A Leray model for the Orlik-Solomon algebra

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$\mathcal{A}$: finite set of hyperplanes in $\mathbb{C}^{\ell+1}$ "arrangement"

$M(\mathcal{A}) = \mathbb{C}^{\ell+1} \setminus \bigcup_{H \in \mathcal{A}} H$ "complement"

$U(\mathcal{A}) = \mathbb{P}^{\ell} \setminus \bigcup_{H \in \mathcal{A}} \mathbb{P}H$ "projective complement"

L : set of intersections $\bigcap_{H \in S} H$, $S \subseteq \mathcal{A}$ "intersection lattice"

partially ordered by reverse inclusion

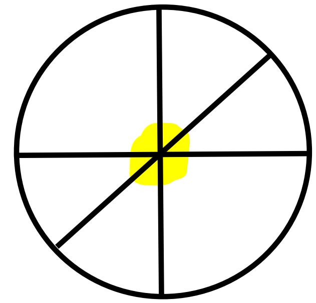
(= lattice of flats of matroid M, a geometric lattice)

Example

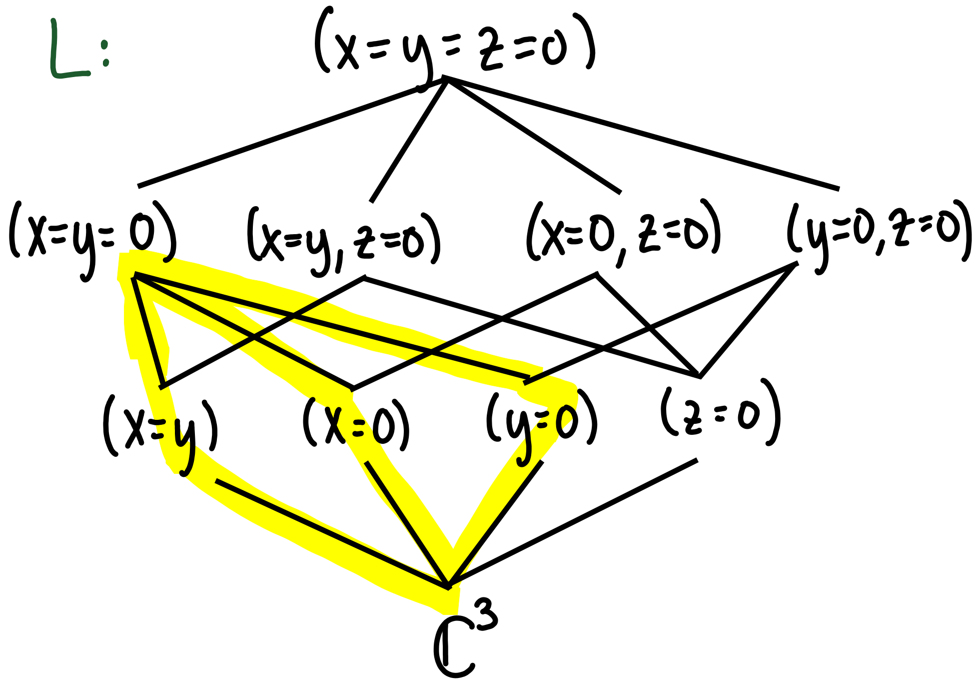
Define $\mathcal{A}$ in $\mathbb{C}^3$ by:

$x = y$
$x = 0$
$y = 0$
$z = 0$

$U(\mathcal{A})$ in $\mathbb{P}^2$:



L:



Note: 3 hyperplanes meeting in codimension 2 corresponds to a linear dependency

$x - y = 0$	$(1,-1)$
$x \quad = 0$	$(1,0)$
$y = 0$	$(0,1)$

E = exterior algebra over $\mathbb{Q}$ generated by e_H where $H \in \mathcal{A}$

$\partial : E \to E$ derivation with $\partial e_H = 1$

$$OS(L) = E \big/ \left(\partial e_C \text{ when } C \text{ is dependent} \right)$$

"Orlik-Solomon algebra"

dependent: $\operatorname{codim}\left(\bigcap_{H \in C} H \right) < |C|$

$\overline{OS}(L)$ = subalgebra of $OS(L)$ generated by $\ker \partial = \langle e_H - e_{H'} \rangle$

"Projective OS algebra"

<u>Theorem</u> [Brieskorn '73, Orlik-Solomon '80, Kawahara '04]

$$H^*(M(\mathcal{A}), \mathbb{Q}) \cong OS(L) \qquad H^*(U(\mathcal{A}); \mathbb{Q}) \cong \overline{OS}(L)$$

The cohomology rings depend only on the combinatorics!

