[COMMENTS, QUESTIONS, JOKES TO ALEXANDER PERLIS, "aprl@math.arizona.edu".]

Noetherian rings

The following equivalent properties characterize a **noetherian** ring:

- The ring is a noetherian module over itself.
- Every ideal is finitely generated.
- Every ascending chain of ideals stabilizes or terminates.
- With respect to set inclusion, every collection of ideals has maximal elements.

A quotient of a noetherian ring is noetherian. (Although trivial, this does not follow from a similar result that any submodule of a noetherian module is noetherian. If the ring A is noetherian, then A is a noetherian A-module, and thus any quotient of A is also a noetherian A-module, but showing that it is a noetherian module over itself requires an argument.)

A subring of a noetherian ring is *not* necessarily noetherian.

If A is noetherian, then $A[X]$ is noetherian; this fact is known as the Hilbert Basis Theorem.

If A is noetherian, then $S^{-1}A$ is noetherian, where $S \subseteq A$ is a multiplicatively closed subset. In particular, if A is noetherian, then the localization $A_{\mathfrak{p}}$ is noetherian, where $\mathfrak{p} \subset A$ is any prime ideal of A.

If A is noetherian, then so is any ring that is ring-finite (finitely generated as a ring) over A.

Corollaries

Every principal ring, in particular every PID, is noetherian.

A ring that is finitely generated over $\mathbb{Z}$, i.e., any quotient of $\mathbb{Z}[X_1, \ldots, X_n]$, is noetherian.

A ring that is finitely generated over a field, i.e., any quotient of $k[X_1, \ldots, X_n]$, is noetherian.

Modules

Herein lies a summary of various facts about modules. Throughout, let A be a ring (commutative with identity), and let M be an A–module. The general theory, which lacks elegance, becomes manageable when additional assumptions are placed on A. The most common cases that occur in practice, in increasing order of simplicity, are that A is noetherian, a PID, or a field. After summarizing the general theory below, I shall summarize the common special cases.

Rank

The underlying set of M is certainly a generating set for the module; however, smaller generating sets often exist. The **rank** of M is the minimum number of generators. The module M is **finitely generated** when $\mathrm{Rank}(M) < \infty$.

Given a module M and a submodule $N \subseteq M$, you can always form the quotient module M/N. Projecting a generating set of M into the quotient gives a generating set (possibly unnecessarily large) of the quotient. Combining a generating set for N with representatives of a generating set for M/N gives a generating set for M. Thus there is the following general inequality that holds for any module M and any submodule $N \subseteq M$:

$$\mathrm{Rk}(M/N) \leq \mathrm{Rk}(M) \leq \mathrm{Rk}(N) + \mathrm{Rk}(M/N).$$

The inequality allows the rank of the submodule N to exceed the rank of the module; this *does* occur. (Make an example where N has larger but still finite rank, and an example where N has infinite rank.) The following relationship holds when A is a PID, but not in general:

$$\mathrm{RANK}(N) \le \mathrm{RANK}(M).$$

Annihilators, torsion

The **annihilator** of an element $x \in M$ is the ideal $\mathrm{ANN}(x) = \{\, a \in A \mid \mathrm{ax} = 0 \,\}$. The element $x \in M$ is a **torsion** element if its annihilator is not trivial. The torsion elements of M compose the **torsion submodule**. If the only torsion element of M is the zero element, or equivalently if the torsion submodule is trivial, then M is **torsion-free**.

The module M/M_τ is always torsion-free, where M_τ denotes the torsion submodule of M.

If M is torsion-free, then any submodule of M is also torsion-free.

Note that, in all books known to me, torsion is discussed only when the ring A is assumed to be an integral domain. I don't know why. There are two reasons a ring A fails to be an integral domain: it may be the zero ring, or it may contain proper zero-divisors. If A is the zero ring, then every A–module is torsion-free.† If A contains proper zero-divisors, then every nonzero A–module has torsion elements: say $ab = 0$ with nonzero $a, b \in A$, and $x \in M$ with $x \neq 0$; if $bx = 0$ then x has torsion; else $a(bx) = 0$ so that bx has torsion.

