PRODUCT MEASURE AND THE FUBINI THEOREM

PRODUCT MEASURE

Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be any two measure spaces. In $X \times Y$ let $\mathcal{R}$ be the collection of all "measurable rectangles", i.e., sets $B \times C$ with $B \in \mathcal{B}$ and $C \in \mathcal{C}$. For such sets define $\rho(B \times C) = \mu(B)\nu(C)$ when it makes sense; otherwise, namely when one of the sets has measure 0 and the other has measure ∞, define $\rho(B \times C) = 0$.

The collection $\mathcal{R}$ is a semiring, and ρ turns out to be countably additive on $\mathcal{R}$; but, ρ is not yet a measure because $\mathcal{R}$ is generally not a σ-algebra. However, any countably additive function on a semiring extends to a measure on the σ-algebra generated by that semiring; but, the extension is generally not unique. Yet the extension must be unique up to a point: the ring $\mathcal{A}$ generated by the semiring $\mathcal{R}$ comprises all finite disjoint unions of sets in $\mathcal{R}$, and the extension of ρ to such sets is unique: $\rho(\bigcup R_i) = \sum \rho(R_i)$. Now, generally speaking, semirings generate rings, rings generate algebras, and algebras generate σ-algebras; in our case, our ring A is already an algebra. Thus the matter at hand concerns the extension of ρ to the σ-algebra generated by $\mathcal{A}$ (also generated directly by $\mathcal{R}$); that σ-algebra is called the *product σ-algebra*, denoted $\mathcal{B} \times \mathcal{C}$. We seek conditions under which ρ has a unique extension to $\mathcal{B} \times \mathcal{C}$; when this is the case, the extension is denoted $\mu \times \nu$ and is called the *product measure* of μ and ν.

If μ and ν are each σ-finite, then the product measure $\mu \times \nu$ exists (i.e., ρ has a unique extension to $\mathcal{B} \times \mathcal{C}$). To calculate $(\mu \times \nu)(E)$ for any $E \in \mathcal{B} \times \mathcal{C}$:

- If you can express E as a finite *disjoint* union of measurable rectangles, then perform a straightforward computation:

$$(\mu \times \nu)\big((B_1 \times C_1) \cup (B_2 \times C_2)\big) = \mu(B_1)\nu(C_1) + \mu(B_2)\nu(C_2).$$

- In any case, you may always calculate the measure using an iterated integral in either order:

$$(\mu \times \nu)(E) = \iint 1_E(x,y)\, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) = \iint 1_E(x,y)\, \mathrm{d}\nu(y)\, \mathrm{d}\mu(x).$$

TONELLI–FUBINI THEOREM

Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ each be σ-finite, and let f be a function on $X \times Y$ that is measurable with respect to the product σ-algebra $\mathcal{B} \times \mathcal{C}$. If f is either nonnegative or integrable, then

$$\int f\, \mathrm{d}(\mu \times \nu) = \iint f(x,y)\, \mathrm{d}\mu(x)\, \mathrm{d}\nu(y) = \iint f(x,y)\, \mathrm{d}\nu(y)\, \mathrm{d}\mu(x).$$

In particular, the inside integrals make sense almost everywhere; that is, $\int f(x,y)\, \mathrm{d}\mu(x)$ is defined for ν-almost all y, and $\int f(x,y)\, \mathrm{d}\nu(y)$ is defined for μ-almost all x.

Having checked that you are working with σ-finite measure spaces, applying the theorem to a particular function comes down to checking the function's measurability, and, in the case that the function takes on negative values, its integrability. For the former, you must employ the usual results: all continuous functions are measurable, limits of sequences of measurable functions are again measurable, and so on. For the latter, here is the key fact:

The function f is integrable if and only if both of the following hold:

- f is measurable (with respect to $\mathcal{B} \times \mathcal{C}$);
- one of the iterated integrals $\iint |f|\, \mathrm{d}\mu\, \mathrm{d}\nu$ or $\iint |f|\, \mathrm{d}\nu\, \mathrm{d}\mu$ is finite.

This key fact is really the following special application of TONELLI–FUBINI. Recall, if a function f is measurable, then so is the function $|f|$. Since $|f|$ is nonnegative, we see from TONELLI–FUBINI that if, say, $\iint |f|\, \mathrm{d}\mu\, \mathrm{d}\nu < \infty$, then $\int |f|\, \mathrm{d}(\mu \times \nu) < \infty$, showing f to be integrable.

Two standard examples (which you should look up, study, and memorize) show that the iterated integrals in the theorem's conclusion may each be finite yet distinct in the two cases that violate the hypotheses: f is nonnegative but not measurable, and f is measurable but neither nonnegative nor integrable.

