Supersolvable Posets & Fiber-Type Arrangements

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Configuration Spaces

Inputs $\begin{cases} X = \text{manifold of dimension} \geq 2 \\ n \in \mathbb{Z}_{>0} \end{cases}$

$$\boxed{\mathrm{Conf}_n(X) = \{(x_1,\dots,x_n) \in X^n : i \neq j \Rightarrow x_i \neq x_j\}}$$

Applications to physics, robotics, ...

Theorem [Fadell-Neuwirth '62] There is a fiber bundle

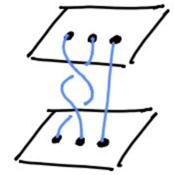
$$\begin{array}{ccc} X - \{n \text{ points}\} \longrightarrow \mathrm{Conf}_{n+1}(X) & & (x_1,\dots,x_n,x_{n+1}) \\ \downarrow & & \downarrow \\ \mathrm{Conf}_n(X) & & (x_1,\dots,x_n) \end{array}$$

"forget the last point"

✳ This is a tool to study topological invariants of configuration spaces (eg. homotopy type, (co)homology, stability)

Motivating Example

$X = \mathbb{C}$



* $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$ is a pure braid group [Fox-Neuwirth '62]
* $\mathrm{Conf}_n(\mathbb{C})$ is $K(\pi,1)$ and rationally $K(\pi,1)$ [Kohno '85]

  LES $\cdots \to \pi_i(\mathbb{C} - \{n \text{ pts}\}) \to \pi_i(\mathrm{Conf}_{n+1}(\mathbb{C})) \to \pi_i(\mathrm{Conf}_n(\mathbb{C})) \to \cdots$
* $\pi_1(\mathrm{Conf}_n(\mathbb{C}))$ is an iterated semidirect product of free groups
* $H^*(\mathrm{Conf}_{n+1}(\mathbb{C})) \cong H^*(\mathrm{Conf}_n(\mathbb{C})) \otimes H^*(\mathbb{C} - \{n \text{ pts}\})$

  as $H^*(\mathrm{Conf}_n(\mathbb{C}))$-modules

  Serre spectral sequence $E_2^{pq} \cong H^p(\mathrm{Conf}_n(\mathbb{C}); \mathcal{H}^q(\mathbb{C} - \{n \text{ pts}\})) \Rightarrow H^{p+q}(\mathrm{Conf}_{n+1}(\mathbb{C}))$

  or Leray-Hirsch Theorem
* The Poincaré polynomial has rational roots: [Arnol'd '69]

$$\sum_{i \geq 0} \dim H^i(\mathrm{Conf}_n(\mathbb{C}))\, t^i = \prod_{j=1}^{n-1} (1 + jt)$$

* For each i, the sequence of S_n-representations

  $\{H^i(\mathrm{Conf}_n(\mathbb{C}))\}_n$ stabilizes [Church-Farb '13]
* and more...

Combinatorics

* view $\boxed{\mathrm{Conf}_n(X) = X^n - \bigcup_{1\le i<j\le n} \Delta_{ij}}$

the complement of an arrangement of submanifolds

$\mathcal{A}_n = \{\Delta_{ij} : 1\le i<j\le n\}$ where $\Delta_{ij} = \{(x_1,\dots,x_n)\in X^n : x_i = x_j\}$

$\boxed{\text{Set of intersections } \bigcap_{\Delta\in S}\Delta,\ S\subseteq \mathcal{A}_n \text{ partially ordered by reverse inclusion}} \xleftrightarrow[\text{isomorphism}]{\text{poset}} \boxed{\text{set of partitions of } \{1,2,\dots,n\} \text{ partially ordered by refinement}}$

$$
\begin{tikzcd}[column sep=small]
 & (x_1=x_2=x_3) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\Delta_{12} \arrow[dr, dash] & \Delta_{13} \arrow[d, dash] & \Delta_{23} \arrow[dl, dash] \\
 & X^3 &
\end{tikzcd}
\quad \cong \quad
\begin{tikzcd}[column sep=small]
 & 123 \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
12/3 \arrow[dr, dash] & 13/2 \arrow[d, dash] & 23/1 \arrow[dl, dash] \\
 & 1/2/3 &
\end{tikzcd}
$$

Question For what arrangements do we obtain a similar fibration?

$$
\begin{tikzcd}
X^{n+1} \arrow[d] & \supseteq & X^{n+1} - \bigcup_{H\in\mathcal{A}} H \\
X^n & &
\end{tikzcd}
$$

Goal Find a combinatorial characterization

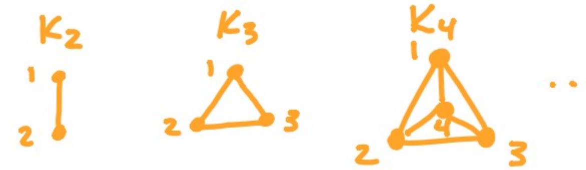
Generalized Configuration Spaces

"Partial configuration space"

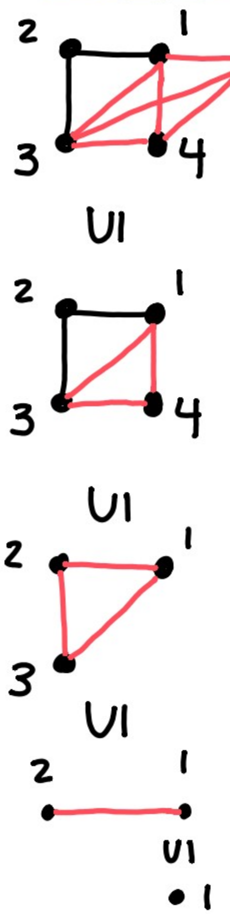
Γ = simple graph on vertices $1, 2, \ldots, n$.

$$\mathrm{Conf}_\Gamma(X) = \{(x_1, \ldots, x_n) \in X^n : x_i \neq x_j \text{ when } \{i,j\} \text{ is an edge of } \Gamma\}$$

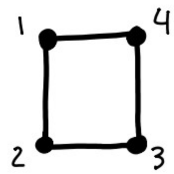
Note $\mathrm{Conf}_{K_n}(X) = \mathrm{Conf}_n(X)$



