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Translated and annotated by Alexander R. Perlis, The University of Arizona, August 1998. Material in brackets and footnotes was added by the translator. Please send comments and corrections to aprl@math.arizona.edu. This is the 25 October 1998 version of this translation. The latest version is available from the translator's webpage: www.math.arizona.edu/~aprl.

[1864] On the hyperosculating planes of space curves

(By Mister *A. Clebsch* at Giessen.)

In his excellent work on the analytic geometry of space, p. 266†, Mister *Salmon* comments that he was unable to put into simple form the equation of the surface that meets the intersection of two surfaces, of degrees m and n, in those points that lie together with the three closest to them in a plane. The osculating planes of these points go through *four* consecutive points, and, due to the analogie with inflectionary tangents of plane curves, are called inflectionary planes [or hyperosculating planes]. The study of the aforementioned work left me the opportunity to take up the present investigation, and the transformations, which I carry out below, might not be without interest for the general theory of surfaces. **[Not translated literally, but is is accurate enough to stand?]**

Let $u = 0$ be a surface of degree m and $v = 0$ a surface of degree n. We shall use u_i, v_i, u_{ik}, v_{ik} to denote the quantities:

$$(1.)\quad u_i = \frac{1}{m}\frac{\partial u}{\partial x_i},\quad v_i = \frac{1}{n}\frac{\partial v}{\partial x_i},\quad u_{ik} = \frac{1}{m.m{-}1}\frac{\partial^2 u}{\partial x_i\,\partial x_k},\quad v_{ik} = \frac{1}{n.n{-}1}\frac{\partial^2 v}{\partial x_i\,\partial x_k},$$

where x_1, x_2, x_3, x_4 denote the coordinates of a point x. One has then identically:

$$(2.)\quad \begin{cases} u_i = \sum\limits_k u_{ik}\,x_k, & u = \sum u_k\,x_k, \\ v_i = \sum\limits_k v_{ik}\,x_k, & v = \sum v_k\,x_k. \end{cases}$$

Mister *Hesse* (this journal, vol. 41, p. 272) showed how to put the equation of the osculating

† Clebsch is referring to the original edition (18??) of George Salmon's *A Treatise on the Analytic Geometry of Three Dimensions*. The second edition (1865) incorporated the results of the present paper into article 357, page 291. In the most recent edition (seventh edition, 1928, reprinted by Chelsea New York, 1958), the results are in article 363, page 380. [CHECK]

plane of a point x, which lies in the curve of intersection of both surfaces, in the form‡

$$(3.)\quad 0 = U(v_1X_1 + v_2X_2 + v_3X_3 + v_4X_4) - V(u_1X_1 + u_2X_2 + u_3X_3 + u_4X_4),$$

where U and V denote the covariants:

$$(4.)\quad U = (m-1)\begin{vmatrix} u_{11} & u_{12} & u_{13} & u_{14} & v_1 \\ u_{21} & u_{22} & u_{23} & u_{24} & v_2 \\ u_{31} & u_{32} & u_{33} & u_{34} & v_3 \\ u_{41} & u_{42} & u_{43} & u_{44} & v_4 \\ v_1 & v_2 & v_3 & v_4 & 0 \end{vmatrix}, \quad V = (n-1)\begin{vmatrix} v_{11} & v_{12} & v_{13} & v_{14} & u_1 \\ v_{21} & v_{22} & v_{23} & v_{24} & u_2 \\ v_{31} & v_{32} & v_{33} & v_{34} & u_3 \\ v_{41} & v_{42} & v_{43} & v_{44} & u_4 \\ u_1 & u_2 & u_3 & u_4 & 0 \end{vmatrix}.$$

First of all, I will again derive the equation (3.), since the method of *Hesse* can perhaps be shortened in some points. The three consecutive points of the curve shall be x, $x+dx$, $x+2\,dx+d^2x$; then the equation§ of the osculating plane is:

$$(5.)\quad 0 = \begin{vmatrix} X_1 & X_2 & X_3 & X_4 \\ x_1 & x_2 & x_3 & x_4 \\ dx_1 & dx_2 & dx_3 & dx_4 \\ d^2x_1 & d^2x_2 & d^2x_3 & d^2x_4 \end{vmatrix}.$$

