A Leray model for the Orlik-Solomon algebra

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\[ \mathcal{A} : \text{finite set of hyperplanes in } \mathbb{C}^{d+1} \quad \text{"arrangement"} \]

\[ M(\mathcal{A}) = \mathbb{C}^{d+1} \setminus \bigcup_{H \in \mathcal{A}} H \quad \text{"complement"} \]

\[ U(\mathcal{A}) = \mathbb{P}^d \setminus \bigcup_{H \in \mathcal{A}} H \quad \text{"projective complement"} \]

\[ L : \text{set of intersections } \bigcap_{H \subseteq \mathcal{A}} H, S \subseteq \mathcal{A} \quad \text{"intersection lattice"} \]

\[ \text{partially ordered by reverse inclusion} \]

\[ (= \text{lattice of flats of matroid } M, \text{ a geometric lattice}) \]
Example

Define \( A \) in \( \mathbb{C}^3 \) by:

\[
\begin{align*}
x &= y \\
x &= 0 \\
y &= 0 \\
z &= 0
\end{align*}
\]

\( L: \)

\[
\begin{align*}
(x = y = 0) & \quad (x = y, z = 0) & \quad (x = 0, z = 0) & \quad (y = 0, z = 0)
\end{align*}
\]

\( U(A) \) in \( \mathbb{P}^2 \):

Note: 3 hyperplanes meeting in codimension 2 corresponds to a linear dependency

\[
\begin{align*}
x - y &= 0 & \quad (1, -1) \\
x &= 0 & \quad (1, 0) \\
y &= 0 & \quad (0, 1)
\end{align*}
\]
$E = \text{exterior algebra over } \mathbb{Q} \text{ generated by } e_H \text{ where } H \in \mathcal{A}$

$\delta : E \to E$ derivation with $\delta e_H = 1$

$\text{OS}(L) = \frac{E}{(\delta e_C \text{ when } C \text{ is dependent})}$

$\text{Codim}_{\text{HeC}}(\wedge H) < 1|C|$

$\overline{\text{OS}}(L) = \text{subalgebra of } \text{OS}(L) \text{ generated by } \ker \delta = \langle e_H - e_{H'} \rangle$

Theorem [Brieskorn '73, Orlik-Solomon '80, Kawahara '04]

$H^*(\text{M}(A); \mathbb{Q}) \cong \text{OS}(L)$  

$H^*(\text{U}(A); \mathbb{Q}) \cong \overline{\text{OS}}(L)$

The cohomology rings depend only on the combinatorics!
$Y(A,G)$ = blowup of $\mathbb{P}^l$ along a "building set" $G \in L$.

Note $U(A) \hookrightarrow Y(A,G)$ as complement of normal crossings divisor

Example $A = \{x_i = x_j \}_{1 \leq i < j \leq n}$ in $\mathbb{C}^n$ "braid arrangement"

$M(A) =$ ordered configuration space $\text{Conf}_n(\mathbb{C})$

$L =$ partitions of $\{1, 2, \ldots, n\}$

$G =$ partitions with one non-singleton block

$Y(A, G) = \overline{M}_{0,n+1}$ Deligne-Mumford compactification of the moduli space of genus 0 curves with $n+1$ marked points
Example

$U(A)$ in $\mathbb{P}^2$:

\[ y(A, \mathcal{G}) = \text{Bl}_p \mathbb{P}^2 \]

$\mathbb{P}^3$
$U(A) \hookrightarrow Y(A, G)$ as complement of normal crossings divisor with components indexed by $G$

$\text{DP}(L, G) =$ commutative $\mathbb{Q}$-algebra generated by $x_q$ where $q \in G$ in degree 2 with relations

- $X^T$ when corresponding divisors don't intersect
- $\sum_{q \geq h} x_q$ where $H \in A$

Theorem [De Concini-Procesi '95, Feichtner-Yuzvinsky '04]

$H^\ast(Y(A, G), \mathbb{Q}) \cong \text{DP}(L, G)$

This depends only on the combinatorics $(L \& G)$
The Leray spectral sequence of the inclusion \( f : U(A) \hookrightarrow Y(d, g) \) relates these cohomologies:

\[
E_2^{pq} = H^p(Y(A, g), R^q f_* \mathcal{O}) \Rightarrow H^{p+q}(U(A), \mathcal{O})
\]

Actually, \( (E_2, d_2) \) is a \( \mathbb{Q} \)-cdga model for \( U(A) \) via a quasi-isomorphism

\[
( H^*(U(A), \mathcal{O}), 0 ) \rightarrow (E_2, d_2)
\]

Both are combinatorial: depend only on \( L \) and \( G \subseteq L \)

Goal: To obtain this quasi-isomorphism combinatorially (for any \( L, G \) - not necessarily realizable by \( A \))
A set $G \subseteq L_{>0}$ is a building set if $\forall x \in L_{>0}$,

$$V : \prod \{\emptyset, g\} \xrightarrow{\cong} \{\emptyset, x\}$$

$\forall x \in G \leq x$

Note. $A = \text{atoms}(L) \subseteq L_{\text{inv}} \subseteq G \subseteq L_{>0}$

Write $G = \text{atoms}(L) \cup \{p_1, p_2, \ldots, p_n\}$ s.t. $p_i \leq p_j$ in $L$ implies $i > j$

A partial building set is a set

$$\mathcal{H} = \text{atoms}(L) \cup \{p_1, p_2, \ldots, p_m\}$$

for some $m \leq n$.

