image is the open set $\mathbb{R}$. Here is a more interesting example: the graph of $y = 1/x$ is closed in $\mathbb{R}^2$, yet its image is the open set $\mathbb{R} \setminus \{0\}$.

(c) In $\mathbb{R}^n$ the compact sets are precisely the closed and bounded sets. (This is the Heine–Borel theorem.) The continuous image of a compact set is always compact. Thus, if F is closed and bounded, then $p_1(F)$ is also closed and bounded. The only closed and bounded set that is open is the empty set. The answer to part (c) is that yes, one can find such a set, but more importantly, the empty set is the only such set.

3. Let X be an infinite set. Say that a subset $U \subset X$ is open if $U = \varnothing$, or $U = X$, or if the complement $X \setminus U$ is finite. Let $\mathcal{U}$ denote the set of all open subsets of X.

(a) Prove that $(X, \mathcal{U})$ is a topological space.

(b) Is this space Hausdorff?

(c) Prove that every subset $C \subset X$ is compact.

(d) Find a necessary and sufficient condition for a sequence $x_n \in X$ to converge to $x \in X$ in the topology $\mathcal{U}$. Are such limits unique?

SOLUTION:

(a) To prove that $(X, \mathcal{U})$ is a topological space, I must show that $\mathcal{U}$ is a topology; that is, I must show that $\mathcal{U}$, in addition to containing the empty set and the entire space, is closed under pairwise (and thus finite) intersection and arbitrary union. We know that $\varnothing \in \mathcal{U}$ and $X \in \mathcal{U}$. Let U, V be open sets. If $U \cap V$ is empty or the entire space, then $U \cap V$ is open; else, certainly neither U nor V is empty and thus each has a finite complement; then the set $(U \cap V)^{\complement} = U^{\complement} \cup V^{\complement}$, being the union of two finite sets, is finite, and thus $U \cap V$ is open. Finally, consider an arbitrary union of open sets. If the union is empty or the entire space, then it is open; else, the union remains unchanged when empty sets are removed from it and thus can be thought of as a union of nonempty open sets; that is, the union consists of sets each of whose complement is finite. The complement of this arbitrary union is the intersection of the complements; that intersection is contained in any of its terms and each term is a finite set. Thus the complement of the arbitrary union is finite, and thus the union is open.

(b) A topological space is T_2 or Hausdorff if any two distinct points can be separated by open sets; separating the points by open sets means to find two open sets, each containing one of the two points, such that the two open sets are disjoint.

In the space $(X, \mathcal{U})$, any two nonempty open sets meet; after all, if U, V are nonempty open sets, then their complements are finite and $U \cap V = (U^{\complement} \cup V^{\complement})^{\complement}$; but the union of two finite sets is finite, and inside the infinite set X, the complement of a finite set is infinite; thus $U \cap V$ is an infinite set and cannot be empty.

The fact that any two nonempty open sets meet tells us that the space $(X, \mathcal{U})$ is not Hausdorff. Given two distinct points, one cannot find an open set containing one and another open set containing the other such that these two sets are disjoint; after all, they will be nonempty, and I just showed that nonempty open sets always meet.

(c) To show that every subset $C \subseteq X$ is compact, I must show that every open cover of C (that is, every cover of C by open sets) contains a finite subcover. If C is empty, then the result is trivial; else there exists some nonempty open set in the cover; call it U. Then $U^{\complement}$ is a finite set, and $U^{\complement} \cap C$ is a (possibly even smaller) finite set. For each point in this finite set, there must be some open set in the cover that covers this point. Then U together with these (finitely many) open sets covers all of C and thus is the sought finite subcover.

(d) For a sequence x_n to converge to x means that the sequence eventually lies in every open neighborhood of x; that is, given any open set containing x, there exists some index N such that x_n lies in the open set whenever $n \geq N$.

Consider the case that the sequence x_n contains no constant subsequence. This is a fancy way of saying that the sequence hits no value infinitely often. This also means that the sequence can hit no finite set infinitely often, for hitting a finite set infinitely often implies hitting at least one of the elements infinitely often. Then given any open neighborhood of x, it is nonempty and thus its complement is finite; the sequence cannot hit this finite complement infinitely often and thus there exists an index after which the sequence is out of the complement and thus is iin the open neighborhood of x. Thus the sequence converges to x. Any point x would have worked in this argument; thus a sequence with no constant subsequence converges to every point.

Consider next the case that the sequence x_n contains precisely one constant subsequence. This is a fancy way of saying that the sequence hits precisely one value, say x_0, infinitely often, but hits no other value infinitely often. If $x \neq x_0$, then the set $X \setminus \{x_0\}$ is an open neighborhood of x that is missed infinitely often by the sequence; thus the sequence does not converge to x. On the other hand, if $x = x_0$, then the previous argument works to tell us that the sequence converges to x; after all, the complement of any open neighborhood of x_0 is finite and does not contain x_0; thus the sequence hits this complement only finitely often and thus eventually lies in the open neighborhood.

Finally consider the case that the sequence x_n contains more than one constant subsequence. This is a fancy way of saying that the sequence hits at least two values, say x_1 and x_2, infinitely often. Then the previous arguments show that the sequence can neither converge to a point distinct from x_1 nor converge to a point distinct from x_2; that is, the sequence cannot converge.

To summarize, the necessary and sufficient condition for a sequence x_n to converge to x is that the sequence either hits no value infinitely often or else hits only x infinitely often. If the sequence hits no value infinitely often, then it converges to every point and thus the limit is not unique.

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4. Determine all the covering spaces of $\mathbb{RP}^2$.

SOLUTION:

For finite graphs (that is, one-simplices with finitely-many edges), we studied correspondences between subgroups of the fundamental group of the graph and covers of the graph; we also studied correspondences between subgroups of the automorphism group of the universal cover and other covers of the graph. Presumably some of these results carry over to other situations but we did not study this.

The space $\mathbb{RP}^2$ consists of homogeneous ordered triples; that is, classes $[x,y,z]$ of nonzero points (x,y,z), where (x,y,z) and $(\lambda x, \lambda y, \lambda z)$, $\lambda \neq 0$, are in the same class. Thus one can think of $\mathbb{RP}^2$ as all classes of the form $[x,y,1]$ together with all classes of the form $[x,1,0]$ together with the class $[1,0,0]$; that is, $\mathbb{RP}^2$ consists of a real plane plus a real line plus a real point; the point is a "point at infinity" which ties up the line into a circle—this circle is a "circle at infinity" which ties up the plane into the object $\mathbb{RP}^2$ (I don't know of a word to describe it visually). Another way to think of $\mathbb{RP}^2$ is that in addition to the real plane we wish to have one point at infinity corresponding to each possible slope of a line including the undefined slope of vertical lines; the various slopes form the extra line, and the undefined slope forms the extra point. Finally, starting with the original definition using homogeneous coordinates, we see that the point classes are precisely lines through the origin in $\mathbb{R}^3$; or, what amounts to the same thing, they are pairs of antipodal points on the sphere S^2 in $\mathbb{R}^3$; thus one can think of $\mathbb{RP}^2$ as the sphre S^2 modulo the identification of antipodal points. We can throw away the bottom half of the sphere to give us the upper half modulo the identification of opposite points on the base circle. If we throw away half the base circle, and flatten out our image, then we are left with a disc that has half of its perimeter missing modulo the identification of the one pair of opposite perimeter points. **[Draw a picture and finish the explanation to determine the fundamental group.]**

The fundamental group $\pi_1(\mathbb{RP}^2)$ is $\mathbb{Z}/2\mathbb{Z}$; it has two subgroups (itself and the trivial subgroup); thus there should be two connected covers of $\mathbb{RP}^2$; they are the universal cover S^2 (where the covering map sends antipodal points to the same class) and the trivial cover $\mathbb{RP}^2$ (where the covering map is the identity). Arbitrary (not necessarily connected) covers consist of finite disjoint unions of these connected covers. For example, $S^2 \cup \mathbb{RP}^2$ is a cover of $\mathbb{RP}^2$ (where the covering map is the antipodal identification on S^2 and is the identity on $\mathbb{RP}^2$).