[Clebsch's next goal is to eliminate the differentials from this equation.] By differentiating $u = 0$, $v = 0$, one obtains the following relation between dx and d^2x:

$$(6.)\quad 0 = \sum u_i\,dx_i, \quad 0 = \sum v_i\,dx_i,$$

$$(7.)\quad \begin{cases} 0 = \sum u_i\,d^2x_i + (m-1)\sum\sum u_{ik}\,dx_i\,dx_k, \\ 0 = \sum v_i\,d^2x_i + (n-1)\sum\sum v_{ik}\,dx_i\,dx_k. \end{cases}$$

‡ As Hesse observes in his article, the osculating plane can be written in the form $\mu\sum v_iX_i - \lambda\sum u_iX_i$. To see this, observe that the curve's osculating plane at any point must contain the curve's tangent line at that point. Now the tangent line is the intersection of the planes $\sum v_iX_i$ and $\sum u_iX_i$, these being the tangent planes to the two surfaces u and v. Any plane containing the tangent line is some member of the "projective pencil" $\mu\sum v_iX_i - \lambda\sum u_iX_i$. Thus the search for the osculating plane comes down to determining two coefficients, or more precisely, to determining their ratio.

§ I'm not entirely certain how Clebsch intends the symbols "dx" and "d^2x" to be interpreted; I shall now explain my own interpretation. I think of the curve of intersection as being parameterized by some parameter t; thus $x = x(t)$ is a vector-valued function of t, so that $\partial x/\partial t$ gives the tangent vector at each point. I believe that Clebsch uses "dx" as shorthand for this tangent vector. Then $x+dx$ denotes the result of moving x in the direction of $\partial x/\partial t$. The parameterized curve $x + dx$ has $(\partial/\partial t)(x + dx) = dx + d^2x$ as its tangent vector, so that the result of moving $x + dx$ in *that* direction is $x + dx + (\partial/\partial t)(x + dx) = x + 2dx + d^2x$.

A different interpretation is to let $\partial x/\partial t = dx/dt$, so that dx is an "infinitesimal" tangent vector. Then $x + dx$ denotes the "next consecutive" point on the curve after x. Similarly, $x + dx + d(x + dx) = x + 2\,dx + d^2x$ denotes the "next consecutive" point after that.

I would be much obliged if the reader would share some insight into the matter above.

Given three linearly independent points in projective space, the plane determined by them comprises all points X that lie in their span. The points x, $x + dx$, $x + 2\,dx + d^2x$ are generically linearly independent; demanding that X lie in their span is equivalent to forcing the determinant

$$\begin{vmatrix} X_1 & X_2 & X_3 & X_4 \\ x_1 & x_2 & x_3 & x_4 \\ x_1 + dx_1 & x_2 + dx_2 & x_3 + dx_3 & x_4 + dx_4 \\ x_1 + 2\,dx_1 + d^2x_1 & x_2 + 2\,dx_2 + d^2x_2 & x_3 + 2\,dx_3 + d^2x_3 & x_4 + 2\,dx_4 + d^2x_4 \end{vmatrix}$$

to vanish identically. By performing some obvious row operations, this determinantal condition is simplified to the equation (5.) given by Clebsch.

Since one may assume† an additional equation

$$k_1x_1 + k_2x_2 + k_3x_3 + k_4x_4 = 1$$

between the homogeneous coordinates, we additionally obtain the following equations:

$$(8.)\quad \sum k_i\,dx_i = 0, \quad \sum k_i\,d^2x_i = 0,$$

and one sees that the quantities dx_i become proportional with the determinants‡ formed from the system

$$(9.)\quad \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ k_1 & k_2 & k_3 & k_4 \end{vmatrix}.$$

If one multiplies the determinant of (5.) with the [transpose of the] determinant

$$\begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{vmatrix},$$

where α, β denote arbitrary quantities, then the product is the determinant

$$\begin{vmatrix} \sum u_i\,X_i & \sum v_i\,X_i & \sum \alpha_i\,X_i & \sum \beta_i\,X_i \\ 0 & 0 & \sum \alpha_i\,x_i & \sum \beta_i\,x_i \\ 0 & 0 & \sum \alpha_i\,dx_i & \sum \beta_i\,dx_i \\ \sum u_i\,d^2x_i & \sum v_i\,d^2x_i & \sum \alpha_i\,d^2x_i & \sum \beta_i\,d^2x_i \end{vmatrix}.$$

Thus, with the removal of an unnecessary factor, the equation (5.) takes the form§

$$\sum v_i\,X_i.\sum u_i\,d^2x_i - \sum u_i\,X_i.\sum v_i\,d^2x_i = 0,$$

or in light of (7.):

$$(10.)\quad 0 = (m-1)\sum\sum u_{ik}\,dx_i\,dx_k.\sum v_i\,X_i - (n-1)\sum\sum v_{ik}\,dx_i\,dx_k.\sum u_i\,X_i.$$

† Forcing such an equation is part of the process of "dehomogenization". Clebsch isn't actually dehomogenizing; instead, he is merely observing that some such additional equation always holds locally.