Γ chordal



not chordal:



$$\mathrm{Conf}_{\square}(X)$$
$$\downarrow$$
$$\mathrm{Conf}_{\llcorner}(X) \ni (x_1, x_2, x_3)$$

if $x_1 = x_3$ fiber is $X - \{x_1\}$
if $x_1 \neq x_3$ fiber is $X - \{x_1, x_3\}$

fibers not homeomorphic
$\Rightarrow$ not a fiber bundle

"orbit configuration space"

$G \circlearrowright X$ free group action

$$\mathrm{Conf}_n^G(X) = \{(x_1, \ldots, x_n) \in X^n : i \neq j \Rightarrow Gx_i \cap Gx_j = \emptyset\}$$

$$\mathrm{Conf}_{n+1}^G(X) \to \mathrm{Conf}_n^G(X)$$ is a fiber bundle

[Xicotencatl '97]

Abelian Arrangements

X = connected abelian Lie group
$\cong (S^1)^d \times \mathbb{R}^v$ where $d+v \geq 2$

my favorite examples: $\mathbb{C}$, $\mathbb{C}^\times$, $S^1 \times S^1$

$$(a_1, \dots, a_n) \in \mathbb{Z}^n \longleftrightarrow \alpha : X^n \to X \quad \rightsquigarrow \quad H_\alpha = \ker(\alpha) \leq X^n$$
$$(x_1, \dots, x_n) \longmapsto \sum_{i=1}^n a_i x_i$$

An **abelian arrangement** is a finite set $\mathcal{A}$ of H_α in X^n.

The topological space:

arrangement complement

$$M(\mathcal{A}) = X^n - \bigcup_{H \in \mathcal{A}} H$$

* we understand $H^*(M(\mathcal{A}))$ better when X is not compact

(even $H^*(\mathrm{Conf}_n(S^1 \times S^1))$ not fully understood)

The combinatorial structure:

poset of layers $P(\mathcal{A})$

connected components of intersections $\bigcap_{H \in S} H$ $(S \subseteq \mathcal{A})$

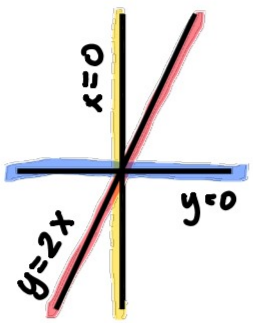
partially ordered by reverse inclusion

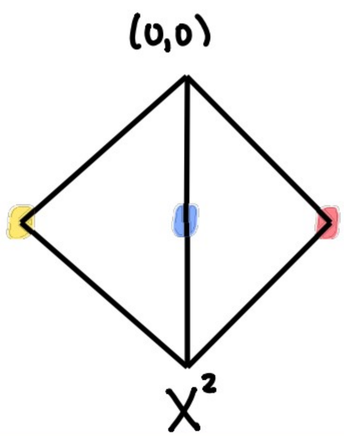
Examples/ $n=2$, $\alpha_1 = (1,0)$, $\alpha_2 = (0,1)$, $\alpha_3 = (2,-1)$

$x = 0$ $\quad$ $y = 0$ $\quad$ $2x - y = 0$

$X = \mathbb{C}$

(drawn $\mathbb{R}$)





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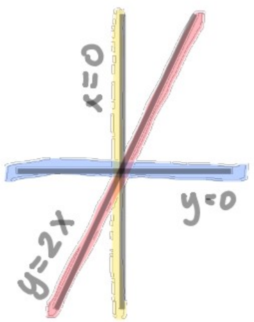
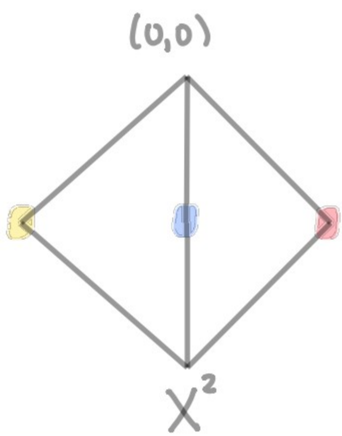
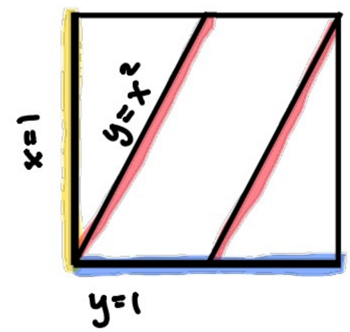
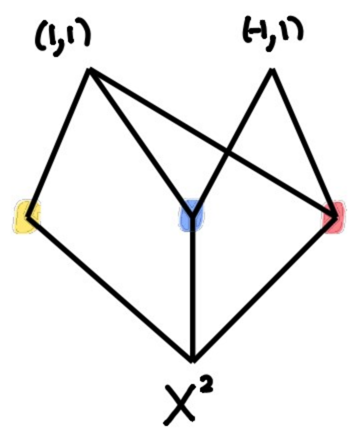




$X = \mathbb{C}^\times$
(drawn S^1)





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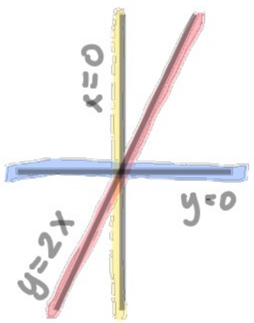

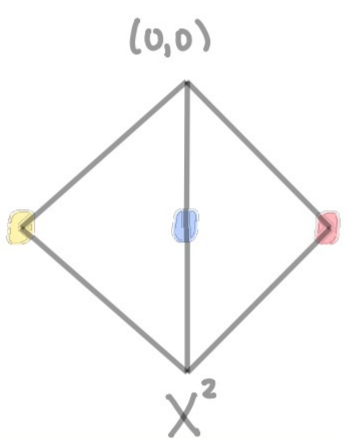

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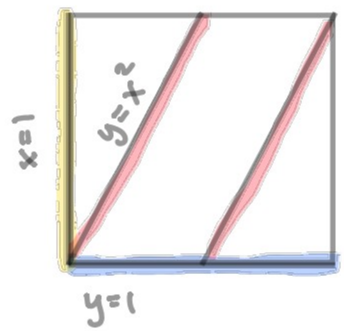