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5. The map $\pi\colon \mathbb{R}^2 \to S^1 \times S^1$ defined by

$$\pi\colon (x,y) \mapsto \pi(x,y) = (e^{ix}, e^{iy})$$

is a covering space map from the plane into the torus.

(a) If $g\colon S^2 \to S^1 \times S^1$ is a continuous map, does it always lift to a continuous map $\tilde{g}\colon S^2 \to \mathbb{R}^2$ so that $g = \pi\tilde{g}$?

(b) Find an example of a continuous map $g\colon S^1 \to S^1 \times S^1$ for which no lift $\tilde{g}\colon S^1 \to \mathbb{R}^2$ with $g = \pi\tilde{g}$ exists.

SOLUTION:

(a) Continuous maps between spaces induce homomorphisms between fundamental groups; specifically, the continuous map g induces a map g_* which sends the homotopy class of some path γ based at the point x to the homotopy class of the path $g\gamma$ based at the point $g(x)$. If a continuous map factors through another continuous map, then the induced homomorphism factors through the other induced homomorphism. This property turns out to be sufficient; namely, if the induced homomorphism factors, then the continuous map factors as well. **[Is this the complete theorem, or does it only apply to certain spaces?]**

In this problem, g_* is the trivial map since S^2 has trivial fundamental group. The map π_* is also trivial since $\mathbb{R}^2$ has trivial fundamental group. Thus g_* does factor through π_* and therefore g factors through π. The answer is to the question is "yes".

(b) Since the space $S^1 \times S^1$ does not have trivial fundamental group, there should **[or must?]** exist maps that do not factor through π; specifically, I seek a map g such that g_* is not trivial. The map $g\colon (x,y) \mapsto (x,1)$ will do. For this map to lift, we must have a $\tilde{g}$ such that $\pi\tilde{g}(x,y) = (x,1)$. Now $\tilde{g}$ is continuous if and only if its two coordinates functions are continuous; call them $\tilde{g}_1$ and $\tilde{g}_2$. Thus we seek continuous functions $\tilde{g}_1$ and $\tilde{g}_2$ such that $\pi\big(\tilde{g}_1(x,y), \tilde{g}_2(x,y)\big) = (x,1)$; equivalently, $(e^{i\tilde{g}_1(x,y)}, e^{i\tilde{g}_2(x,y)}) = (x,1)$. The only possible values for $\tilde{g}_2(x,y)$ come from the discrete set $\{\, 2k\pi : k \in \mathbb{Z}\,\}$ and thus the requirement that $\tilde{g}_2$ be continuous forces it to be a constant map $\tilde{g}_2(x,y) = 2k\pi$ for some fixed $k \in \mathbb{Z}$. The condition $e^{i\tilde{g}_1(x,y)} = x$ tells us that $\tilde{g}_1(x,y)$ must be the angle of the point $x \in S^1$; but this map cannot be continuous for to be so it would have to send the origin of the unit circle to both 0 and 2π, among other values. Since no continuous $\tilde{g}_1$ can exist, no continuous $\tilde{g}$ can exist.

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6. The space $\mathbb{C}^2$ becomes an inner product space when we define

$$\langle a, b\rangle = a_1\overline{b_1} + a_2\overline{b_2}.$$

Let $U(2)$ denote the set of 2×2 complex matrices

$$[u,v] = \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}, \qquad \text{where } \langle u,u\rangle = \langle v,v\rangle = 1 \text{ and } \langle u,v\rangle = 0.$$

Note that $U(2)$ is called the unitary group on $\mathbb{C}^2$. Observe that the set of $u \in \mathbb{C}^2$ with $\langle u,u\rangle = 1$ can be identified with S^3 and the set of $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ can be identified with S^1.

(a) Show that the map $\phi\colon S^3 \times S^1 \to U(2)$ defined by

$$\phi\colon (u,\lambda) \to \begin{bmatrix} u_1 & -\lambda\overline{u_2} \\ u_2 & \lambda\overline{u_1} \end{bmatrix}$$

is a homeomorphism. Here $u \in \mathbb{C}^2$ with $\langle u,u\rangle = 1$ is regarded as an element of S^3 and $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ is regarded as an element of S^1.

(b) Compute the fundamental group $\pi_1(U(2))$.

(c) Suppose that $\gamma\colon [0,1] \to U(2)$ is a continuous closed path in $U(2)$ which starts and ends at the identity matrix. If the map $t \to \text{DET}\,\gamma(t)$ is contractible to a point in S^1, does this imply that $\gamma(t)$ is homotopically trivial in $U(2)$?

SOLUTION:

(a) To show that ϕ is a homeomorphism, I must show that ϕ is invertible (that is, it is one-to-one and onto) and furthermore that both ϕ and its inverse are continuous. If $\phi(u,\lambda) = \phi(u',\lambda')$, that is, if

$$\begin{bmatrix} u_1 & -\lambda\overline{u_2} \\ u_2 & \lambda\overline{u_1} \end{bmatrix} = \begin{bmatrix} u_1' & -\lambda'\overline{u_2'} \\ u_2' & \lambda'\overline{u_1'} \end{bmatrix},$$

then clearly $u_1 = u_1'$ and $u_2 = u_2'$ and thus $u = u'$; furthermore, the condition $\langle u,u\rangle = 1$ implies $u \neq 0$; thus at least one of u_1, u_2 is nonzero, so that at least one of $\overline{u_1}, \overline{u_2}$ is nonzero, and thus either $\lambda\overline{u_1} = \lambda'\overline{u_1}$ or $\lambda\overline{u_2} = \lambda'\overline{u_2}$ implies $\lambda = \lambda'$. The map ϕ is one-to-one. Notice that

$$\text{DET}\begin{bmatrix} u_1 & -\lambda\overline{u_2} \\ u_2 & \lambda\overline{u_1} \end{bmatrix} = \lambda(u_1\overline{u_1} + u_2\overline{u_2}) = \lambda(|u_1|^2 + |u_2|^2) = \lambda \cdot \langle u,u\rangle = \lambda.$$

Furthermore, given any matrix in $U(2)$, its determinant has length $|u_1v_2 - u_2v_1| = one$? **PROBLEM**. I propose to show that ϕ is onto by exhibiting a candidate for the inverse map and verifying that it works. The candidate ϕ^{-1} sends a matrix in $U(2)$ to $\big((u_1,u_2), u_1v_2 - u_2v_1\big)$. Since $\text{DET}\,\phi(u,\lambda) = \lambda$, it is clear that $\phi^{-1}\phi$ is the identity on $S^3 \times S^1$. To show that $\phi\phi^{-1}$ is the identity on $U(2)$, I must show

$$\begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix} \stackrel{?}{=} \begin{matrix} u_1 & (u_1v_2 - u_2v_1)\overline{u_2} \\ u_2 & (u_1v_2 - u_2v_1)\overline{u_1} \end{matrix};$$

but the equality holds because $(u_1v_2 - u_2v_1)\overline{u_2} = (oops)$ **[WHY DOES IT HOLD?]**. The maps ϕ and ϕ^{-1} are continuous if and only if their various component **(or coordinate? word appears a few more times below)** maps are continuous; since each coordinate map consists of compositions of continuous maps (identities, conjugations, multiplications), each coordinate map is continuous. Thus the map ϕ is continuous and has a continuous inverse; that is, ϕ is a homeomorphism.