‡ Here Clebsch is referring to the 3×3 subdeterminants together with appropriate choice of signs; these determinants are certain cofactors of a 4×4 determinant, as follows. Equations (6.) and (8.) state that dx is orthogonal to u, v, and k; thus dx is orthogonal to the plane spanned by u, v, and k, which is the plane comprising all points X satisfying

$$0 = \begin{vmatrix} X_1 & X_2 & X_3 & X_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ k_1 & k_2 & k_3 & k_4 \end{vmatrix} = X_1 \begin{vmatrix} u_2 & u_3 & u_4 \\ v_2 & v_3 & v_4 \\ k_2 & k_3 & k_4 \end{vmatrix} - X_2 \begin{vmatrix} u_1 & u_3 & u_4 \\ v_1 & v_3 & v_4 \\ k_1 & k_3 & k_4 \end{vmatrix} + X_3 \begin{vmatrix} u_1 & u_2 & u_4 \\ v_1 & v_2 & v_4 \\ k_1 & k_2 & k_4 \end{vmatrix} - X_4 \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ k_1 & k_2 & k_3 \end{vmatrix}.$$

Letting C_i denote the cofactors obtained by expanding the 4×4 determinant along the top row, as shown above, we have $0 = \sum X_i C_i$, which says that the projective point $C = [C_1, C_2, C_3, C_4]$ is orthogonal to each point X in the plane. Thus the projective points C and dx must be one and the same. Said another way, the quantities dx_i are proportional to the quantities C_i.

§ Since the extra determinant introduced by Clebsch is generically nonzero, equation (5.) is not affected by its introduction. The "unnecessary factor" referred to by Clebsch is the generically nonzero 2×2 subdeterminant $\begin{vmatrix} \Sigma\,\alpha_i\,x_i & \Sigma\,\beta_i\,x_i \\ \Sigma\,\alpha_i\,dx_i & \Sigma\,\beta_i\,dx_i \end{vmatrix}$. To see how this factor appears, either expand the 4×4 determinant down the left column, or more easily, swap rows and use the following fact for block upper triangular matrices: $\begin{vmatrix} A & B \\ 0 & D \end{vmatrix} = |A| \cdot |D|$.

If we introduce in place of dx_i, dx_k values proportional to them, then the following determinants take the place of the double sums:†

$$(11.)\quad \begin{vmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_1 & v_1 & k_1 \\ u_{21} & u_{22} & u_{23} & u_{24} & u_2 & v_2 & k_2 \\ u_{31} & u_{32} & u_{33} & u_{34} & u_3 & v_3 & k_3 \\ u_{41} & u_{42} & u_{43} & u_{44} & u_4 & v_4 & k_4 \\ u_1 & u_2 & u_3 & u_4 & 0 & 0 & 0 \\ v_1 & v_2 & v_3 & v_4 & 0 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 & 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} v_{11} & v_{12} & v_{13} & v_{14} & v_1 & u_1 & k_1 \\ v_{21} & v_{22} & v_{23} & v_{24} & v_2 & u_2 & k_2 \\ v_{31} & v_{32} & v_{33} & v_{34} & v_3 & u_3 & k_3 \\ v_{41} & v_{42} & v_{43} & v_{44} & v_4 & u_4 & k_4 \\ v_1 & v_2 & v_3 & v_4 & 0 & 0 & 0 \\ u_1 & u_2 & u_3 & u_4 & 0 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 & 0 & 0 & 0 \end{vmatrix}.$$

To transform these, one must merely multiply the first four rows and columns with $x_1, x_2, \ldots$, and subtract from the fifth row and column. Then the rows and columns with u_i in the first matrix, with v_i in the second matrix, disappear, and in place of the zeros [in the bottom right corner] one has the systems

$$\begin{matrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 0, \end{matrix}$$

so that the determinants (11.) turn into $-\frac{U}{m-1}$, $-\frac{V}{n-1}$. The equation (10.) reduces itself this way immediately into the form given by *Hesse*.

