A Serre Spectral Sequence for Moduli Spaces of Tropical Curves

Christin Bibby (LSU)
Melody Chan (Brown)
Nir Gadish (Michigan)
Claudia He Yun (Michigan)

BCGY1
arXiv: 2109.03302
(Experimental Math)

BCGY2
arXiv: 2307.01960

github.com/ClaudiaHeYun/BCGY

Modern Developments in the Theory of Configuration Spaces
JMM 2024

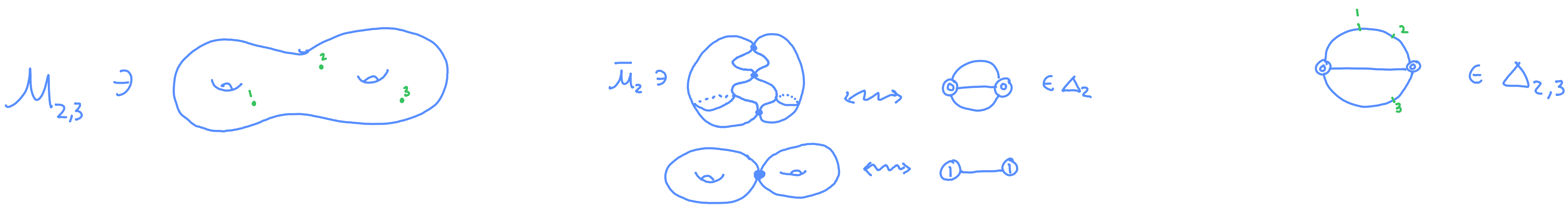
motivation

$$Gr^{W}_{6g-6+2n} H^{6g-6+2n-k}(\mathcal{M}_{g,n};\mathbb{Q}) \cong \widetilde{H}_{k-1}(\Delta_{g,n};\mathbb{Q})$$

moduli space of genus g n-marked algebraic curves

[Abramovich – Caporaso–Payne '15]
identify $\Delta_{g,n}$ with the boundary complex of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$

combinatorial moduli space of genus g n-marked tropical curves

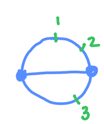


Goal

Compute $\widetilde{H}_*(\Delta_{g,n};\mathbb{Q})$ as an S_n-representation

Approach

use graph configuration spaces



$\in \mathrm{Conf}_3(\Theta)$

$$\mathrm{Conf}_n(G) = \{(x_1,x_2,\dots,x_n)\in G^n \mid i\neq j \Rightarrow x_i \neq x_j\}$$

What we know / $\tilde{H}_*(\Delta_{g,n};\mathbb{Q})$ as an S_n-representation $(2g-2+n>0)$

Formulas:

* S_n-equivariant Euler characteristic
  [conj. Zagier, proved Chan-Faber-Galatius-Payne '19]
* $g=0$: character formula [Robinson-Whitehouse '96]
  wedge of spheres
* $g=1$: character of $\mathcal{M}_{1,n}$ [Getzler '99, Chan-Galatius-Payne '21]

Calculations:

* multiplicity of sign & trivial [Khoroshkin-Willwacher-Živković '17]
  via hairy graph complexes
* $g=2$ and $n\le 8$ [Yun '21]
  $n \le 11$ [BCGY1]
  for $n=8$ computer took ≈ 1 week vs <24 hours
  see github.com/ClaudiaHeYun/BCGY
* $g=3$ and $n\le 9$ [BCGY2]
  & partial data for larger n
* $g>3$ ...

Moduli space of tropical curves

A **tropical curve** is a vertex-decorated metric graph

(G, w, m, ℓ) where

- G = connected finite graph
- $w : V(G) \to \mathbb{Z}_{\geq 0}$ weight function
- $m : \{1, 2, \ldots, n\} \to V(G)$ marking
- $\ell : E(G) \to \mathbb{R}_{>0}$ edge lengths