COMPLETION AND LEBESGUE MEASURE

Even when μ and ν are each complete (which means that all subsets of measure 0 sets are measurable), the product measure $\mu \times \nu$ generally fails to be complete: let $\mu(B) = 0$ with $B \in \mathcal{B}$ nonempty, and let $E \subset C \in \mathcal{C}$, where E is *not* measurable with respect to ν; then $B \times E$ is a nonmeasurable subset of the set $B \times C$, which has product measure 0. Now given any measure space that is not complete, one may "complete" the space by enlarging the σ-algebra to the smallest σ-algebra containing both the old σ-algebra and all subsets of measure 0 sets.

Usual 1-dimensional LEBESGUE measure λ is complete. Some authors define k-dimensional LEBESGUE measure to be the product of lower-dimensional Lebesgue measures: $\lambda^k = \lambda^{k-1} \times \lambda$. Many authors define k-dimensional LEBESGUE measure as the *completion* of that product. The reason for preferring the latter is that it agrees with the direct development of planar and spatial LEBESGUE measure in which one starts with the outer measure for area or for volume, and then uses the CARATHÉODORY criterion to restrict to a suitable class of measurable sets. (The measure obtained by this process is *always* complete.)

SUBTLETIES

Completing a measure space introduces a twist: because you are increasing the number of measurable sets, functions that were nonmeasurable before might now be measurable. That might sound desirable, but the conclusions of some theorems are affected by this in subtle ways. In fact, the TONELLI–FUBINI theorem has this affliction: the hypotheses and conclusions (and even the name of the theorem!) do not agree from author to author. *In the form stated here, the theorem remains correct when $\mathcal{B} \times \mathcal{C}$ and $\mu \times \nu$ denote the completed product σ-algebra and the completed product measure.*

To illustrate the subtleties, consider the first of the two iterated integrals in the TONELLI–FUBINI theorem. When working with the product σ-algebra, for each fixed y, the inner function $f(x, y)$ is measurable (as a function of x alone), whereas when working with the completion of the product σ-algebra, the inner function is measurable only for almost all y. Hence, in the case of the completed product σ-algebra, for a few fixed y, the inner integral $\int f(x, y)\, d\mu(x)$ is nonsense; in general, the inner integral may make sense (because f is measurable for that particular y) but nonetheless be infinite. Regardless, the statement of TONELLI–FUBINI, in the form given here, is correct: after tossing out a set of measure 0, for each remaining y the inner integral makes sense, and furthermore the entire inner integral is always measurable (as a function of y), so that the outer integral always makes sense (but might be infinite).

HISTORY

LEBESGUE (1904) proved that the order of integration does not matter for bounded measurable functions (with respect to LEBESGUE measure?). Supposedly this had been known since CAUCHY's time for the integration of continuous functions with respect to LEBESGUE measure. FUBINI (1907) stated more general conditions under which the order of integration with respect to LEBESGUE measure does not matter; FUBINI's proof is supposedly incorrect. TONELLI (1909) gave a correct proof of FUBINI's result (and extended it to measures other than LEBESGUE?).

LITERATURE

Unfortunately, many texts disagree on definitions and disagree on names and statements of important theorems. Your best bet is to learn the subtleties of each approach. The following guide might help.

ROYDEN's development of product measure is questionable because he discusses "the" product measure even when μ and ν are not assumed to be σ-finite. Furthermore, he considers only completed product measures: his "product measure $\mu \times \nu$" is really the *completion* of our $\mu \times \nu$ here. Thus, in his usage, the product of LEBESGUE measure with itself gives back higher-dimensional LEBESGUE measure. He handles the TONELLI–FUBINI theorem separately for nonnegative measurable f and for integrable f, so his setup is ever-so-slightly more general but somewhat confusing. His discussion [ROYDEN, page 309] is worth reading.

RUDIN's development is somewhat technical: his *definition* of $(\mu \times \nu)(E)$ is the very same integral that is given here as a last-resort calculation technique. Because he covers all bases and squeezes out the strongest possible results, he has two versions of TONELLI–FUBINI, one for the product σ-algebra, another for its completion, and the conclusions of each theorem depend slightly on whether f is nonnegative or integrable. His counterexamples [RUDIN, page 166] are worth studying.

DUDLEY's development is almost identical with the one here, but of course he includes proofs and examples. Having read this, his development [DUDLEY, §4.4, page 102] is worth digesting. Presumably to avoid confusion with cartesian products, he uses "$\mathcal{B} \otimes \mathcal{C}$" and "$\mu \otimes \nu$" for the product σ-algebra and product measure. He differs from ROYDEN and RUDIN (and likely many others) in defining higher-dimensional LEBESGUE measure as the (uncompleted) product of LEBESGUE measure with itself. His historical notes [DUDLEY, page 113] are delightful, and are the basis for my summary above.

BIBLIOGRAPHY

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ROYDEN, H.L. *Real analysis, third edition.* Macmillan, 1988.
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