(b) If two spaces are homeomorphic, then they have the same fundamental group. Thus the fundamental group $\pi_1(U(2))$ is just $\pi_1(S^3 \times S^1) = \pi_1(S^3) \times \pi_1(S^1) = 1 \times \mathbb{Z} = \mathbb{Z}$.

(c) If we identify $U(2)$ with $S^3 \times S^1$ via the map ϕ from part (a), then the path γ is a continuous closed path $[0,1] \to S^3 \times S^1$; each component map is a continuous closed path. The path γ is contractible to a point if and only if each component map is contractible to a point. The first coordinate map is necessarily contractible to a point because S^3 is simply connected; we are given that the second component map is contractible to a point. The answer to the question is "yes".

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7. Determine all the homology groups $H_q(S^2 \times \mathbb{R})$ for $q = 0, 1, 2, 3, \ldots$

SOLUTION:

Since $S^2 \times \{0\}$ is a deformation retract of $S^2 \times \mathbb{R}$, and since $S^2 \times \{0\}$ is homeomorphic to S^2, I conclude that $H_*(S^2 \times \mathbb{R}) = H_*(S^2)$. As for the homology of S^2, since S^2 is connected, $H_0(S^2) = \mathbb{Z}$, and since it is a compact oriented manifold, duality tells me $H_2(S^2) = \mathbb{Z}$. Finally, it is simply connected, hence $\pi_1(S^2) = 1$, and thus its abelianization is $H_1(S^2) = 0$. Since S^2 is a 2-dimensional manifold, the homologies H_3 and higher vanish.

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8. Let S^1 denote the circle of radius 1 in $\mathbb{R}^2$. Let X denote the space

$$X = \{\, (x,y) : (x+1)^2 + y^2 = 1 \,\} \cup \{\, (x,y) : (x-1)^2 + y^2 = 1 \,\}.$$

Define a continuous map $f\colon S^1 \to X$ by

$$f(e^{i\theta}) = \begin{cases} e^{2i\theta} - 1, & \text{for } \theta \in [0,\pi] \\ 1 - e^{2i\theta}, & \text{for } \theta \in [\pi, 2\pi]. \end{cases}$$

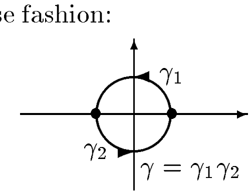
We have taken the liberty of identifying $\mathbb{R}^2$ with $\mathbb{C}$ in the usual fashion. Give S^1 and X the structures of finite regular graphs so that the map f takes vertices to vertices. Use this graph structure to determine explicit generators for $H_1(S^1)$ and $H_1(X)$. Compute the map

$$f_*\colon H_1(S^1) \to H_1(X)$$

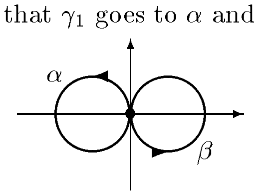
explicitly in terms of these generators.

SOLUTION:

Let γ_1 denote the path $e^{i\theta}$ as $0 \le \theta \le \pi$, and let γ_2 denote the path $e^{i\theta}$ as $\pi \le \theta \le 2\pi$, so that $\gamma = \gamma_1\gamma_2$ is a loop that goes once around S^1 in a counterclockwise fashion:



Now S^1 is being mapped onto the graph X so that γ_1 goes to α and γ_2 goes to β, as shown here:



Now $H_1(S^1)$ is infinite cyclic, freely generated by the single loop γ, whereas $H_1(X)$ is freely generated by the two loops α and β. The induced homomorphism $H_1(S^1) \to H_1(X)$ sends γ to the element $\alpha\beta$.

9. Show that if σ is a closed 1-form on S^2, then σ must be exact.

SOLUTION:

More generally, I'll show that if σ is a closed 1-form on any simply connected oriented manifold M of arbitrary dimension, then σ must be exact. Fix $P_0 \in M$ and consider the function $h: M \to \mathbb{R}$ defined by $h(P) = \int_{P_0}^{P} \sigma$. (The integral is well-defined, i.e., independent of path, since the difference of any two paths connecting P_0 to P is a loop, and M being simply connected says that this loop is a boundary, and then Stokes says that any closed form integrates to zero on a boundary.) Finally, we have $\mathrm{d}h = \sigma$, which follows by choosing local coordinates and applying the fundamental theorem of calculus. I'll give an example of this last step for the case $M = S^2$. Say that $\sigma = f(\phi, \theta)\,\mathrm{d}\phi + g(\phi, \theta)\,\mathrm{d}\theta$. (Even though ϕ and θ aren't smooth coordinates on the entire sphere, it nonetheless makes sense to write the above, noting that this implies f and g to be doubly-periodic functions.) Since the path used in the integral expression for $h(P)$ does not matter, in considering $(\partial h/\partial\phi)(P)$, we may consider paths from P_0 to P that are constant with respect to θ in a neighborhood of P, i.e., change only in the ϕ direction. And then we see that $h(P + \Delta\phi) - h(P) = \int_{P}^{P+\Delta\phi} f(\phi, \theta)\,\mathrm{d}\phi$, so that dividing by $\Delta\phi$ and taking the limit as $\Delta\phi \to 0$, by the fundamental theorem of calculus gives $(\partial h/\partial\phi)(P) = f(P)$. Similarly, $(\partial h/\partial\theta)(P) = g(P)$. And thus $\mathrm{d}h = (\partial h/\partial\phi)\,\mathrm{d}\phi + (\partial h/\partial\theta)\,\mathrm{d}\theta = f\,\mathrm{d}\phi + g\,\mathrm{d}\theta = \sigma$, as desired.

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN SPRING 1993

1. Suppose that $f(z)$ is an analytic function defined in a neighborhood of $z = 0$ with $f(0) = 0$. Suppose that for some sufficiently small neighborhood of $z = 0$ the map $z \mapsto f(z)$ is injective. Is it true that $f(z)/z$ must be analytic and *non-zero* near $z = 0$? Give a proof or a counter-example.

SOLUTION:

I shall give a proof. Around $z = 0$, we have $f(z) = a_1 z + a_2 z^2 + \cdots$. (There are no negative power terms since f is analytic, and there is no constant term since $f(0) = 0$.) If a_1 were 0, then f would locally be a 2-to-1 map (proof below), contrary to assumption; hence $a_1 \neq 0$. Now consider $f(z)/z$; it looks like $a_1 + a_2 z + \cdots$, which is analytic and non-zero near $z = 0$.

I shall prove that $f(z) = a_n z^n + \cdots$ is locally n-to-1. I will accept as known that z^n is locally n-to-1 (it wraps the complex plane around the origin n times). Factor out the power of z, writing $f(z) = z^n u(z)$, where $u(0) \neq 0$. Since $u(0) \neq 0$, there is an analytic choice of nth root of $u(z)$ in a neighborhood of $z = 0$, i.e., $u(z) = (v(z))^n$; observe that $v(0) \neq 0$. Thus $f(z) = (zv(z))^n$ in a neighborhood of 0. The function $h(z) = zv(z)$ vanishes to first order at the origin; thus its derivative is nonzero, and the inverse function theorem applies to show that $h(z)$ is locally invertible. Hence $f(z)$ is the nth power map composed with a locally invertible map, and thus $f(z)$ is locally n-to-1. [Thanks to Minhyong Kim for helping with this argument.]