Stable

$$2w(v) + \mathrm{val}(v) + |m^{-1}(v)| > 2 \quad \forall v \in V(G)$$

unit volume $\sum_{e \in E(G)} \ell(e) = 1$

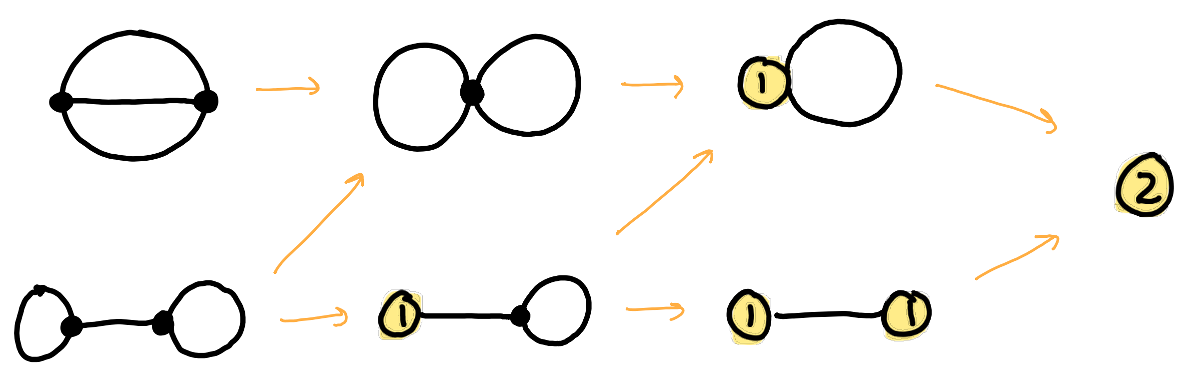
genus

$$g = |E(G)| - |V(G)| + 1 + \sum_{v \in V(G)} w(v)$$

$\Delta_{g,n}$ = moduli space of genus g, n-marked tropical curves

* Glue together simplices that parametrize edge lengths on (G, w, m)

$g=2$
$n=0$

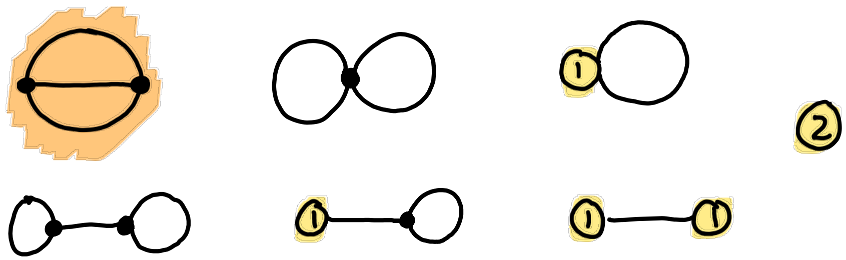


We don't need every graph

useful fact [Chan–Galatius–Payne '21-22]

$\Delta_{g,n}$ has a very large contractible subcomplex

genus 2



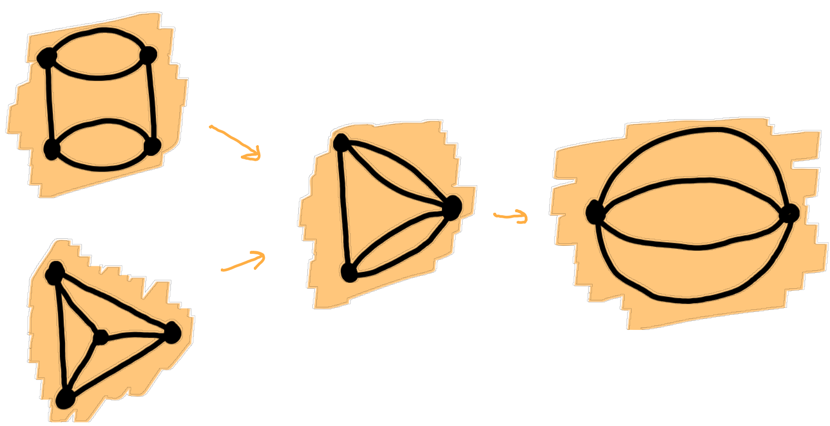
Theorem [BCGY1] There is an S_n-equivariant isomorphism

$$\tilde{H}_i(\Delta_{2,n};\mathbb{Q}) \cong \left(\mathrm{sgn}_3 \otimes \tilde{H}_{i-1}(\mathrm{Conf}_n(\Theta)^+;\mathbb{Q})\right)_{\mathrm{Iso}\,\Theta}$$

$\mathrm{Iso}\,\Theta \cong S_2 \times S_3$ coinvariants

Proof idea: $$\left[\left((\Delta^2)^\circ \times \mathrm{Conf}_n(\Theta)\right)/\mathrm{Iso}\,\Theta\right]^+ \cong \left(\Delta^{\Theta}_{2,n}\right)^+ \simeq \Delta_{2,n}$$

genus 3

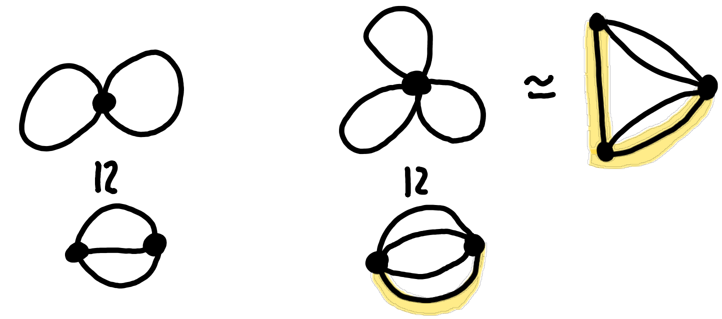


genus 4

17 important graphs

Actually, we only need a rose graph

$$R_g = \bigvee_{i=1}^{g} S^1$$



$G \simeq R_g$ $\rightsquigarrow$ $H_*(\mathrm{Conf}_n(G)^+; \mathbb{Q}) \cong H_*(\mathrm{Conf}_n(R_g)^+; \mathbb{Q})$