Linear independence

The elements $v_1, \dots, v_n \in M$ are **linearly independent** when $a_1 v_1 + \cdots + a_n v_n = 0$ implies $a_1 = \cdots = a_n = 0$, where $a_i \in A$. The elements $v_1, \dots, v_n \in M$ are a **spanning set** or **generating set**, or **span** or **generate** the module M, when they generate M, i.e., every element of M is a linear combination (not necessarily unique) of $v_1, \dots, v_n$. The elements $v_1, \dots, v_n \in M$ are a **basis** when they are a linearly independent spanning set; this is equivalent to every element of M being a unique linear combination of $v_1, \dots, v_n$. Note that a module does not necessarily have a basis—see the next section.

The concepts from the previous paragraph extend to infinite collections. The subset $S \subseteq M$ is linearly independent when every *finite* subset of S is linearly independent. The subset S spans when it generates M, i.e., every element of M is a *finite* linear combination (not necessarily unique) of elements of S. The subset S is a basis when it is linearly independent and spans; this is equivalent to every element of M being a unique *finite* linear combination of elements of S.‡

In determining the rank of a module, one often constructs generating sets and shows, somehow, that no other generating set can be smaller, so that the size of the constructed generating set must be the rank of the module. If the generating set is linearly independent, then one has also found a basis. Various questions come to mind; for example, could there be a generating set of a certain size, but a linearly independent generating set of a larger size? (The answer is no.) The following fact, and the results in the

† Discussing modules over the zero ring is mostly nonsense in the first place, unless the module is trivial. In working with a module over a commutative ring with identity, one usually assumes the module to be unitary, i.e., $1 \cdot x = x$ for all x in the module. But if $1 = 0$ and $x \neq 0$, then there is a contradiction.

‡ There are other ways to generalize definitions from finite situations to infinite situations, especially when additional structure is available. For example, if some means of defining convergence of series is available, then an infinite basis could be a set of elements so that any element is a unique infinite series of the basis. This approach is not taken in algebra.

next section, help to answer such questions.

If $S \subseteq M$ contains more elements than $\textsc{Rank}(M)$, then S is linearly dependent.

Free modules

A module that has a basis is **free**, and the trivial module, even though it has no basis, is also considered to be free. Free modules may possess many bases, and certainly may possess even more generating sets. Fortunately no generating set is smaller than a basis, and every basis has the same number of elements, namely whatever the rank of the module is. For free modules, the rank is also called the **dimension**.

TO FIGURE OUT: I don't know if any spanning set with $\textsc{Rank}(M)$ elements is a basis, or whether any set of $\textsc{Rank}(M)$ linearly independent elements is a basis. (This is what happens with vector spaces.)

There is no relationship, in general, between a module being free and/or torsion-free. (But see the section on integral domains.)

[PUT HERE AN EXAMPLE OF A FREE BUT NOT TORSION-FREE MODULE, AND AN EXAMPLE OF A TORSION-FREE BUT NOT FREE MODULE.]

A submodule of a free module is not free, in general. However, over a PID, a submodule of a free module is free.

If M is a free A–module and $\mathfrak{a} \subseteq A$ is an ideal, then $\mathfrak{a}M$ is a free $(A/\mathfrak{a})$–module.

A module is free when its elements possess as few relations as possible (under the constraint that the rank and underlying ring do not change). Any other module, whose elements possess various relations, may be obtained by starting with a free module and introducing the appropriate relations. The precise statement is that every A–module M is a quotient of a free A–module F, where F may be chosen so that its rank is no larger than the rank of M. Let $K \subseteq F$ be the kernel of the quotient map. The following exact sequence, which always exists, is the **free presentation** of M:

$$0 \to K \to F \to M \to 0.$$

Projective modules

The module M is **projective** when $M \oplus M'$ is free for some M'. There are other equivalent characterizations; the most important is that M is projective if and only if a short exact sequence with M in the last position splits.