Let us move on to the search of those points x, whose three consecutive points $x + dx$, $x + 2\,dx + d^2x$, $x + 3\,dx + 3\,d^2x + d^3x$ lie in one plane, the inflectionary [or hyperosculating] plane. First of all, this problem can be handled with direct use of the equation (3.), by searching for a point x in the curve of intersection, whose osculating plane is the same as the osculating plane of the next point $x + dx$. If the equation of the osculating plane at the point x is

$$0 = \sum X_i\,(Uv_i - Vu_i),$$

then the next osculating plane is:‡

$$0 = \sum X_i\,\big[(U + dU)(v_i + dv_i) - (V + dV)(u_i + du_i)\big];$$

and the requirement, that these planes shall be identical, is stated, with help of an arbitrary differential dt, by the four equations:§

$$(12.)\quad v_i\,dU + U\,dv_i - u_i\,dV - V\,du_i = (Uv_i - Vu_i)\,dt.$$

† Clebsch is claiming, for example, that $\sum\sum u_{ik}\,dx_i\,dx_k$ may be replaced with the horrendous determinant on the left. Here is one way to show this. Think of the horrendous determinant as a linear form on the 16-dimensional space dual to the one spanned by the symbols u_{ik}. (To see this, expand the determinant using the permutation sum definition; there is precisely one u_{ik} in each surviving term.) To obtain the coefficient of some u_{ik}, say of u_{23}, set the remaining u_{ik} to zero. Thus, for example, expanding the horrendous determinant along the second row (or down the third column), one obtains a certain 6×6 block diagonal determinant, where each block is one of the cofactors defined in a previous footnote. Thus the coefficient of u_{23} is $C_2\,C_3$, and the horrendous determinant as a whole is $\sum\sum u_{ik}\,C_i\,C_k$. But the C_i are proportional to the dx_i, say $C_i = \alpha\,dx_i$, so that the horrendous determinant is $\alpha^2 \sum\sum u_{ik}\,dx_i\,dx_k$. Similarly, the horrendous detereminant on the right is $\alpha^2 \sum\sum v_{ik}\,dx_i\,dx_k$. Hence the double sums in (10.) may be replaced with the two horrendous determinants, giving an equation equivalent to (5.) but with all differentials eliminated. The final step is to simplify the horrendous determinants, both to eliminate the k_i and to recover the equation (3.) given by Hesse.

‡ As before, think of all quantities, x, U, v_i, etc., as depending on a parameter t. Thus, whereas U is the value of a certain covariant at the point x, $U + dU$ is the value of that same covariant at the "next consecutive" point $x + dx$.

§ Two planes in projective space are identical precisely when the coefficients of their defining equations are proportional. The "arbitrary differential dt" is most likely simply an arbitrary nonzero constant; it is used as such in the sequel. But where does (12.) come from? We wish to demand that the two vectors from the previous two displays be linearly dependent; we

These are four linear equations for the five quantities dx_1, dx_2, dx_3, dx_4, dt, between which in addition also the equations (6.), (8.) stand. Thus, by eliminating the five quantities one is led to three equations, which completely determine the position of the sought point x.† But of these three equations, two of them are the equations $u = 0$, $v = 0$, which one may see thereby, that with the help of these two, the four equations (12.) reduce to only two distinct [i.e., independent] equations. Namely, since from the equations

$$\sum u_i\, dx_i = 0, \quad \sum v_i\, dx_i = 0$$

also these two others

$$\sum x_i\, du_i = 0, \quad \sum x_i\, dv_i = 0$$

follow;‡ and thus, if one multiplies the equations (12.) with x_1, x_2, x_3, x_4 and adds, the resulting sum disappears. But if one instead multiplies with dx_1, etc., and adds, then comes:§

$$\begin{aligned} 0 &= U \sum dx_i\, dv_i - V \sum dx_i\, du_i \\ &= U \sum\sum (n-1)\, v_{ik}\, dx_i\, dx_k - V \sum\sum (m-1)\, u_{ik}\, dx_i\, dx_k. \end{aligned}$$

But the double sums are, like the earlier ones [from (10.)], proportional to U and V, and thus also this combination disappears.

Since, as shown, two combinations of the equations (12.) give nothing new, one can replace these equations with an arbitrary pair of [independent] combinations of them. Such combinations are now obtained in the following manner.† Expand U and V in terms of the quantities u_i, v_i,

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may replace the second by the result of subtracting the first, so that it becomes:

$$0 = \sum X_i(U\, dv_i - V\, du_i + dU\, v_i - dV\, u_i + dU\, dv_i - dV\, du_i).$$

It is not clear how the higher differential terms $dU\, dv_i$ etc. disappear to give us (12.).