Geometrically: $M(A) \hookrightarrow \text{Bl}_{p_m} \cdots \text{Bl}_{p_1}(\mathbb{C}^{l+1})$

complement is divisor $\cup D_g$, where nonempty intersections $g e \mathcal{H}$

$\bigcap D_g$ (since $g e \mathcal{H}$) form a locally geometric semilattice
Let $\mathcal{L}$ be a locally geometric semilattice, $p \in \mathcal{L}_{>0}$.

The "blowup" $Bl_p \mathcal{L}$ is defined as:

$$\{ x \in \mathcal{L} \mid x \neq p \} \cup \{(p, x) \mid x \in \mathcal{L}, x \neq p, px \in \mathcal{L}\}$$

$Bl_p \mathcal{L}$ is a locally geometric semilattice via:

$$y > x \text{ in } \mathcal{L} \Rightarrow y > x, (p, y) > (p, x), (p, y) > x$$

**Example**

$U(A)$ in $\mathbb{P}^2$:

$$y(A, G) = Bl_p \mathbb{P}^2$$
Example (continued)
For a partial building set $\mathcal{H} = \text{atoms}(L) \cup \{p_1, \ldots, p_m\}$ in a geometric lattice $L$, define

$$L(L, \mathcal{H}) = B_{p_m} \cdots B_{p_2} B_{p_1} L$$

This is a locally geometric semilattice with atoms $\mathcal{H}$.

Define $B(L, \mathcal{H})$ as the graded-commutative $\mathbb{Q}$-algebra with generators:

- $e_g$, $x_g$ for $g \in \mathcal{H}$
- Indegrees: $(0,1)$, $(2,0)$

and relations:

1. $e_s x_t$ if $V(SUT) \notin L(L, \mathcal{H})$
2. $\partial_e c$ if $c$ is dependent in some $[0, x] \subseteq L(L, \mathcal{H})$
3. $\sum_{g \geq n} x_g$ if $h \in \text{atoms}(L)$.

And differential:

- $d(e_g) = x_g$
- $d(x_g) = 0$
Theorem [B - Denham - Feichtner] If $i \in \mathcal{H}$ there are isomorphisms

$$\text{OS}(L) \cong H^*(B(L,\mathcal{H}), d)$$
$$\overline{\text{OS}}(L) \cong H^*(\frac{B(L,\mathcal{H})}{e_i}, d)$$

induced by an injection

$$e_h \mapsto \sum_{g \geq h} e_g \quad h \in \text{atoms}(L)$$

Remarks

1. If $L$ is realizable over $\mathbb{C}$, then $(B(L,\mathcal{G}), d) \cong (E_2, d_2)$

   [De Concini - Procesi '95]

2. The subalgebra of $B(L,\mathcal{G})$ generated by $x_g$'s is $D\mathcal{P}(L,\mathcal{G})$

3. The subalgebra of $B(L,\mathcal{H})$ generated by $e_g$'s is $\text{OS}(L(L,\mathcal{H}))$

4. $B(L, \text{atoms}(L)) \cong \text{OS}(L)$
Idea of the Proof.

1. one step at a time:

\[ B(L, \mathcal{H}) \xrightarrow{\varphi} B(L, \mathcal{H} \cup \{p\}) \]

\[ e_g \quad \begin{cases} e_g & \text{if } g \not\leq p \\ e_g + e_p & \text{if } g \leq p \end{cases} \]

\[ x_g \quad \begin{cases} x_g & \text{if } g \not\leq p \\ x_g + x_p & \text{if } g \leq p \end{cases} \]

2. upgrade relations to a Gröbner basis to obtain a monomial basis and prove that \( \varphi \) is injective.
③ Dualize $B$, using Poincaré duality of $DP(L, \mathcal{H})$ — the subalgebra generated by $x_\mathfrak{g}$'s — and local versions modeling cohomology of strata.

④ Upgrade the dual to a sheaf of algebras on the poset $\{(x,y) \mid x, y \in \mathcal{L}(L, \mathcal{H}), x \leq y\}$, whose global sections are $\mathcal{B}(L, \mathcal{H})$, use this to establish the quasi-isomorphism.

Thank you!