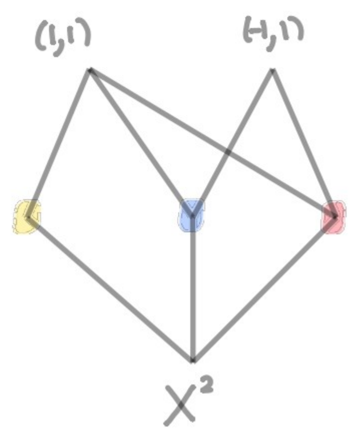

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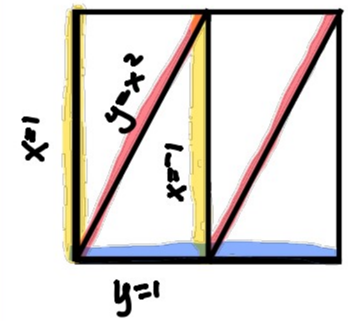

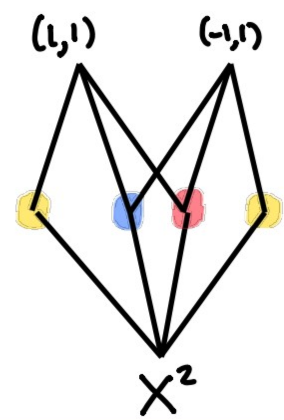

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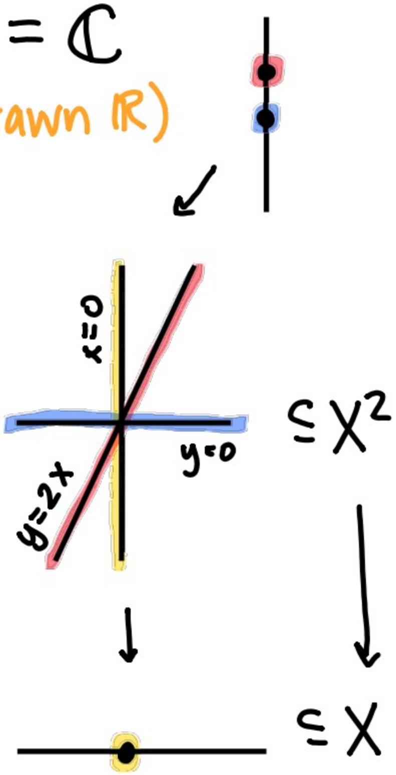

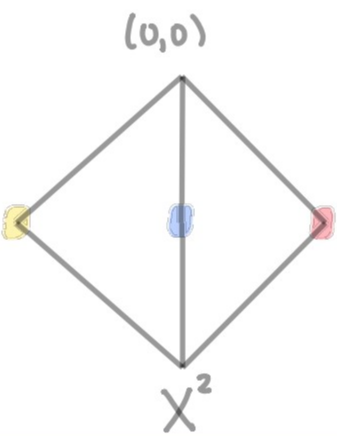

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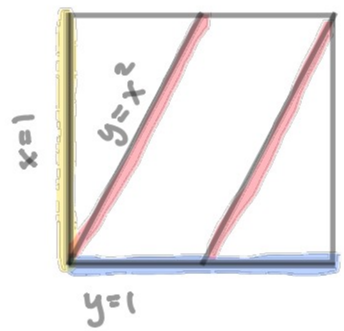

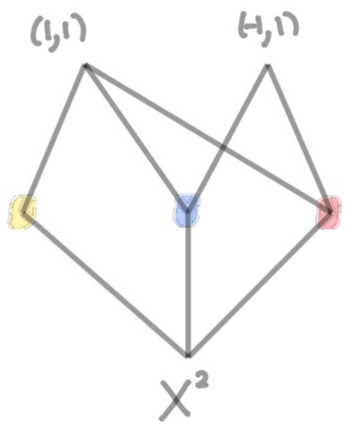

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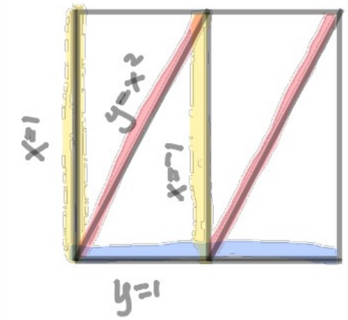

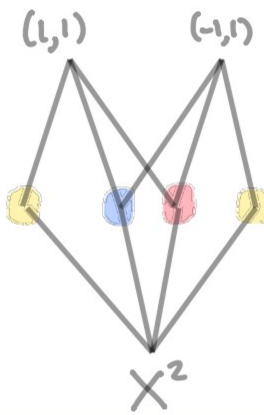

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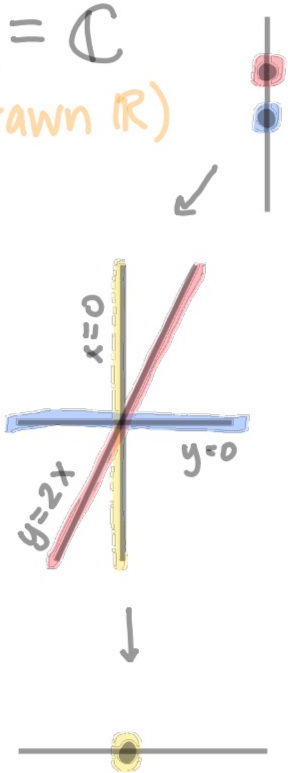
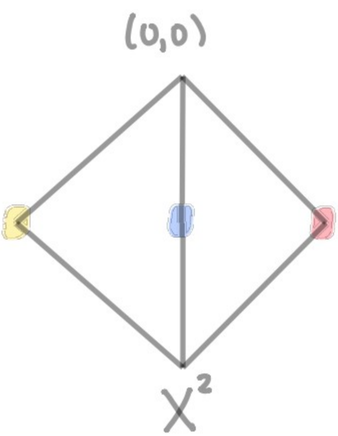