Here follows a different approach. We write down the limit that we want to have vanish, and figure it out. **[HOW DID IT GO AGAIN?]**

† Clebsch is referring to the two equations in (6.), the first equation in (8.), and the unnumbered equation $\sum k_i x_i = 1$ that appears immediately prior to (8.); combining these with the four equations in (12.) makes for a total of eight equations involving the quantities dx_i and dt. When Clebsch asserts the existence of a certain number of equations, it is to be understood that he means *independent* equations. The three remaining equations involve only the x_i and thus determine a finite collection of points. As Clebsch is about to show, the equations $u = 0$, $v = 0$ are a consequence of the four equations in (12.), and thus this finite collection of points lies on the curve of intersection. **[BUT WHY DO WE GET ONLY THE POINT x?]**

‡ I believe that Clebsch starts with $u = \sum u_i\, x_i$ from (2.), substitutes $u = 0$, and differentiates to obtain $0 = \sum du_i\, x_i + \sum u_i\, dx_i$, which with (6.) becomes $0 = \sum x_i\, du_i$.

§ From (1.), $dv_i = \sum_k (\partial v_i / \partial x_k)\, dx_k = (n-1) \sum v_{ik}\, dx_k$. **[ON THE OTHER HAND, FROM (2.), WE HAVE $dv_i = \sum dv_{ik}\, x_k + \sum v_{ik}\, dx_k$. SO THIS SAYS THAT $\sum dv_{ik}\, x_k = (n-2) \sum v_{ik}\, dx_k$; IS THAT GOOD FOR ANYTHING?]** As Clebsch indicates in the displayed equation, the sum is already zero because the right side of (12.) vanishes upon multiplying by dx_1, etc., and adding. I'm not certain why Clebsch points out again, by a separate observation, that the displayed equation vanishes.

† I don't completely understand Clebsch's wording, but here is the point: the four equations in (12.) were originally independent **[I'M NOT SURE WHY, ACTUALLY]**; upon introducing $u = 0$ and $v = 0$, we obtained two independent relations between the four. This is why, upon eliminating the dx and dt with (6.), (8.), and (12.), the resulting three equations contain the information $u = 0$ and $v = 0$. So if one can somehow extract the remaining equation, then that must be the equation of some surface that cuts the curve in the points of hyperosculation. Keeping $u = 0$ and $v = 0$ throughout the remaining discussion, Clebsch will now go about writing down two useful combinations of the equations in (12.), the point being that these will be nonzero combinations and will be equivalent to the information in (12.). Combining these two with (6.) and (8.) makes six equations, and eliminating dx and dt gives a single equation (which is the one we seek!).

which form one row in the determinant form, and thereby set:

$$(13.)\quad U = \sum v_k\, A_k, \quad V = \sum u_k\, B_k.$$

[As explained next,] the coefficients A, B satisfy the requirements:

$$(14.)\quad \begin{cases} \sum A_i\, u_i = 0, & \sum A_i\, du_i = 0, \\ \sum B_i\, v_i = 0, & \sum B_i\, dv_i = 0. \end{cases}$$

Now these combinations distinguish themselves from the determinants (4.) only insofar as that in place of the row of v_i in U either the u_i or the du_i appear, and that in place of the row of u_i in V either the v_i or the dv_i appear. One immediately establishes the disappearance of the resulting determinants.‡

If one now multiplies the equations (12.) respectively with A_1, A_2, ..., or with B_1, B_2, ..., and adds, then the following new combinations result:

$$(15.)\quad \begin{cases} dU + \sum A_i\, dv_i = U\, dt, \\ dV + \sum B_i\, du_i = V\, dt, \end{cases}$$

where in the first equation only a factor U and in the second equation only a factor V were removed.

Let us set, in accordance with our earlier notation,

$$U_i = \frac{1}{3m+2n-8}\frac{\partial U}{\partial x_i}, \quad V_i = \frac{1}{3n+2m-8}\frac{\partial V}{\partial x_i},$$

then the equations (15.) also turn into:

$$\begin{aligned} \sum dx_i\big((3m+2n-8)\,U_i + (n-1)\sum A_i\, v_{ik}\big) &= U\, dt, \\ \sum dx_i\big((3n+2m-8)\,V_i + (m-1)\sum B_k\, u_{ik}\big) &= V\, dt. \end{aligned}$$