2. Suppose that X is a set and that $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologies for X. If $(X, \mathcal{F}_1)$ and $(X, \mathcal{F}_2)$ are each compact Hausdorff spaces and furthermore $\mathcal{F}_2 \subseteq \mathcal{F}_1$, is it true that $\mathcal{F}_1 = \mathcal{F}_2$? Give a proof or a counter-example.

SOLUTION:

[Solution by Steve Phillips, based on ideas of "Jay", July 1998.] I will give a proof. Throughout, I shall consider the "identity" map $(X, \mathcal{F}_1) \to (X, \mathcal{F}_2)$ given by $x \mapsto x$. The assumption $\mathcal{F}_2 \subseteq \mathcal{F}_1$ tells me that any subset of X that is open in the $\mathcal{F}_2$-topology is also open in the $\mathcal{F}_1$-topology; hence the map is continuous. To show that $\mathcal{F}_1 \subseteq \mathcal{F}_2$ (which completes the proof), I must show that this map has a continuous inverse, which amounts to showing that it is an open map. Since it is bijective, it is compatible with the taking of complements, and so I may instead show that it is a closed map. So let $A \subseteq X$ be a subset that is closed in the $\mathcal{F}_1$-topology. Any closed subset of a compact space is again compact; continuous maps send compact sets to compact sets; compact subsets of a Hausdorff space are closed. Hence X is closed in the $\mathcal{F}_2$-topology. This completes the proof.

Note: I really needed to know only that $(X, \mathcal{F}_1)$ is compact and that $(X, \mathcal{F}_2)$ is Hausdorff; as a consequence I discover that my map is a homeomorphism, so that both spaces must be both compact and Hausdorff.

3. Suppose that X and Y are metric spaces and that $f: X \to Y$ is a map from X to Y. Suppose that f maps every convergent sequence x_k in X into a convergent sequence $f(x_k)$ in Y. Is f continuous? Give a proof or a counterexample.

SOLUTION:

[Solution by Pasaad Kongtaln.] I shall give a proof. First, note that with a continuous map of arbitrary topological spaces, if $x_n \to x$, then $f(x_n) \to f(x)$. After all, each open neighborhood of x contains all but finitely many terms of the sequence x_n; let U be an arbitrary open neighborhood of $f(x)$; then $f^{-1}(U)$ is an open neighborhood of x; all but finitely many terms of x_n lie in $f^{-1}(U)$, and so all but finitely many terms of $f(x_n)$ lie in U. Since we have $x_n \to x$ implying $f(x_n) \to f(x)$, in particular we get that if x_n converges, then also $f(x_n)$ converges. With metric spaces, one has a converse: if f maps convergent sequences to convergent sequences, then f is continuous, so that, in particular, if $x_n \to x$, then $f(x_n) \to f(x)$. (Note the latter isn't obvious; just because $f(x_n)$ is convergent, it isn't a priori clear that $f(x_n) \to f(x)$.)

So let f have the property that convergent sequences map to convergent sequences. First I claim that if $x_n \to x$, then $f(x_n) \to f(x)$. The sequence $x_1, x, x_2, x, x_3, x, \ldots$ also converges to x, so that $f(x_1), f(x), f(x_2), f(x), \ldots$ must converge; thus any subsequence converges and has the same limit(s); metric spaces are Hausdorff, thus convergent sequences have unique limits; the subsequence $f(x), f(x), \dots$ has the limit $f(x)$; hence $x_1, f(x), x_2, f(x), \dots$ also has the limit $f(x)$, and therefore the subsequence $x_1, x_2, x_3, \dots$ (our original sequence) also has the limit $f(x)$.

The usual definition of continuity is that f is continuous if and only if f^{-1} is an open mapping, or equivalently, since f^{-1} commutes with complementation, that f^{-1} is a closed mapping. I shall prove another characterization of continuity: The map f is continuous if and only if $f(\overline{A}) \subseteq \overline{f(A)}$ for all subsets A. To prove one direction, assume that f is continuous. Let $x \in \overline{A}$. Fix an open neighborhood U of $f(x)$. Then $f^{-1}(U)$ is an open neighborhood of x, hence meets A, so that U meets $f(A)$. Since x was an arbitrary limit point of A, I conclude that $f(\overline{A}) \subseteq \overline{f(A)}$. For the other direction, assume that $f(\overline{A}) \subseteq \overline{f(A)}$. I shall show that f^{-1} is a closed mapping. Let $B = \overline{B}$ be closed, and let x be a limit point of $A = f^{-1}(B)$. By assumption, $f(x)$ is a limit point of B, hence $f(x) \in \overline{B} = B$, so that $x \in A$, showing that A is closed.

Now assume that whenever $x_n \to x$, then $f(x_n) \to f(x)$. I shall show that f is continuous. Let A be arbitrary, and fix $x \in \overline{A}$. Each neighborhood of x meets A. Using the metric structure, I can construct a sequence $x_n \to x$ with each $x_n \in A$ by choosing points from A in a family of shrinking balls about x. Then $f(x_n) \to f(x)$, showing that $f(x)$ is a limit point of $f(A)$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$, and thus f is continuous.

Let me summarize. If f is continuous, then f preserves convergent sequences, in fact f preserves the limit of such sequences. In a space with unique limits, for example a Hausdorff space, if an arbitrary map f preserves convergent sequences, then it preserves the limit of such sequences. Finally, in a space in which, having fixed a limit point of a set, one can construct a sequence in the set converging to the given limit point, then a map f that preserves limits of convergent sequences is necessarily continuous. The two crucial properties (unique limits and ability to construct sequences converging to limit points) hold in metric spaces, thus solving the problem at hand.

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4. Let $\mathrm{GL}(2,\mathbb{R})$ denote the group of 2×2 real invertible matrices, and let $\mathrm{SL}(2,\mathbb{R})$ denote the subgroup of $\mathrm{GL}(2,\mathbb{R})$ composed by the matrices of determinant 1.

- Show that $\mathrm{SL}(2,\mathbb{R})$ is a differentiable manifold.
- Show that $\mathrm{GL}(2,\mathbb{R})$ is not connected, and determine how many components it has. HINT: Consider the map

$$\mathrm{DET}\colon \mathrm{GL}(2,\mathbb{R}) \to \mathbb{R} \setminus \{\, 0 \,\}.$$

SOLUTION:

I shall identify 2×2 matrices with $\mathbb{R}^4$ in the usual manner: $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \leftrightarrow (a,b,c,d)$. The determinant map $\mathbb{R}^4 \to \mathbb{R}$ is the map $(a,b,c,d) \mapsto ad - bc$; hence its derivative is the 1×4 matrix $[d, -c, -b, a]$. If $ad - bc \neq 0$, then at least one of a, b, c, d must be nonzero, so that the derivative matrix has full rank; in other words, any nonzero $\alpha \in \mathbb{R}$ is a regular value of this map. (Note that 0 is not even in the image. If we consider DET on all of $\mathbb{R}^4$, then 0 is not a regular value, but if we were to restrict the map to $\mathbb{R}^4 \setminus \{0\}$, then 0 would be a regular value.) Since 1 is a regular value, $\mathrm{DET}^{-1}(1) = \mathrm{SL}(2,\mathbb{R})$ is an embedded submanifold of $\mathbb{R}^4$. Since our manifolds are smooth and DET is smooth, in fact $\mathrm{SL}(2,\mathbb{R})$ is a smooth (= differentiable) submanifold.