$\subseteq X^2$

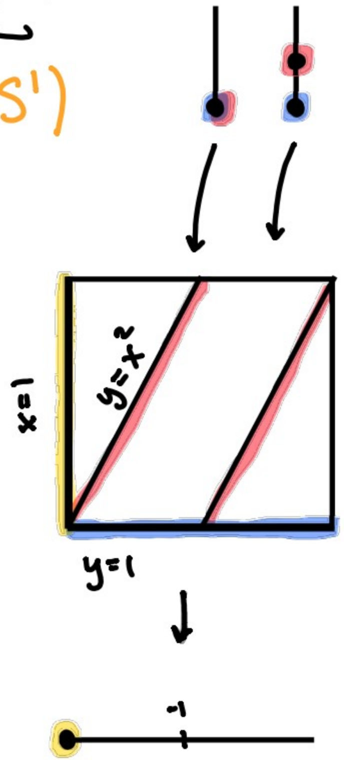
$\subseteq X$



$X = \mathbb{C}^\times$

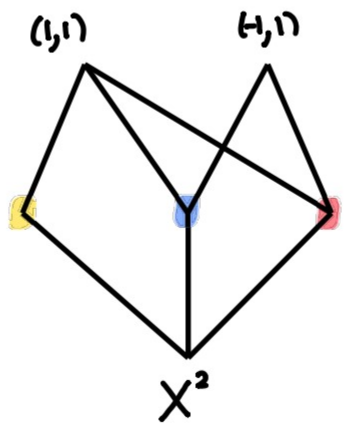
(drawn S^1)

$X^2 \supseteq$



$X \supseteq$

not a fiber bundle



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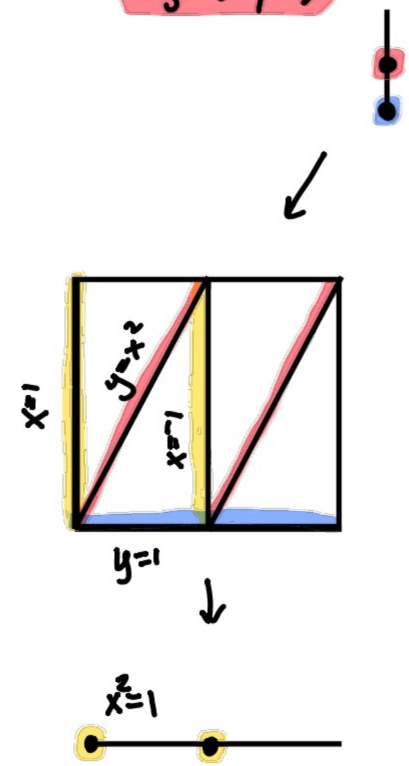
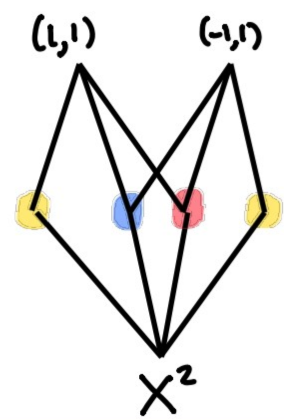