If one now eliminates the dx, dt from these equations in connection with (6.), (8.), then one next obtains§ the determinant:

$$\begin{vmatrix} (3m+2n-8)\,U_1 + (n-1)\sum A_k\, v_{1k} & (3n+2m-8)\,V_1 + (m-1)\sum B_k\, u_{1k} & u_1 & v_1 & k_1 \\ (3m+2n-8)\,U_2 + (n-1)\sum A_k\, v_{2k} & (3n+2m-8)\,V_2 + (m-1)\sum B_k\, u_{2k} & u_2 & v_2 & k_2 \\ (3m+2n-8)\,U_3 + (n-1)\sum A_k\, v_{3k} & (3n+2m-8)\,V_3 + (m-1)\sum B_k\, u_{3k} & u_3 & v_3 & k_3 \\ (3m+2n-8)\,U_4 + (n-1)\sum A_k\, v_{4k} & (3n+2m-8)\,V_4 + (m-1)\sum B_k\, u_{4k} & u_4 & v_4 & k_4 \\ U & V & 0 & 0 & 0 \end{vmatrix} = 0.$$

But when one multiplies the first four rows respectively with x_1, x_2, x_3, x_4 and subtracts them from the last row, after multiplying it by $3m + 3n - 9$, then all elements of the last row except the last

‡ Consider the matrix of U in (4.) with the bottom row replaced with the u_i. The sum of the first four rows, multiplied with x_1, x_2, x_3, x_4, is identical to the bottom row, and thus the determinant vanishes. Now consider the matrix with the bottom row replaced with the du_i. Multiplying this time with dx_1, etc., again the determinant is seen to vanish.

§ Here we have a system of 5 homogeneous equations in 5 unknowns; we have the two equations just established, together with $\sum u_i\, dx_i = 0$, $\sum v_i\, dx_i = 0$, and $\sum k_i\, dx_i = 0$. Demanding a nontrivial solution is equivalent to demanding that the coefficient matrix have zero determinant. This is equivalent to demanding the same of the transpose of the coefficient matrix, and furthermore one may negate all the dt coefficients. The result is the condition given by Clebsch.

disappear, and one can therefore replace the above equation with the following:

$$(16.)\quad \begin{vmatrix} (3m{+}2n{-}8)\,U_1 + (n{-}1)\sum A_k\,v_{1k} & (3n{+}2m{-}8)\,V_1 + (m{-}1)\sum B_k\,u_{1k} & u_1 & v_1 \\ (3m{+}2n{-}8)\,U_2 + (n{-}1)\sum A_k\,v_{2k} & (3n{+}2m{-}8)\,V_2 + (m{-}1)\sum B_k\,u_{2k} & u_2 & v_2 \\ (3m{+}2n{-}8)\,U_3 + (n{-}1)\sum A_k\,v_{3k} & (3n{+}2m{-}8)\,V_3 + (m{-}1)\sum B_k\,u_{3k} & u_3 & v_3 \\ (3m{+}2n{-}8)\,U_4 + (n{-}1)\sum A_k\,v_{4k} & (3n{+}2m{-}8)\,V_4 + (m{-}1)\sum B_k\,u_{4k} & u_4 & v_4 \end{vmatrix} = 0.$$

This is the equation of a surface of degree $(6m+6n-20)$, which cuts the curve $u=0$, $v=0$ in the $nm(6m+6n-20)$ points where it is met by inflectionary [or hyperosculating] planes.

If both given surfaces are of the second degree, then the quantities $\sum A_k\,v_{ik}$, $\sum B_k\,u_{ik}$ are not different† from U_i, V_i, and the equation (16.) reduces itself to the simpler‡ form:

$$\sum \pm\, U_1\,V_2\,u_3\,v_4 = 0.$$

One notes, that the entire study relies heavily on the establishment of equation (15.). But one can also derive this equation in a more direct way, in which one makes no use of the equation of the osculating plane. Namely the requirement, that four neighboring points of the curve $u=0$, $v=0$ lie in one plane, is

$$\sum \pm\, x_1\,dx_2\,d^2x_3\,d^3x_4 = 0.$$

If we multiply this equation with $\sum \pm u_1 v_2 \alpha_3 \beta_4 = 0$, where the α, β denote arbitrary quantities, then some members of the resulting determinants disappear on the left side, and that same determinant splits into two factors, one of which is alien to the problem. With removal of the same, one then finds:

$$(17.)\quad \begin{vmatrix} \sum u_i\,d^2x_i & \sum u_i\,d^3x_i \\ \sum v_i\,d^2x_i & \sum v_i\,d^3x_i \end{vmatrix} = 0.$$

