Quadratic-linear duality and rational homotopy theory of chordal arrangements

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Graphic Arrangements

$C=$ $\mathbb{C}$, $\mathbb{C}^*$, or complex projective curve of positive genus

$\Gamma=(\mathcal{V},\mathcal{E})$ ordered simple graph

$\forall e\in\mathcal{E}$, label vertices $h(e)>t(e)$ and let

$$H_e := \{\underline{x}\in C^{\mathcal{V}} \mid x_{h(e)} = x_{t(e)}\}$$

$\mathcal{A}(\Gamma,C) := \{H_e \mid e\in\mathcal{E}\}$ is a graphic arrangement in $C^{\mathcal{V}}$

with complement $X_{\mathcal{A}} := C^{\mathcal{V}} \setminus \bigcup_{e\in\mathcal{E}} H_e$

(Note: $\mathcal{A}(\Gamma,\mathbb{C})$ is a hyperplane arrangement)

Example: A Configuration space

$C = \mathbb{C}, \mathbb{C}^*$, complex projective curve of positive genus

$\Gamma = K_n$ (complete graph)

$\forall\, 1 \le i < j \le n$, have $H_{ij} = \{(x_1, \dots, x_n) \in C^n \mid x_i = x_j\}$

$A(K_n, C)$ is the braid arrangement, whose complement X_A is an ordered configuration space of n points on C.

(Note: Bezrukavnikov '94 studied rational homotopy theory of ordered configuration spaces on positive genus complex projective curves)

DGA Model for $X_{\mathcal{A}}$ ($C = \mathbb{C}$)

Theorem (Orlik-Solomon '80, Brieskorn '73)

Let $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C})$. Then

$X_{\mathcal{A}}$ is formal and

$$H^\bullet(X_{\mathcal{A}}; \mathbb{Q}) \cong \frac{\Lambda(g_e \mid e \in \mathcal{E})}{\left(\sum_{j=1}^{k} (-1)^j g_{e_1} \cdots \hat{g}_{e_j} \cdots g_{e_k} \text{ if } e_1 < \cdots < e_k \text{ is a cycle} \right)}$$

DGA Model for $X_{\mathcal{A}}$ $(C = \mathbb{C}^\times)$

Theorem (DeConcini–Procesi '05)

Let $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C}^\times)$. Then

$X_{\mathcal{A}}$ is formal and

$$H^*(X_{\mathcal{A}}; \mathbb{Q}) \cong \frac{\Lambda(x_v, g_e \mid v \in \mathcal{V}, e \in \mathcal{E})}{\left(\begin{array}{l} \text{(i)} \ \sum_{j=1}^{k} (-1)^j g_{e_1} \cdots \hat{g}_{e_j} \cdots g_{e_k} + \text{other stuff} \\ \qquad \text{if } e_1 < \dots < e_k \text{ is a cycle} \\ \text{(ii)} \ (x_{h(e)} - x_{t(e)}) g_e \quad \text{for } e \in \mathcal{E} \end{array} \right)}$$

DGA Model for $X_{\mathcal{A}}$ (C projective)

Theorem (Dupont '13)

Let $\mathcal{A} = \mathcal{A}(\Gamma, C)$ where C is a complex projective curve of positive genus.

Then
$$A(\mathcal{A}) = \frac{H^*(C^{\nu};\mathbb{Q})[g_e \mid e \in \mathcal{E}]}{\left(\begin{array}{ll}\text{(i)} \ \sum_{j=1}^{k}(-1)^j g_{e_1}\cdots\hat{g}_{e_j}\cdots g_{e_k} & \text{if } e_1 < \dots < e_k \text{ is a cycle} \\ \text{(ii)} \ \left(p_{h(e)}^*(x) - p_{t(e)}^*(x)\right) g_e & \text{if } e \in \mathcal{E},\ x \in H^1(C,\mathbb{Q}), \\ & p_v : C^{\nu} \to C \text{ projection}\end{array}\right)}$$

with differential $d: g_e \longmapsto [H_e] \in H^2(C^{\nu};\mathbb{Q})$

is a model for $X_{\mathcal{A}}$.

Chordal Implies Koszul

Γ is chordal if every cycle with more than 3 vertices has a chord.

Theorem (Shelton-Yuzvinsky '97)

If Γ is chordal and $\mathcal{A} = \mathcal{A}(\Gamma, \mathbb{C})$, then $H^*(X_{\mathcal{A}}, \mathbb{Q})$ is Koszul.

Theorem (Bibby-Hilburn)

Let Γ be chordal. and $\mathcal{A} = \mathcal{A}(\Gamma; C)$.

- If $C = \mathbb{C}^\times$ then $H^*(X_{\mathcal{A}}; \mathbb{Q})$ is Koszul.
- If C is complex projective curve of positive genus, then $A(\mathcal{A})$ is Koszul.

Rational homotopy theory results

Theorem (Bibby-Hilburn)

Let $\mathcal{A} = \mathcal{A}(\Gamma, C)$ with Γ chordal and C complex projective curve of positive genus.

- The Koszul dual to $(A(\mathcal{A}), d)$ is $U(L)$ for some Lie algebra L.
- The completion of L with respect to bracket length is the Malcev Lie algebra of $\pi_1(X_{\mathcal{A}})$.

  That is, it determines the $\mathbb{Q}$-nilpotent completion of $\pi_1(X_{\mathcal{A}})$
- The (graded) standard complex of L is the minimal model of $X_{\mathcal{A}}$

Rational homotopy theory results

(theorem continued)

- $X_\mathcal{A}$ is rationally $K(\pi,1)$.

(Note: The analogous theorem in formal case is due to Papadima-Yuzvinsky '99)

Corollary If Γ is chordal and not a tree, and C is a complex elliptic curve, then $X_\mathcal{A}$ is not formal.

(Special case of a result by Berceanu-Macinic-Papadima-Popescu '15)

Questions. When is it formal? Coformal?

Thank you