fiber bundle



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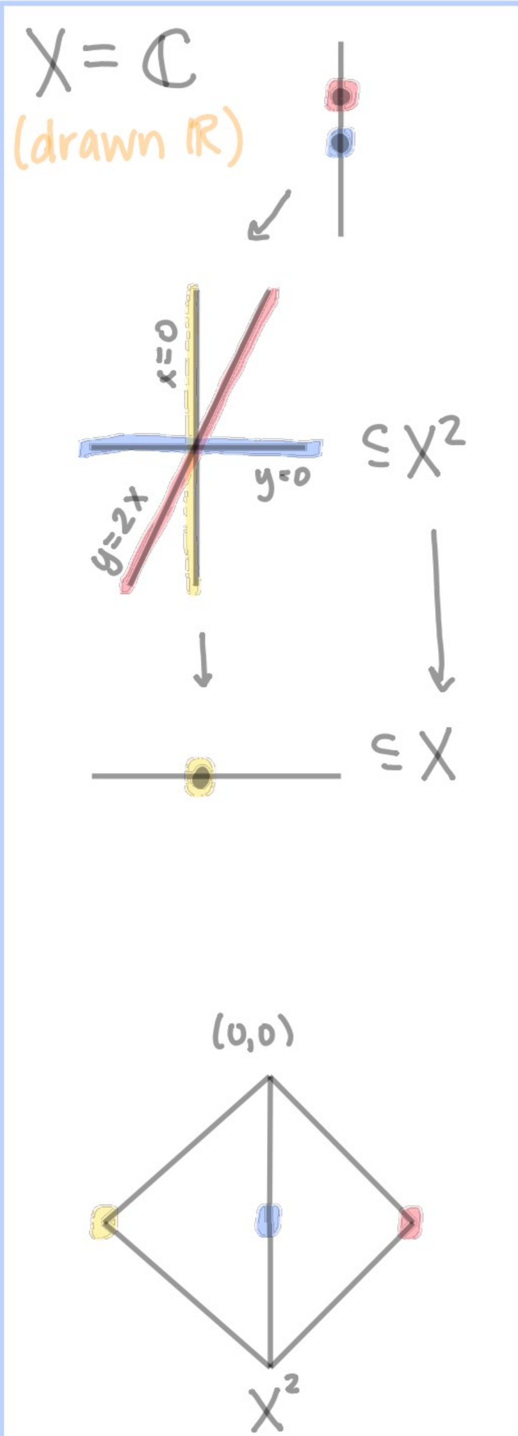
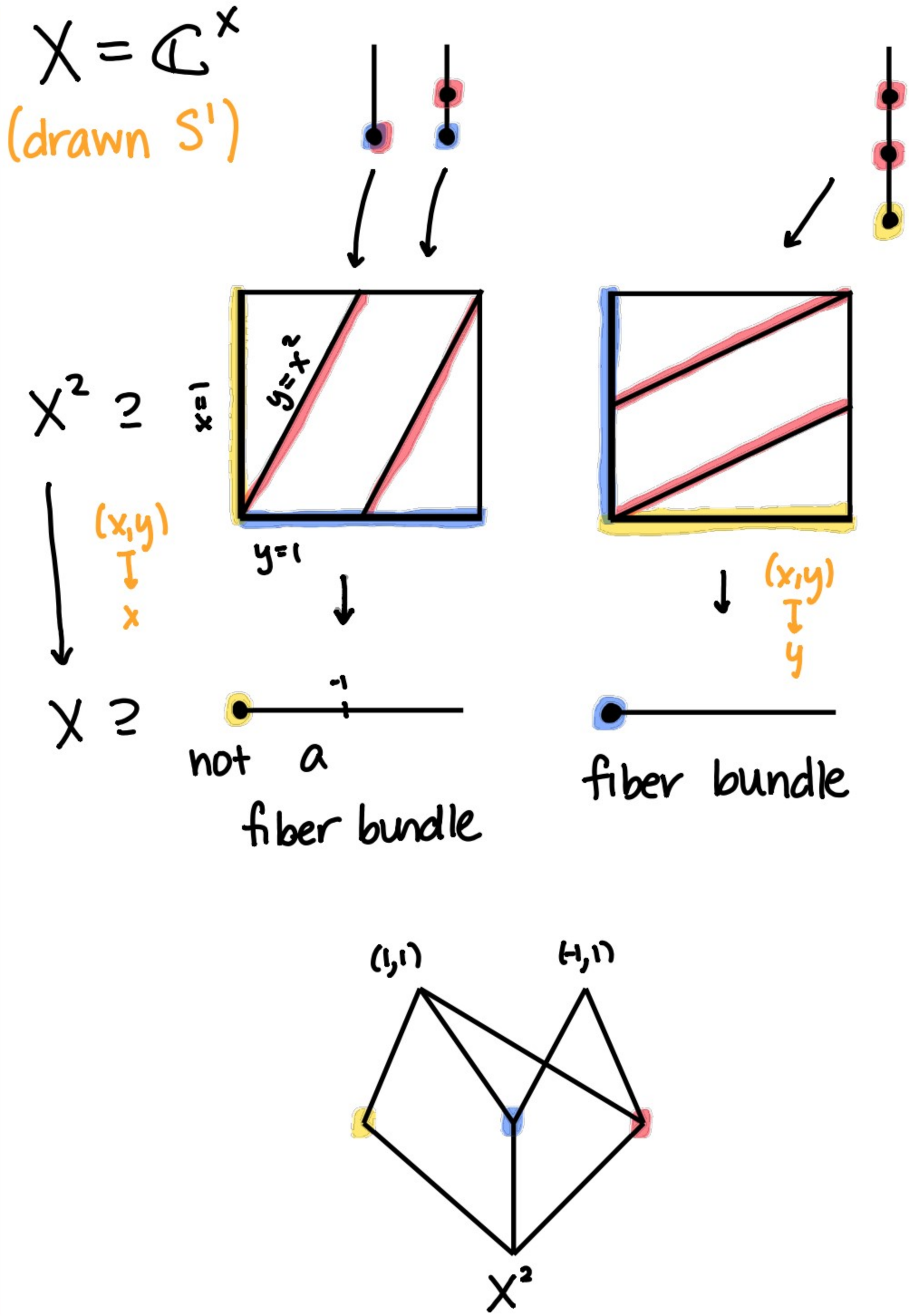
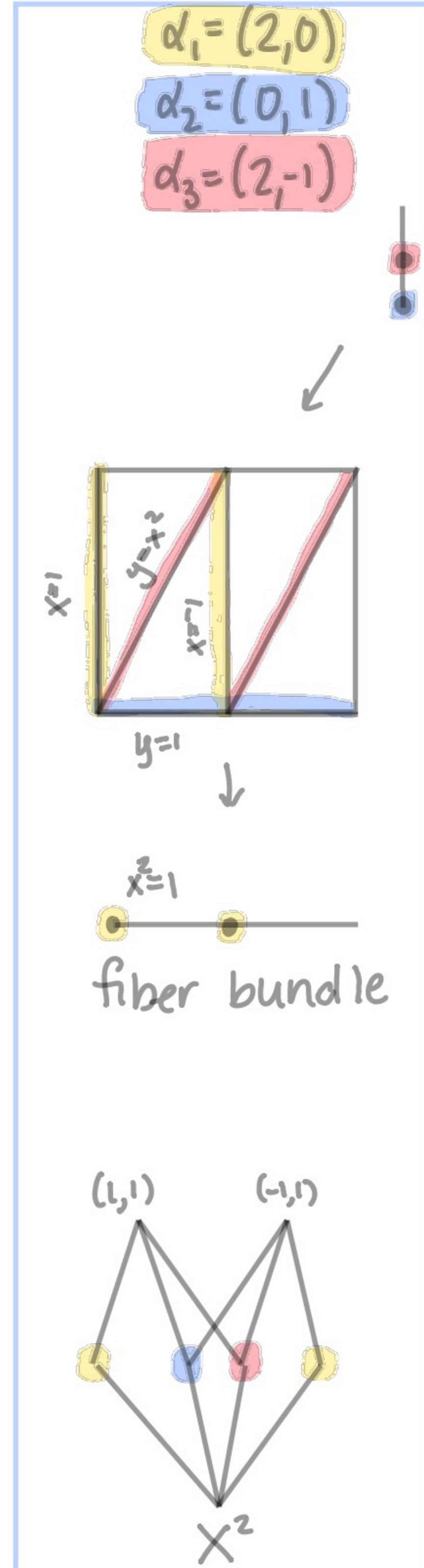