The determinant map is polynomial hence continuous. Continuous maps preserve connectedness. Since the image of DET is the two-component set $\mathbb{R} \setminus \{0\}$, we must have that $\mathrm{GL}(2,\mathbb{R})$ has at least two components. I claim that it has precisely two components. Since we are in a metric space, connectedness is the same as path connectedness. Thus I must show that any two matrices with both positive or both negative determinant can be connected by a path. Two things must be shown: each submanifold $\mathrm{DET}^{-1}(\alpha)$ is connected, and given two such submanifolds corresponding to two positive or two negative values of α, some path exists between them.

[This argument is due to Yuko Fukatsu, July 1998.] Fix two submanifolds, $\mathrm{DET}^{-1}(\alpha)$ and $\mathrm{DET}^{-1}(\beta)$, where $\alpha\beta > 0$ (i.e., neither is zero and they have the same sign). Fix an arbitrary matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{DET}^{-1}(\alpha)$. Then $ad - bc = \alpha$, so at least one of the two diagonals must have both entries nonzero. By continuously varying one of those two entries without making it zero, we can change the determinant to any number agreeing in sign with α; in particular, we have some path that connects us with a matrix of determinant β.

[This argument was inspired by Yuko Fukatsu, but it took many false starts on my part to finally come up with the right way of breaking into cases to present a nice argument.] Now we must show that each $\mathrm{DET}^{-1}(\alpha)$ is connected. To do this, I will exhibit a path from any matrix of determinant α to a particular matrix of determinant α, namely $\sqrt{\alpha} I$ when $\alpha > 0$, and $\left[\begin{smallmatrix} 0 & \sqrt{\alpha} \\ -\sqrt{\alpha} & 0 \end{smallmatrix}\right]$ when $\alpha < 0$. (I just picked these guys out of the blue; there is no real reason to prefer one matrix over another.) The two cases are argued almost identically, so I'll just argue the case $\alpha < 0$. (If there were any concern of one case being more complicated than the other, surely it would be the one I picked; of course, it is not really harder mathematically, only we are less used to working with negatives.) So let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ have determinant α, i.e., $ad - bc = \alpha$. Observe that we must have at least one of the diagonals with two nonzero entries. I will break into cases.

- If $ad = 0$, $bc \neq 0$, and $b < 0$, then $c = (ad - \alpha)/b$, so we may continuously adjust a and d to each be nonzero, letting c take up the slack to keep the determinant constant. This puts us into the next case.
- If $ad \neq 0$, then $a = (\alpha + bc)/d$, so we can adjust b and c so that $b > 0$ and $c \neq 0$, letting a take up the slack to keep the determinant constant. This puts us into the last case.
- If $bc \neq 0$ and $b > 0$, then $c = (ad - \alpha)/b$, so we may continuously change a and d into zero, letting c take up the slack. Then we have $c = \alpha/b$, and as we continuously change b to $\sqrt{\alpha}$, the value of c changes to $-\sqrt{\alpha}$, and we end up (finally!) with the desired matrix.

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5. Let

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} : \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

It is a group under matrix multiplication. Consider the subgroup

$$Z = \{ \gamma, \gamma^2, \gamma^3 \},$$

where

$$\gamma = \begin{pmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{pmatrix}.$$

Let X be the coset space $\mathrm{SU}(2)/Z$, with the quotient topology from $\mathrm{SU}(2)$.

- Prove that $\mathrm{SU}(2) \cong S^3$.
- Calculate $\pi_1(X)$.
- Briefly indicate how you would construct a space X such that $\pi_1(X)$ is the cyclic group of order n.
- For $X = \mathrm{SU}(2)/Z$ as before, prove that any map $f\colon X \to \mathbb{RP}(2)$ lifts to a map $\overline{f}\colon X \to S^2$ such that $p\overline{f} = f$, where $p\colon S^2 \to \mathbb{RP}(2)$ is the obvious projection.

SOLUTION:

The correspondences

$$\begin{array}{ccccc} \mathrm{SU}(2) & \hookrightarrow & \mathbb{C}^2 & \leftrightarrow & \mathbb{R}^4 \\ \mathrm{SU}(2) & \leftrightarrow & \{ (\alpha,\beta) : |\alpha|^2 + |\beta|^2 = 1 \} & \leftrightarrow & S^3 = \{ (x,y,z,w) : x^2+y^2+z^2+w^2 = 1 \} \\ \begin{bmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{bmatrix} & \leftrightarrow & (\alpha, \beta) & \leftrightarrow & (\Re(\alpha), \Im(\alpha), \Re(\beta), \Im(\beta)) \end{array}$$

are clearly bijective and continuous in each direction, hence compose to give a homeomorphism $\mathrm{SU}(2) \cong S^3$.

The finite subgroup Z acts on the continuous group $\mathrm{SU}(2)$ by left multiplication; since Z is finite, this action is necessarily properly discontinuous, which means that each point in $\mathrm{SU}(2)$ has a neighborhood so that the orbit of that neighborhood under the action consists of pairwise disjoint copies of the neighborhood. Thus $\mathrm{SU}(2)$, together with the canonical projection map p, is a regular cover of the quotient $X = \mathrm{SU}(2)/Z$, and furthermore, Z is the cover automorphism group: $Z = \textsc{aut}(\mathrm{SU}(2), p)$. Since the cover is regular, we have that $p_*(\pi_1(\mathrm{SU}(2)))$ is a normal subgroup of $\pi_1(X)$. In general, the cover automorphism group is obtained by projecting the fundamental group of the cover down and taking its quotient inside its normalizer. [Check this in Massey for precision, I haven't thought about it in a while.] Here the normalizer will be all of $\pi_1(X)$, and so we have

$$Z = \textsc{aut}(\mathrm{SU}(2), p) = \frac{\pi_1(X)}{p_*(\pi_1(\mathrm{SU}(2)))},$$

but $\pi_1(\mathrm{SU}(2)) = 1$ since $\mathrm{SU}(2) \cong S^3$ is simply connected, hence the denominator above is trivial, and we have $Z = \pi_1(X)$. Note that Z was isomorphic to $\mathbb{Z}/3\mathbb{Z}$, hence this is $\pi_1(X)$.

There was nothing special about the number 3 in the work above. If we let γ be the matrix involving a primitive nth root of unity $e^{2\pi i/n}$, then we will have $Z = \mathbb{Z}/n\mathbb{Z}$, and everything above goes through to give us a quotient with fundamental group $\mathbb{Z}/n\mathbb{Z}$.

Now let f be a map $X \to \mathbb{RP}^2$; does f factor through the cover $p\colon S^2 \to \mathbb{RP}^2$? This happens if and only if $f_*(\pi_1(X)) \leq p_*(\pi_1(S^2))$, i.e., if and only if these two subgroups of $\pi_1(\mathbb{RP}^2)$ satisfy the stated containment. Now $\pi_1(S^2) = 1$ while $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$, so f factors through S^2 if and only if $f_*(\pi_1(X)) = 1$. Since there are no nontrivial homomorphisms $\mathbb{Z}/3\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, the necessary and sufficient condition holds, and I conclude that all maps f factor through S^2. Note that if we change γ so that $Z = \mathbb{Z}/n\mathbb{Z}$, then for some values of n I can no longer conclude that maps will factor.