The members of the first column were already found above [combine display (3.) with the one preceding (10.)] to be proportional with U, V; thus, if one denotes the determinants that result from the system (9.) with $\frac{dx_1}{ds}$, $\frac{dx_2}{ds}$ etc., one gets§

$$(18.)\quad \begin{cases} \sum u_i\,d^2x_i = -U\,ds^2, \\ \sum v_i\,d^2x_i = -V\,ds^2. \end{cases}$$

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† I haven't found a simple way of seeing this. I only know to verify this by the following procedure. Since u, v are of degree 2, we may write $u = x'\mathbf{u}x$, $v = x'\mathbf{u}x$, where $\mathbf{u}$ and $\mathbf{u}$ are 4×4 symmetric matrices, and in fact turn out to be identical to the upper left 4×4 matrices in (4.) that define U and V. Letting $\mathbf{U}$ and $\mathbf{V}$ denote the classical adjoint (or adjugate) matrices of $\mathbf{u}$ and $\mathbf{u}$, we may verify that (4.) says $U = x'\mathbf{u}\mathbf{U}\mathbf{u}x$ and $V = x'\mathbf{u}\mathbf{V}\mathbf{u}x$. Hence $U_i = \sum[\mathbf{u}\mathbf{U}\mathbf{u}]_{ik}x_k$, so that the coefficient of x_k is row i of $\mathbf{u}$ times column k of $\mathbf{U}\mathbf{u}$. On the other hand, $\sum A_k v_{ik}$ is the determinant U from (4.) with the bottom row replaced by row i from $\mathbf{u}$, so that's $e_i'\mathbf{u}\mathbf{U}\mathbf{u}x$. This is a linear form, whose coefficient of x_k is row i of $\mathbf{u}$ times column k of $\mathbf{U}\mathbf{u}$. Thus the two forms in question are seen to be the same.

There must be an easier way to see the equality. Starting with (13.), we see that $U_i = \sum v_{ik}A_k + \sum v_k \partial A_k/\partial x_i$. Now one must show that $\sum v_k \partial A_k/\partial x_i$ vanishes. (In fact, for Clebsch's purposes, it would be sufficient to show the two supposedly equal quantities to merely be proportional, so one could show that $\sum v_k \partial A_k/\partial x_i$ is a multiple of $\sum v_{ik}A_k$.) I have no idea how to do this without bringing in the ingredients from the previous paragraph.

‡ This notation is shorthand for the following sum over all permutations σ in the symmetric group on four symbols: $\sum_\sigma \mathrm{SGN}(\sigma)\,U_{\sigma 1}\,V_{\sigma 2}\,u_{\sigma 3}\,v_{\sigma 4}$. In other words, it is the determinant $|U_i\,V_i\,u_i\,v_i|$.

§ I don't understand the notation, but just think of $-ds^2$ as being the constant of proportionality. Clebsch even points out below that we must view it as a constant.

If one now differentiates these equations yet again, in so doing viewing ds as a constant differential, then one finds:

$$(19.)\quad \begin{cases} \sum u_i\, d^3x_i = -(m-1)\sum\sum u_{ik}\, d^2x_i\, dx_k - ds^2\, dU, \\ \sum v_i\, d^3x_i = -(n-1)\sum\sum v_{ik}\, d^2x_i\, dx_k - ds^2\, dV. \end{cases}$$

I CAN'T FOLLOW THE REST. I DON'T UNDERSTAND THIS ds BUSINESS.

Now since the $\frac{dx_i}{ds}$ were the partial determinants that came out of the system (9.), thus the quantities $\frac{d^2x_i}{ds^2}$ are the sums of the partial determinants formed from the two systems

$$\begin{vmatrix} \frac{du_1}{ds} & \frac{du_2}{ds} & \frac{du_3}{ds} & \frac{du_4}{ds} \\ v_1 & v_2 & v_3 & v_4 \\ k_1 & k_2 & k_3 & k_4 \end{vmatrix}, \quad \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ \frac{dv_1}{ds} & \frac{dv_2}{ds} & \frac{dv_3}{ds} & \frac{dv_4}{ds} \\ k_1 & k_2 & k_3 & k_4 \end{vmatrix},$$

and thus one has:

$$(m-1)\sum\sum u_{ik}\,\frac{d^2x_i}{ds^2}\,\frac{dx_k}{ds} = \begin{vmatrix} \frac{du_1}{ds} & \frac{du_2}{ds} & \frac{du_3}{ds} & \frac{du_4}{ds} \\ u_1 & u_2 & u_3 & u_4 \\ \frac{dv_1}{ds} & \frac{dv_2}{ds} & \frac{dv_3}{ds} & \frac{dv_4}{ds} \\ k_1 & k_2 & k_3 & k_4 \end{vmatrix}$$

$$= -(m-1)\begin{vmatrix} u_{11} & u_{12} & u_{13} & u_{14} & u_1 & v_1 & k_1 \\ u_{21} & u_{22} & u_{23} & u_{24} & u_2 & v_2 & k_2 \\ u_{31} & u_{32} & u_{33} & u_{34} & u_3 & v_3 & k_3 \\ u_{41} & u_{42} & u_{43} & u_{44} & u_4 & v_4 & k_4 \\ u_1 & u_2 & u_3 & u_4 & 0 & 0 & 0 \\ \frac{dv_1}{ds} & \frac{dv_2}{ds} & \frac{dv_3}{ds} & \frac{dv_4}{ds} & 0 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 & 0 & 0 & 0 \end{vmatrix}$$

$$(n-1)\sum\sum v_{ik}\,\frac{d^2x_i}{ds^2}\,\frac{dx_k}{ds} = \begin{vmatrix} \frac{dv_1}{ds} & \frac{dv_2}{ds} & \frac{dv_3}{ds} & \frac{dv_4}{ds} \\ \frac{du_1}{ds} & \frac{du_2}{ds} & \frac{du_3}{ds} & \frac{du_4}{ds} \\ v_1 & v_2 & v_3 & v_4 \\ k_1 & k_2 & k_3 & k_4 \end{vmatrix}$$

$$= -(n-1)\begin{vmatrix} v_{11} & v_{12} & v_{13} & v_{14} & u_1 & v_1 & k_1 \\ v_{21} & v_{22} & v_{23} & v_{24} & u_2 & v_2 & k_2 \\ v_{31} & v_{32} & v_{33} & v_{34} & u_3 & v_3 & k_3 \\ v_{41} & v_{42} & v_{43} & v_{44} & u_4 & v_4 & k_4 \\ \frac{du_1}{ds} & \frac{du_2}{ds} & \frac{du_3}{ds} & \frac{du_4}{ds} & 0 & 0 & 0 \\ v_1 & v_2 & v_3 & v_4 & 0 & 0 & 0 \\ k_1 & k_2 & k_3 & k_4 & 0 & 0 & 0 \end{vmatrix}.$$

If one now proceeds with these determinants in the usual way, to destroy the row u_i in the first, the row v_i in the second, then one obtains in their place:

$$(m-1)\begin{vmatrix} u_{11} & u_{12} & u_{13} & u_{14} & v_1 \\ u_{21} & u_{22} & u_{23} & u_{24} & v_2 \\ u_{31} & u_{32} & u_{33} & u_{34} & v_3 \\ u_{41} & u_{42} & u_{43} & u_{44} & v_4 \\ \frac{dv_1}{ds} & \frac{dv_2}{ds} & \frac{dv_3}{ds} & \frac{dv_4}{ds} & 0 \end{vmatrix} = A_1\frac{dv_1}{ds} + A_2\frac{dv_2}{ds} + A_3\frac{dv_3}{ds} + A_4\frac{dv_4}{ds},$$

$$(n-1)\begin{vmatrix} v_{11} & v_{12} & v_{13} & v_{14} & u_1 \\ v_{21} & v_{22} & v_{23} & v_{24} & u_2 \\ v_{31} & v_{32} & v_{33} & v_{34} & u_3 \\ v_{41} & v_{42} & v_{43} & v_{44} & u_4 \\ \frac{du_1}{ds} & \frac{du_2}{ds} & \frac{du_3}{ds} & \frac{du_4}{ds} & 0 \end{vmatrix} = B_1\frac{du_1}{ds} + B_2\frac{du_2}{ds} + B_3\frac{du_3}{ds} + B_4\frac{du_4}{ds}.$$

If one introduces this next to (18.), (19.) into the equation (17.), then this equation takes on the form:

$$\begin{vmatrix} dU + \sum A_i\,dv_i & U \\ dV + \sum B_i\,dv_i & V \end{vmatrix} = 0,$$

which apparently is identical with the to-be-established equation (15.). Thus in this direct way the equations are derived, on which previously the derivation of the equation (16.) depended.

Carlsruhe, the 3rd of September, 1862.