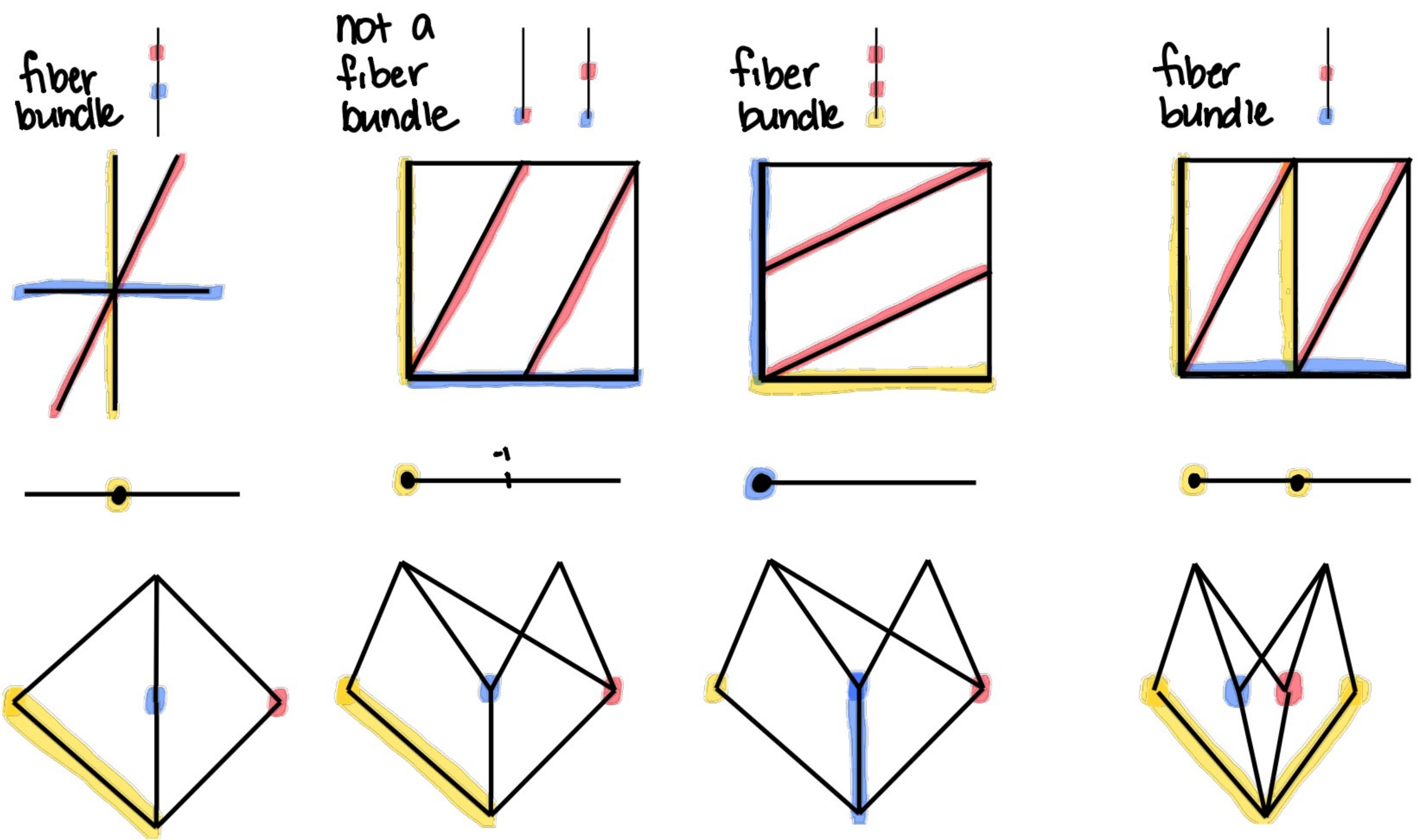
Supersolvability

[Stanley '72 for lattices]

$\mathcal{A}$ = abelian arrangement

$P = P(\mathcal{A})$ = poset of layers

atoms(P) = connected components of $H_\alpha \in \mathcal{A}$



Supersolvability

[Stanley '72 for lattices]

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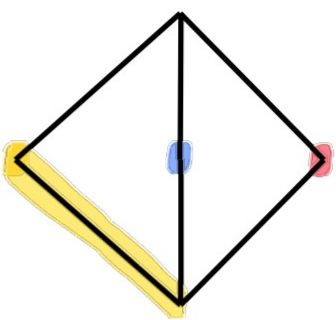
$Q \subseteq P$, join-closed & downward-closed, is an **m-ideal** if for any $H_1, H_2 \in \text{atoms}(P) - Q$ and $u \in \min\{x \in P : x \geq H_1 \text{ \& } x \geq H_2\}$ there is an $H_3 \in \text{atoms}(Q)$ such that $u > H_3$.

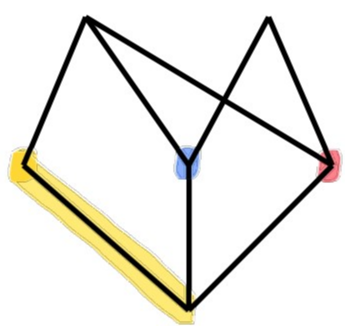
An m-ideal $Q \subseteq P$ is a **TM-ideal** if for any $H \in \text{atoms}(P) - Q$ and $y \in Q$, $H \cap y$ is connected

Say P is **strictly supersolvable** if there is a chain of TM-ideals

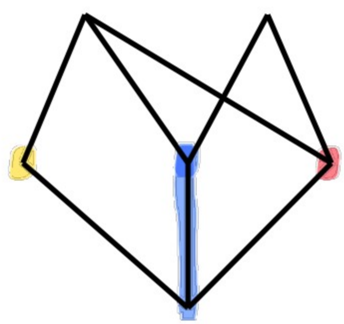
$$\{\min P\} = Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_{r-1} \subsetneq Q_{r = rk(P)} = P$$