Every free module is projective, but the converse is not true in general. However, over a PID, every projective module is free, so that the notions coincide.

A direct sum of modules is projective if and only if every term of the sum is projective.

Noetherian modules

The following equivalent properties characterize a **noetherian** module:

- Every submodule is finitely generated.
- Every ascending chain of submodules stabilizes or terminates.
- With respect to set inclusion, every collection of submodules has maximal elements.

Let $0 \to M' \to M \to M'' \to 0$ be exact; then M is noetherian if and only if M', M'' are noetherian.

Modules over integral domains

Over an integral domain, if M is free, then M is torsion-free. Over a PID, the converse holds as well.

[EXAMPLE OVER INTEGRAL DOMAIN OF TORSION-FREE BUT NOT FREE.]

Modules over noetherian rings

If M is finitely generated over a noetherian ring, then M itself is noetherian.

Modules over PIDs

Although, in general, there is no control over the rank of a submodule, over a PID the rank of a submodule can only shrink.

Although, in general, there is no control over the freeness of a submodule, over a PID a submodule of a free module is free.

Over a PID, the concepts "free" and "projective" are equivalent.

Over a PID, a finitely generated module is free if and only if it is torsion-free; this follows from the structure theorem, below. Thus, over a PID, the concepts "free" and "torsion-free" are almost equivalent. [WHAT ARE THE RESULTS FOR NON-FINITELY-GENERATED MODULES?]

Finitely generated modules over a PID may be described as follows. They are a direct sum of cyclic submodules (meaning rank one), where the annihilators of the cyclic submodules form an increasing chain of ideals. The cyclic submodule Ax is isomorphic, as a module over A, to $A/\text{ANN}(x)$, and thus a finitely generated module over a PID may be written as

$$M \cong \frac{A}{\mathfrak{a}_1} \oplus \frac{A}{\mathfrak{a}_2} \oplus \cdots \frac{A}{\mathfrak{a}_n},$$

where the ideals $\mathfrak{a}_i$ satisfy $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_n$. So long as all the ideals are proper (so that unnecessary zero terms do not appear in the direct sum), the rank of M is n. Some of the ideals may be trivial, thus separating the cyclic submodules into two types, namely the free ones and the torsion ones. After separating the free part and torsion part, a finitely generated module over a PID may be written as

$$M \cong A^d \bigoplus \frac{A}{\mathfrak{a}_1} \oplus \frac{A}{\mathfrak{a}_2} \oplus \cdots \oplus \frac{A}{\mathfrak{a}_k},$$

where the $\mathfrak{a}_i$ are nonzero proper ideals satisfying $\mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots \subseteq \mathfrak{a}_k$, and $\text{RANK}(M) = n = d + k$. The number d is the **free rank** of M.

Modules over fields

A module over a field is also called a **vector space**.

Every module over a field is free; that is, every vector space has a basis.

Corollaries (or Exercises)

Many simple facts follow immediately from the inequalities and statements given earlier. Listing these facts in context unnecessarily inflates the general theory, making it appear too complicated. I list these facts below. Proving each of the facts below in terms of the facts above is, perhaps, a good way to solidify your mental organization of the general theory.

Let $N \subseteq M$. If N and M/N are finitely generated, then M is finitely generated.

A finitely generated torsion-free module over a PID is free.

A finitely generated module over a PID may be decomposed as $M \cong M_\tau \oplus (M/M_\tau)$.

Quotients and submodules of noetherian modules are noetherian.

Let $M = \bigoplus M_i$. If M is free, then each M_i is projective. If each M_i is free, then M is projective.

Over a PID, the direct sum $M = \bigoplus M_i$ is free if and only if each M_i is free.

If F is free, then the exact sequence $0 \to K \to M \to F \to 0$ splits.

If the module M has 18 linearly independent elements, then the rank of M is at least 18.

If the submodule $F \subseteq M$ is free, then $\text{DIM}(F) \leq \text{RANK}(M)$.