6.

- Prove that there is no continuous map $f\colon \mathbb{RP}(2) \to S^2$ such that $pf = \mathrm{ID}_{\mathbb{RP}(2)}$. (As usual, $p\colon S^2 \to \mathbb{RP}(2)$ is the obvious projection.)
- Is there a map f as above such that $fp = \mathrm{ID}_{S^2}$? Justify your answer.

SOLUTION:

A map $f\colon \mathbb{RP}(2) \to S^2$ with $pf = \mathrm{ID}_{\mathbb{RP}(2)}$ would be a lift (or a factoring through the cover) of the map $\mathrm{ID}\colon \mathbb{RP}(2) \to \mathbb{RP}(2)$. Such a map exists if and only if $\mathrm{ID}_*(\pi_1(\mathbb{RP}(2))) \leq p_*(\pi_1(S^2))$, i.e., if and only if $p_*(\pi_1(S^2)) = \pi_1(\mathbb{RP}(2)) = \mathbb{Z}/2\mathbb{Z}$. But that isn't the case, since S^2 is simply connected and thus $\pi_1(S^2) = 1$.

There cannot exist a map $f\colon \mathbb{RP}(2) \to S^2$ with $fp = \mathrm{ID}_{S^2}$; that is to say, there cannot exist any such map whatsoever, continuous or not. After all, p is not injective, so how can its composition with some other map be injective? Specifically: $p(x) = p(-x)$, so where does f send $p(x)$?

7.

- Let $X = \mathbb{R}^2 - (0,0)$. Show that the differential form

$$\omega = \frac{x\,\mathrm{d}x + y\,\mathrm{d}y}{x^2 + y^2}$$

  is closed.
- Show that ω is not exact. (HINT: If it were, what would its integral around the unit circle be?)
- Let $f\colon X \to X$ be given by $f(x,y) = (x^2 - y^2, 2xy)$. Calculate $f^*(\omega)$.

SOLUTION:

[Problem incorrect as stated. Probably they wanted $\mathrm{d}\theta$, which is $(x\,\mathrm{d}y - y\,\mathrm{d}x)/(x^2+y^2)$.]

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8. Let X be a differentiable manifold and let D be a closed disc around a point $x \in X$. Let X' be X/D, the space X with D squeezed to a point. Prove that $H_i(X') = H_i(X)$.

SOLUTION:

[This solution isn't really right. In general, the long exact sequence in homology can be formed with either usual homology or reduced homology. Somehow the excision property gives you information about X/D, but Massey is unclear on this. Or perhaps it is that relative homology $H(X,D)$ is almost by definition the same thing as $H(X/D)$. I don't know.]

The special Mayer–Vietoris sequence with reduced homology in this case is the long exact sequence

$$\to H_3(D) \to H_3(X) \to H_3(X/D) \to H_2(D) \to H_2(X) \to H_2(X/D) \to H_1(D) \to H_1(X) \to H_1(X/D) \to \tilde{H}_0(D) \to \tilde{H}_0(X) \to \tilde{H}_0(X/D) \to 0,$$

where, for any manifold M, $\tilde{H}_0(M) \oplus \mathbb{Z} = H_0(M)$. Since D contracts to a point, each $H_i(D) = 0$ for $i \geq 1$, and since D has a single component, $H_0(D) = \mathbb{Z}$, so that $\tilde{H}_0(D) = 0$. Thus every third entry in the long exact sequence is 0, showing the remaining entries to be isomorphic in pairs, i.e., $H_i(X) \cong H_i(X/D)$ for all i.

9.

- Let $X = T^2 = S^1 \times S^1$. Let z_1 be a base point on the first circle, and let z_2 be a base point on the second circle. Remove $S^1 \times z_2 = C$ from X to get $X' = X - C$. Calculate $H_*(X')$.
- Show that there is a subset $D \cong S^1$ in X, such that $H_1(X - D; \mathbb{Z}) \cong \mathbb{Z}^2$.

SOLUTION:

First of all, $H_*(S^1 \times S^1)$ is:

$$H_0(S^1 \times S^1) = \mathbb{Z},$$
$$H_1(S^1 \times S^1) = \mathbb{Z}^2,$$
$$H_2(S^1 \times S^1) = \mathbb{Z}.$$

Now the given circle C is one of the two generators of $H_1(S^1 \times S^1)$. If we remove it, the remaining surface is homeomorphic to a cylinder, which may be contracted to a circle S^1. Hence $H_*(X')$ is

$$H_0(X') = \mathbb{Z},$$
$$H_1(X') = \mathbb{Z}.$$

Let D be the boundary of a small disc in $S^1 \times S^1$, i.e., a circle that contracts to a point. The result of removing D is a disconnected space with two components: one component is a disc; the other component deform retracts onto a figure eight double loop. Hence we get two generators in 0-homology, coming from the two components, and we get two generators in 1-homology, both living on the figure eight component. Hence

$$H_0(X - D) = \mathbb{Z}^2,$$
$$H_1(X - D) = \mathbb{Z}^2.$$

PROBLEMS FROM EXAM GIVEN AT UNIVERSITY OF ARIZONA IN FALL 1992

1. Compute:

$$\int_0^\infty \frac{\cos(x)}{(1 + x^2)^2} \, \mathrm{d}x$$

2. Can a function be analytic in a neighborhood of $z = 0$ and satisfy

$$f(1/n) = f(-1/n) = 1/n^2$$

for all positive integers $n = 1, 2, \ldots$? How about

$$f(1/n) = f(-1/n) = 1/n^3$$

for all positive integers $n = 1, 2, \ldots$?

SOLUTION:

The function $z \mapsto z^2$ is analytic around $z = 0$ and has the property $f(1/n) = f(-1/n) = 1/n^2$. However, I claim there is no function with $f(1/n) = f(-1/n) = 1/n^3$. Let us say that $f(z)$ were such a function, analytic in some domain containing $z = 0$. The function $z \mapsto z^3$ is also analytic in that domain. These two functions agree on the convergent sequence $1/n$; thus, by the "rigidity of complex analysis", they must be the same function, contradicting the assumption that $f(-1/n) = 1/n^3$.

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3. Suppose that an arc-wise connected space X is the union of two open subsets U and V that are each contractible and such that the intersection $U \cap V$ is arc-wise connected (and non-empty). Show that the fundamental group $\pi_1(X)$ is trivial. Use this result to show that $\pi_1(S^2)$ is trivial, i.e., $\pi_1(S_2) = \{0\}$.

SOLUTION:

This is an application of the SEIFERT & VAN KAMPEN theorem. From the obvious inclusions, we get induced group homomorphisms

$$\begin{tikzcd} & \pi_1(U) \arrow[dr] & \\ \pi_1(U \cap V) \arrow[ur] \arrow[dr] \arrow[rr] & & \pi_1(X) \\ & \pi_1(V) \arrow[ur] & \end{tikzcd}$$

where $\pi_1(X)$ satisfies the following universal property: if G is any group G that can take the place of $\pi_1(X)$ and fit into a diagram as shown, then there is a unique map $\pi_1(X) \to G$ making the combination of the two diagrams commute. [Okay, there *are* more elegant ways to state the property, but really it's never pretty, is it?] The gist is that $\pi_1(X)$ is the "freeest possible group that can fit into the above diagram" without inflating things unnecessarily by tossing in extra free generators. Without requiring the diagram to commute, the freeest such object would be the free product $\pi_1(U) * \pi_1(V)$; but commutativity of the diagram forces us to introduce relations into this free product so that the image of generators of $\pi_1(U \cap V)$ in $\pi_1(U)$ and in $\pi_1(V)$ get identified by these relations. In short, you take generators for $\pi(U \cap V)$, express them in terms of $\pi_1(U)$ and also in terms of $\pi_1(V)$, then formally set corresponding expressions equal and introduce those relations into $\pi_1(U) * \pi_1(V)$, i.e., form the quotient of the free group modulo the normal subgroup generated by the relations. The SEIFERT & VAN KAMPEN theorem asserts that the result is $\pi_1(X)$.