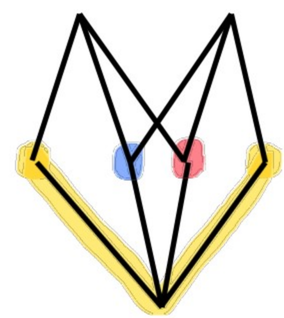
TM-ideal



not an m-ideal

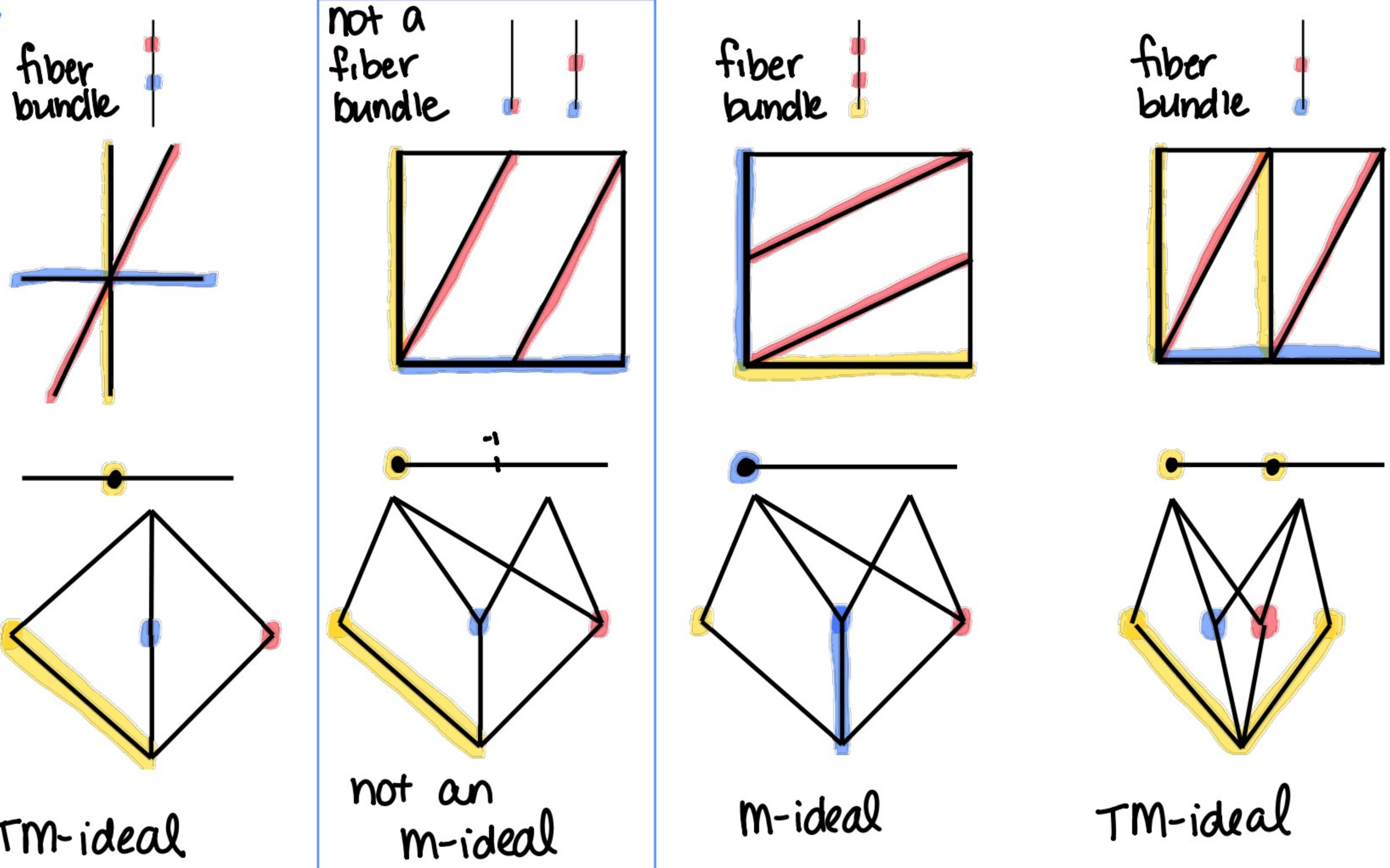


m-ideal



TM-ideal

Arrangement Bundles

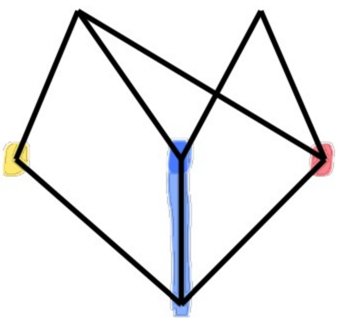


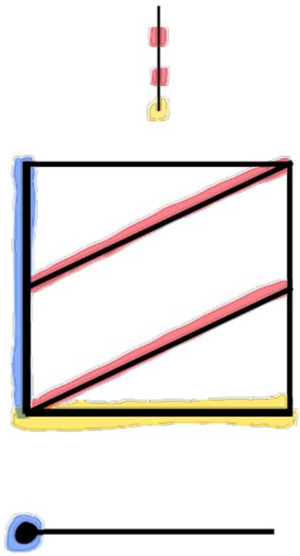
Theorem [B-Delucchi '22, Terao '86 case $X = \mathbb{C}$]
Let $\mathcal{A}$ be an abelian arrangement. There is a choice of coordinates so that $X^{n+1} \to X^n$ restricts to a fiber bundle $M(\mathcal{A}) \to M(\mathcal{B})$ if and only if $P(\mathcal{A})$ has a corank-one M-ideal $(Q \cong P(\mathcal{B}))$

* Fiber is X with k points removed

* tower of fibrations $\longleftrightarrow$ supersolvable

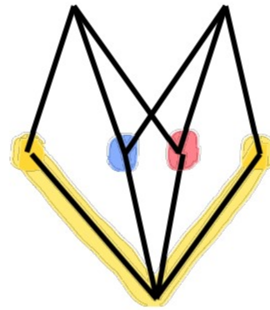
Pullback FN-bundles



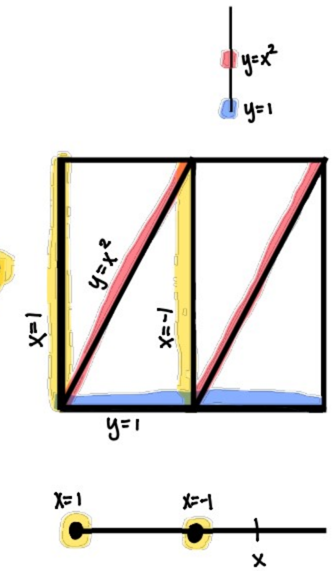


m-ideal

not TM-ideal $\rightsquigarrow$ nontrivial monodromy



TM-ideal



$$\begin{tikzcd} M(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Conf}_3(\mathbb{C}^\times) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Conf}_4(\mathbb{C}) \arrow[d] \\ M(B) \arrow[r] & \mathrm{Conf}_2(\mathbb{C}^\times) \arrow[r] & \mathrm{Conf}_3(\mathbb{C}) \end{tikzcd}$$

$$x \longmapsto (1, x^2) \longmapsto (1, x^2, 0)$$

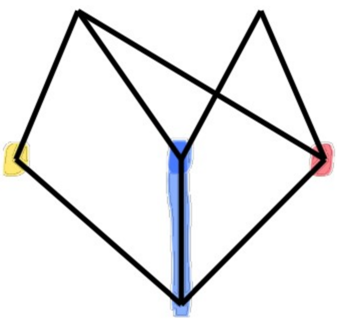
Theorem [B-Cohen-Delucchi]