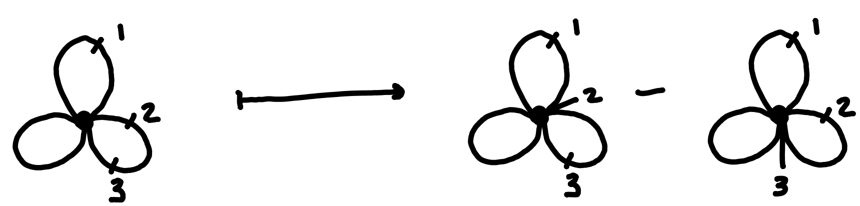
graph of genus g

$$\mathrm{Iso}(G) \leq \mathrm{Out}(F_g)$$

$\mathrm{Conf}_n(R_g)^+$ has S_n-equivariant & combinatorial cell structure $\rightsquigarrow$

$$\mathbb{Z}[S_n]^{\binom{n+g-1}{g-1}} \xrightarrow{\ \partial\ } \mathbb{Z}[S_n]^{\binom{n+g-2}{g-1}}$$

degree n degree $n-1$



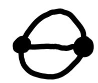
Compute Higher Genus with a Spectral sequence

Theorem [BCGY2] For $g \geq 2, n \geq 0$

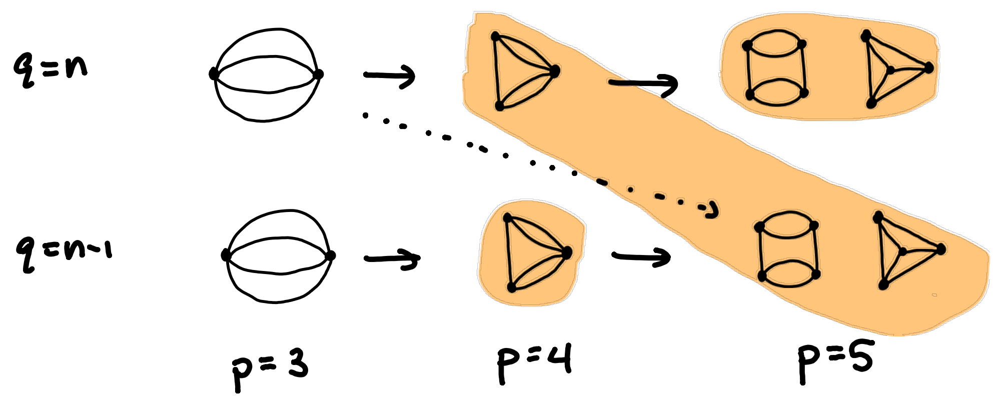
there is a spectral sequence of S_n-representations

$$E_1^{pq} = \bigoplus_{\substack{\text{2-connected} \\ \text{genus-}g\text{ graphs } G \\ \text{with } p+1 \text{ edges}}} \underbrace{\left(H_c^q(\mathrm{Conf}_n(G); \mathbb{Q}) \otimes \det(E(G)) \right)^{\mathrm{Aut}(G)}}_{\text{understand by last slide}} \Rightarrow \tilde{H}^{p+q}(\Delta_{g,n}; \mathbb{Q})$$

* It is supported in rows $q = n-1$ & $q = n$

* Genus $g = 2$: only one graph  $\rightsquigarrow$ supported in column $p=2$ & degenerates at E_1

* Genus $g = 3$: degenerates at E_2



* Genus $g > 3$: degenerates at E_3 & more graphs to manage

It's a Tropical Serre Spectral Sequence

$$
\begin{tikzcd}
\sigma_G^\circ \times \mathrm{diag}_n(G) \arrow[r] \arrow[d, phantom, "\wr"] & \mathcal{U}_{g,n}^{(2,r)} \arrow[r] \arrow[d, phantom, "\wr"] & \Delta_{g,n}^{(2,r)} \arrow[d, phantom, "\wr"] \\
\sigma_G^\circ \times G^n \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U}_{g,n}^{(2)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \Delta_{g,n}^{(2)} \arrow[d, "\text{forget marked points}"] \\
\sigma_G^\circ \arrow[r, "\iota_{(G,h)}"'] & CV_g^{(2)} \arrow[r, "/\mathrm{Out}(F_g)"'] & \Delta_g^{(2)}
\end{tikzcd}
$$

Culler-Vogtmann's outer space

Reimagine the rows of E_1 as graph complexes so we have

$$E_2^{pq} \cong H_c^p\left(\Delta_g^{(2)};\ H_c^q(\mathrm{Conf}_n(G);\mathbb{Q})\right) \Longrightarrow \tilde{H}^{p+q}(\Delta_{g,n};\mathbb{Q})$$

Thank you

BCGY1 = arXiv: 2109.03302
(Experimental Mathematics)

BCGY2 = arXiv: 2307.01960

github.com/ClaudiaHeYun/BCGY