Let's do that in the present case. We are not told what $U \cap V$ looks like, so $\pi_1(U \cap V)$ has *some* set of generators. But these generators vanish in $\pi_1(U)$ and in $\pi_1(V)$, since each of the latter are trivial. The free product $\pi_1(U) * \pi_1(V)$ is also trivial, and the trivial group modulo the normal subgroup generated by trivial relations is once again the trivial group. Hence X is simply connected.

We may apply this to the sphere S^2 as follows: let U be the sphere minus the south pole, and let V be the sphere minus the north pole. Then U and V each deform retract to a point, hence are simply connected. Their intersection deform retracts to the equator circle S^1, which is connected, and thus the results above apply, showing $\pi_1(S^2)$ to be trivial.

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4. Suppose that U is an open subset of the metric space X. Define

$$U^\delta = \{\, x \in U : d(x, X \setminus U) > \delta \,\},$$

which is the set of points in U that are further than δ from the complement of U. Show that if $\{U_1, U_2\}$ is an open covering of a compact metric space X, then for some $\delta > 0$, the sets U_1^δ and U_2^δ still cover X.

SOLUTION:

[Solution by Mike Yarbrough, July 1998.] Since U_1, U_2 are open, we may write $U_1 = \bigcup_{n=1}^\infty U_1^{1/n}$ and $U_2 = \bigcup_{n=1}^\infty U_2^{1/n}$. (To see this, consider a point $x \in U_1$; since U_1 is open, a ball of some radius $r > 0$ centered at x lies entirely inside U_1, and thus x lies inside U_1^r; choosing n so that $1/n < r$, we also have $x \in U_1^{1/n}$.)

We have $X = U_1 \cup U_2 = (\bigcup U_1^{1/n}) \cup (\bigcup U_2^{1/n})$. Since X is compact, a finite number of sets in the latter cover will also cover X, say $X = U_1^{1/n_1} \cup \cdots \cup U_1^{1/n_j} \cup U_2^{1/m_1} \cup \cdots \cup U_2^{1/m_k}$, where $n_1 < \cdots < n_j$ and $m_1 < \cdots < m_k$. But $U_1^{1/n_1} \subset \cdots \subset U_1^{1/n_j}$, similarly for the other collection, so that we have $X = U_1^{1/n_j} \cup U_2^{1/m_k}$. Finally, setting δ to be the smaller of $1/n_j$ and $1/m_k$, we have $X = U_1^\delta \cup U_2^\delta$, as was to be shown.

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5. Calculate $H_*(\mathbb{CP}^n)$ and justify your answer.

6.

- Suppose that $f\colon X \to Y$ is continuous, and $p\colon \tilde{Y} \to Y$ is a covering space projection. Under what circumstances is it possible to lift f to a continuous map $\tilde{f}\colon X \to \tilde{Y}$, i.e., for which $f = p\tilde{f}$?
- Let the projection $p\colon \mathbb{R} \to S^1$ be given by $p(t) = (\cos(t), \sin(t))$. Show that any continuous map $f\colon \mathbb{RP}^2 \to S^1$ lifts to a map $\tilde{f}\colon \mathbb{RP}^2 \to \mathbb{R}$.

SOLUTION:

I can answer this problem only under the assumption that all the spaces (X, Y, and $\tilde{Y}$) are pathwise connected and locally pathwise connected. (Note: most books consider "pathwise" and "arcwise" to be equivalent; others reserve "arc" for paths that are one-to-one; when this is done, requiring arcwise connectivity places unnecessary restrictions on a space. See the book *Counterexamples in topology* for more information on the relationships between these concepts.)

Assume that the lift $\tilde{f}$ exists, so that, for any $x_0 \in X$, we have a commutative diagram:

$$(X, x_0) \xrightarrow{\tilde{f}} (\tilde{Y}, \tilde{y}_0) \xrightarrow{p} (Y, y_0)$$

where $\tilde{y}_0$ and y_0 are the images under these maps of x_0. These maps induce a commutative diagram at the level of fundamental groups:

ditto

Furthermore, the covering space map p is known to induce an injective map p_* at the level of fundamental groups. The commutativity of the diagram implies

$$f_*(\pi_1(X, x_0)) \subseteq p_*(\pi_1(\tilde{Y}, \tilde{y}_0)).$$

The relevant theorem now is that the above containment condition is not only necessary but also sufficient for the existence of the lift $\tilde{f}$. Certainly, if we have the above containment, then there is a unique choice for the map $\tilde{f}_*$, which is easily shown to be a homomorphism. The key then is that there exist a continuous map $X \to \tilde{Y}$ that lifts f and induces $\tilde{f}_*$.

Let $f\colon \mathbb{RP}^2 \to S^1$ be an arbitrary continuous map. Then f lifts to (or factors through) a map $\mathbb{RP}^2 \to \mathbb{R}$ if and only if $f_*(\pi_1(\mathbb{RP}^2)) \le p_*(\pi_1(\mathbb{R}))$. Now $\pi_1(\mathbb{R}) = 1$ and $\pi_1(\mathbb{RP}^2) = \mathbb{Z}/2\mathbb{Z}$. Hence the map f lifts if and only if $f_*(\mathbb{Z}/2\mathbb{Z})$ is the trivial subgroup of $\mathbb{Z}$. But the only homomorphism $\mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ is the trivial one, hence the necessary and sufficient condition holds, showing that f lifts.

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7. Let $[0,1]/\{0,1\}$ denote the quotient space obtained by starting with the usual unit interval and identifying the two endpoints. Let $t\colon [0,1]/\{0,1\} \to C$ denote a diffeomorphism $t \mapsto \big(x(t), y(t)\big)$ onto a closed curve C that lies in the plane.

- Show that the area enclosed by this curve is given by the line integral:

$$\frac{1}{2}\int_0^1 \big(x(t)y'(t) - x'(t)y(t)\big)\,\mathrm{d}t$$

- Given the above, find a 1-form $\omega \in \Omega^1(\mathbb{R}^2)$ such that

$$\int_C i^*(\omega) \neq 0,$$

  where i is the inclusion map $C \hookrightarrow \mathbb{R}^2$.
- Show that $i^*(\omega)$ is a closed 1-form and that $\mathrm{H}^1_{\mathrm{DR}}(C)$ is isomorphic to $\mathbb{R}\cdot[i^*(\omega)]$; that is, show that the first de Rham cohomology is generated by the class of $i^*(\omega)$.