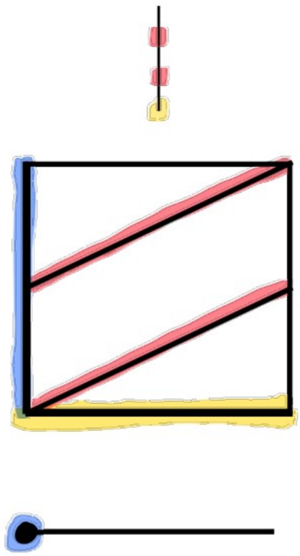
If $X - \{k \text{ pts}\} \to M(A) \xrightarrow{\pi} M(B)$ is a fiber bundle associated to a Tm-ideal $Q \subseteq P(A)$, then we have a pullback diagram

$$\begin{tikzcd} M(A) \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathrm{Conf}_{k+1}(X) \arrow[d] \\ M(B) \arrow[r] & \mathrm{Conf}_k(X) \end{tikzcd}$$

$x \longmapsto$ punctures of $\pi^{-1}(x)$

Pullback FN-bundles





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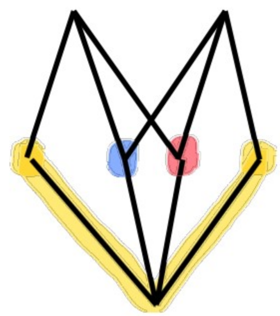
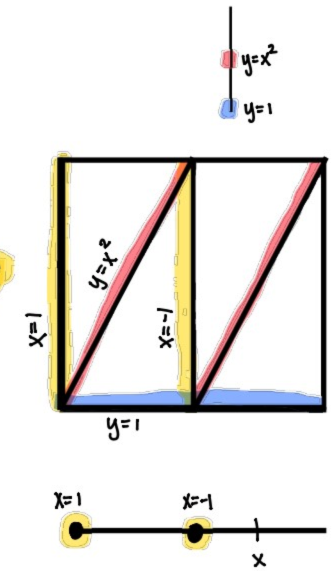


TM-ideal



$$
\begin{tikzcd}
M(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Conf}_3(\mathbb{C}^\times) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Conf}_4(\mathbb{C}) \arrow[d] \\
M(B) \arrow[r] & \mathrm{Conf}_2(\mathbb{C}^\times) \arrow[r] & \mathrm{Conf}_3(\mathbb{C})
\end{tikzcd}
$$

$$x \longmapsto (1, x^2) \longmapsto (1, x^2, 0)$$

Theorem [B-Cohen-Delucchi]

If $X - \{k \text{ pts}\} \to M(\mathcal{A}) \xrightarrow{\pi} M(\mathcal{B})$ is a fiber bundle associated to an m-ideal $Q \subseteq P(\mathcal{A})$, then we have a pullback diagram

$$
\begin{tikzcd}
M(\mathcal{A}) \arrow[r] \arrow[d, "\pi"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Conf}_{k+1}(X)/\Sigma_k \times \Sigma_1 \arrow[d] \\
M(\mathcal{B}) \arrow[r] & \mathrm{Conf}_k(X)/\Sigma_k
\end{tikzcd}
$$

$x \longmapsto$ punctures of $\pi^{-1}(x)$

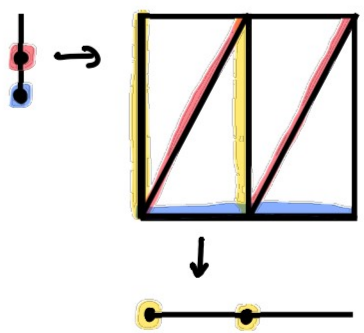
Corollaries

$X = \mathbb{C}^\times$ and $P(\mathcal{A})$ strictly supersolvable:

* $\pi : M(\mathcal{A}) \to M(\mathcal{B})$ has a section
* $\pi_1(M(\mathcal{B})) \circlearrowright H^*(X - \{k \text{ pts}\})$ trivial

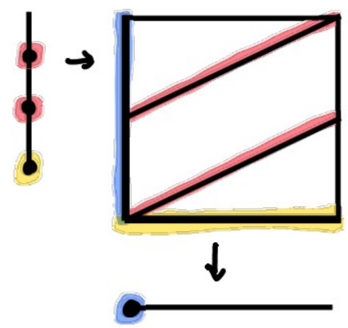
} pull back from FN-bundles

* $H^*(M(\mathcal{A})) \cong H^*(M(\mathcal{B})) \otimes H^*(X - \{k \text{ pts}\})$



$= \mathbb{C} - \{0, \pm 1\}$

Poincaré polynomial $(1+3t)(1+3t)$



Poincaré polynomial $1 + 5t + 7t^2$

* $\pi_i(M(\mathcal{A})) = 0$ for $i > 1$ and $\pi_1(M(\mathcal{A})) \cong \rtimes_i F_{k_i}$

LES + induction (works also for $X = S^1 \times S^1$, non-strict)

$$\cdots \to \pi_i(X - \{k \text{ pts}\}) \to \pi_i(M(\mathcal{A})) \to \pi_i(M(\mathcal{B})) \to \cdots$$

LCS Formula

Theorem [B-Delucchi '22]

[Falk-Randell '85 case $X=\mathbb{C}$, Kohno '85 case $\mathrm{Conf}_n(\mathbb{C})$]

$X = \mathbb{C}^\times$

chain of TM-ideals

$$P(\mathcal{A}) \supsetneq Q_{n-1} \supsetneq \cdots \supsetneq Q_1 \supsetneq \{\hat{0}\}$$

tower of fibrations

$$\begin{tikzcd}
\mathbb{C}-\{k_n \text{ pts}\} \arrow[r] & M(\mathcal{A}) \arrow[d] \\
\mathbb{C}-\{k_{n-1} \text{ pts}\} \arrow[r] & M(\mathcal{A}_{n-1}) \arrow[d] \\
 & \vdots \arrow[d] \\
 & M(\mathcal{A}_1)
\end{tikzcd}$$

$$k_i = 1 + \#\,\mathrm{atoms}(Q_i) - Q_{i-1}$$

The lower central series of $G := \pi_1(M(\mathcal{A}))$

$$G = G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots \qquad \text{where} \qquad G_j = [G_{j-1}, G]$$

* Each $G(j) := G_j / G_{j+1}$ is a free abelian group and

* $$\prod_{j=1}^{\infty} (1-t^j)^{\operatorname{rank} G(j)} = \prod_{i=1}^{n} (1 - k_i t) = \mathrm{Poin}_{M(\mathcal{A})}(-t)$$

Poincaré polynomial

Thank you