SOLUTION:

- When the 1-form $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is defined on and inside the closed loop C, GREEN's theorem tells us that $\oint_C P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint (Q_x - P_y)\,\mathrm{d}A$, where the area integral is over the interior enclosed by the loop C. (I remember this theorem using the differential forms version of STOKES' theorem: $\int_{\partial W}(P\,\mathrm{d}x + Q\,\mathrm{d}y) = \int_W \mathrm{d}(P\,\mathrm{d}x + Q\,\mathrm{d}y)$.) For any of these integrals to compute the area enclosed by the loop C, I simply need $Q_x - P_y$ to identically equal the constant 1. There are two obvious ways of accomplishing this: let the original 1-form $P\,\mathrm{d}x + Q\,\mathrm{d}y$ either be $x\,\mathrm{d}y$ or $-y\,\mathrm{d}x$. Thus, the area enclosed by C is computed both by the path integral $\oint_C x\,\mathrm{d}y$ and by the path integral $\oint_C -y\,\mathrm{d}x$. When we parameterize the path using t, these path integrals turn into the real integrals $\int_0^1 x(t)y'(t)\,\mathrm{d}t$ and $\int_0^1 -y(t)x'(t)\,\mathrm{d}t$. Adding them gives twice the area, so we may also compute the area as half the sum of the integrals, as was to be shown.
- The form $\omega = x\,\mathrm{d}y - y\,\mathrm{d}x$ is a 1-form on $\mathbb{R}^2$ whose pullback $i^*(\omega)$ is $(x(t)y'(t) - x'(t)y(t))\,\mathrm{d}t$ in terms of the coordinate t on C. We just saw that $i^*(\omega)$ integrates to the area enclosed by C. This area is nonzero, for otherwise C couldn't be homeomorphic to a circle.
- Since C is 1-dimensional, the 1-form $i^*(\omega)$ *must* be closed (all top-dimensional forms are closed). We can also calculate this directly: in taking "d" of the form, we will get $\mathrm{d}t \wedge \mathrm{d}t$ appearing in each term, and this is 0. Observe that $\mathrm{d}\omega = 2\,\mathrm{d}x \wedge \mathrm{d}y$ is *not* zero (as a form on $\mathbb{R}^2$), but becomes zero when pulled back to C. If I show that $\mathrm{H}^1_{\mathrm{DR}}(C)$ is 1-dimensional, then it will be generated by any nonzero class, in particular by the class of $i^*(\omega)$ (which must be nonzero since its integral on C is nonzero). I will repeatedly show $\mathrm{DIM}(\mathrm{H}^1_{\mathrm{DR}}(C)) = 1$ by a variety of methods.

— Since C is topologically a circle, we could simply appeal to the fact that $\mathrm{DIM}(\mathrm{H}^1_{\mathrm{DR}}(S^1)) = 1$. But the latter can be shown using a MAYER–VIETORIS sequence. Let U and V denote upper and lower half-circles that overlap a little in a pair of segments. Then we have

$$0 \to \mathrm{H}^0(S^1) \to \mathrm{H}^0(U) \oplus \mathrm{H}^0(V) \to \mathrm{H}^0(U \cap V) \to \mathrm{H}^1(S^1) \to \mathrm{H}^1(U) \oplus \mathrm{H}^1(V) \to \cdots$$

Counting components tells us that the dimensions in the first three positions are 1, 2, and 2. Since U and V are each contractible to a point, their cohomology in dimension 1 vanishes. The exactness of the sequence then forces the dimension of $\mathrm{H}^1_{\mathrm{DR}}(S^1)$ to be 1.

— I will prove directly that $\mathrm{DIM}(\mathrm{H}^1_{\mathrm{DR}}(C)) = 1$ by proving that we have an exact sequence

$$0 \to \text{EXACT 1-FORMS} \to \text{CLOSED 1-FORMS} \to \mathbb{R} \to 0.$$

The latter map is defined by $\eta \mapsto \int_C \eta$. Since we have previously exhibited a form integrating to a nonzero value, by observing $\alpha\eta \mapsto \alpha \int_C \eta$ we see that the map is surjective. We know that exact forms are closed, so the first map, being simply the inclusion map, is injective. By STOKES' theorem, exact forms integrate to 0. The final step is to show that any form that integrates to 0 must be exact. So let $\eta = f(t)\,\mathrm{d}t$ be some fixed but arbitrary closed 1-form with $\int_C \eta = 0$. I claim that η is exact. Define the function $g(t) = \int_0^t f(t)\,\mathrm{d}t$. By the fundamental theorem of calculus, we have $g'(t) = f(t)$. Hence taking "d" of the 0-form $g(t)$ gives me $\mathrm{d}(g(t)) = g'(t)\,\mathrm{d}t = f(t)\,\mathrm{d}t = \eta$, and thus η is exact.

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8. Define $\mathbb{RP}^2$ and give an open covering of $\mathbb{RP}^2$ for which the transition functions are C^∞. Conclude that $\mathbb{RP}^2$ is a manifold.

SOLUTION:

The real projective plane is obtained by starting with the manifold S^2 and forming the quotient modulo the following equivalence relation: antipodal points are defined to be equivalent. Thus each point $[x, y, z] \in \mathbb{RP}^2$ is an equivalence class comprising the two points $(x, y, z) \in S^2$ and $(-x, -y, -z) \in S^2$; note, in particular, that $x^2 + y^2 + z^2 = 1$.

For any point $[x, y, z] \in \mathbb{RP}^2$, at least one of x, y, z must be nonzero. I will cover $\mathbb{RP}^2$ with three open sets, namely the three loci for which, respectively, $x \neq 0$, $y \neq 0$, and $z \neq 0$. Here are the three charts:

$$\begin{aligned}
\mathbb{R}^2 &\to S^2 && \to \mathbb{RP}^2 \\
(y, z) &\mapsto (\sqrt{1 - y^2 - z^2}, y, z) && \mapsto [\sqrt{1 - y^2 - z^2}, y, z] \\
(x, z) &\mapsto (x, \sqrt{1 - x^2 - z^2}, z) && \mapsto [x, \sqrt{1 - x^2 - z^2}, z] \\
(x, y) &\mapsto (x, y, \sqrt{1 - x^2 - y^2}) && \mapsto [x, y, \sqrt{1 - x^2 - y^2}]
\end{aligned}$$

In the above, there were really three different copies of coordinate space $\mathbb{R}^2$. In each case, the map is defined on the open unit disc; so, for example, the first map is defined when $y^2 + z^2 < 1$. One has to be careful in describing the inverse maps, because a careful choice must be made. The inverse of the first map shown above is:

$$\mathbb{RP}^2 \to \mathbb{R}^2$$
$$[x, y, z] \mapsto \begin{cases} (y, z), & \text{if } x > 0; \\ (-y, -z), & \text{if } x < 0. \end{cases}$$

The others are similar. Now let us look at the transition functions. The intersection of the first two cover sets is the locus defined by $x \neq 0 \neq y$. The map from (y, z)-space to (x, z)-space is given by:

$$(y, z) \mapsto [\sqrt{1 - y^2 - z^2}, y, z] \mapsto \begin{cases} (\sqrt{1 - y^2 - z^2}, z), & \text{if } y > 0; \\ (-\sqrt{1 - y^2 - z^2}, -z), & \text{if } y < 0. \end{cases}$$

This map is clearly a smooth diffeomorphism from $\{ (y, z) : y^2 + z^2 < 1, \, y \neq 0 \}$ onto $\{ (x, z) : x^2 + z^2 < 1, \, x \neq 0 \}$. The other transition functions are similar. Having exhibited a differential structure on $\mathbb{RP}^2$, I conclude that $\mathbb{RP}^2$ is a differentiable manifold.