$Y(\mathcal{A}, \mathcal{G})$ = blowup of $\mathbb{P}^{\ell}$ along a "building set" $\mathcal{G} \subseteq L$.

"wonderful compactification"

Note $U(\mathcal{A}) \hookrightarrow Y(\mathcal{A}, \mathcal{G})$ as complement of normal crossings divisor

Example $\mathcal{A} = \{x_i = x_j\}_{1 \le i < j \le n}$ in $\mathbb{C}^n$ "braid arrangement"

$M(\mathcal{A})$ = ordered configuration space $\mathrm{Conf}_n(\mathbb{C})$

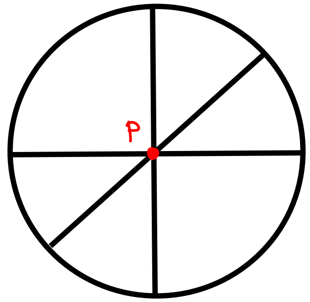
L = partitions of $\{1, 2, \ldots, n\}$

$\mathcal{G}$ = partitions with one non-singleton block

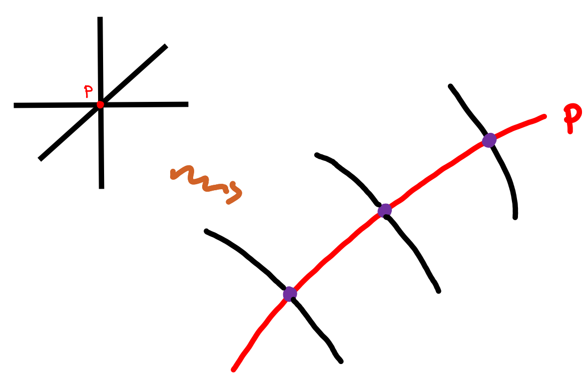
$Y(\mathcal{A}, \mathcal{G}) = \overline{M_{0,n+1}}$ Deligne-Mumford compactification of the moduli space of genus 0 curves with $n+1$ marked points

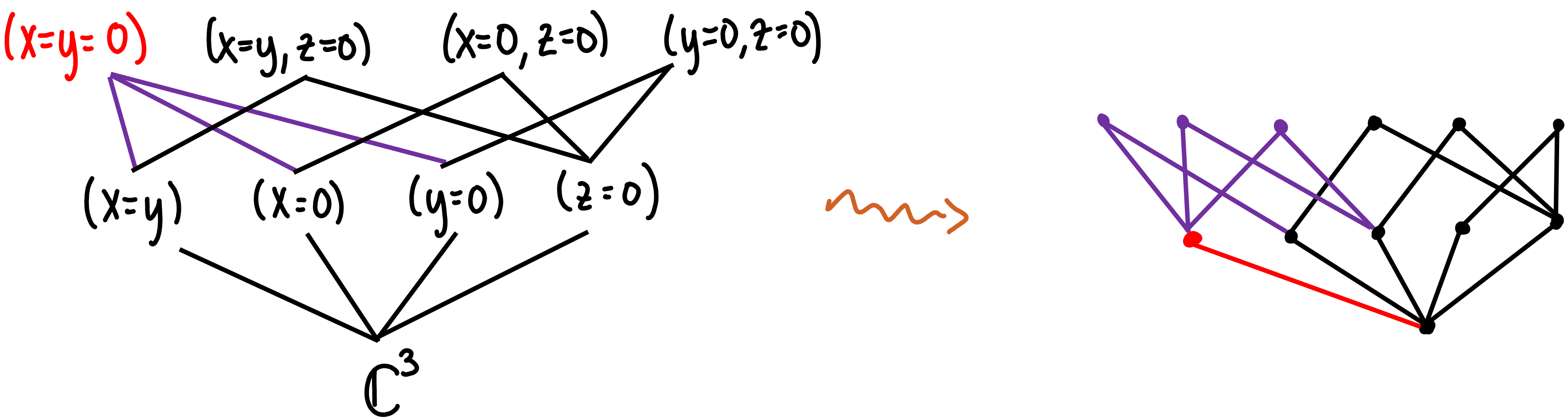
Example

$\mathcal{U}(\mathcal{A})$ in $\mathbb{P}^2$:



$\mathcal{Y}(\mathcal{A},\mathcal{G}) = Bl_p \, \mathbb{P}^2$





$U(\mathcal{A}) \hookrightarrow Y(\mathcal{A}, \mathcal{G})$ as complement of normal crossings divisor with components indexed by $\mathcal{G}$

$DP(L, \mathcal{G})$ = commutative $\mathbb{Q}$-algebra generated by x_g where $g \in \mathcal{G}$ in degree 2 with relations

- x_T when corresponding divisors don't intersect
- $\sum_{g \geq H} x_g$ where $H \in \mathcal{A}$

can be described by a simplicial complex of "nested sets"

<u>Theorem</u> [De Concini-Procesi '95, Feichtner-Yuzvinsky '04]

$$H^*(Y(\mathcal{A}, \mathcal{G}); \mathbb{Q}) \cong DP(L, \mathcal{G})$$

This depends only on the combinatorics (L & $\mathcal{G}$)

The Leray spectral sequence of the inclusion $f: \mathcal{U}(\mathcal{A}) \hookrightarrow \mathcal{Y}(\mathcal{A},\mathcal{G})$ relates these cohomologies:

$$E_2^{pq} = H^p(\mathcal{Y}(\mathcal{A},\mathcal{G}), R^q f_* \mathbb{Q}) \Rightarrow H^{p+q}(\mathcal{U}(\mathcal{A}), \mathbb{Q})$$

Actually, (E_2, d_2) is a $\mathbb{Q}$-cdga model for $\mathcal{U}(\mathcal{A})$

via a quasi-isomorphism

$$(H^*(\mathcal{U}(\mathcal{A}), \mathbb{Q}), 0) \longrightarrow (E_2, d_2)$$

Both are combinatorial : depend only on

$$L \ \& \ \mathcal{G} \subseteq L$$

geometric lattice

building set

<u>Goal</u>: To obtain this quasi-isomorphism combinatorially (for any $L, \mathcal{G}$ – not necessarily realizable by $\mathcal{A}$)

A set $\mathcal{G} \subseteq L_{>\hat{0}}$ is a building set if $\forall x \in L_{>\hat{0}}$,

$$V: \prod_{g \in \max \mathcal{G}_{\leq x}} [\hat{0}, g] \xrightarrow{\cong} [\hat{0}, x]$$

Note $\mathcal{A} = \text{atoms}(L) \subseteq L_{\text{irr}} \subseteq \mathcal{G} \subseteq L_{>\hat{0}}$

Write $\mathcal{G} = \text{atoms}(L) \cup \{p_1, p_2, \ldots, p_n\}$ s.t. $p_i \leq p_j$ in L $\Rightarrow i > j$

A partial building set is a set
$\mathcal{H} = \text{atoms}(L) \cup \{p_1, p_2, \ldots, p_m\}$ for some $m \leq n$

Geometrically: $M(\mathcal{A}) \hookrightarrow Bl_{p_m} \circ \cdots \circ Bl_{p_1}(\mathbb{C}^{\ell+1})$

complement is divisor $\bigcup_{g \in \mathcal{H}} D_g$, where nonempty intersections

$\bigcap_{g \in S} D_g$ $(S \subseteq \mathcal{H})$ form a locally geometric semilattice "combinatorial blowup"

$\mathcal{L}$ = locally geometric semilattice, $p \in \mathcal{L}_{>\hat{0}}$

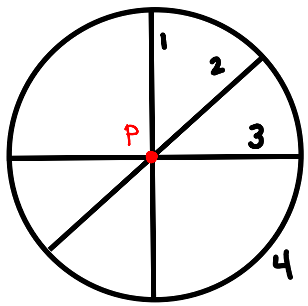
$Bl_p\mathcal{L} := \{x \in \mathcal{L} \mid x \not\geq p\} \cup \{(p,x) \mid x \in \mathcal{L},\ x \not\geq p,\ p \vee x \in \mathcal{L}\}$

$x \not\geq p$ — x intersects p

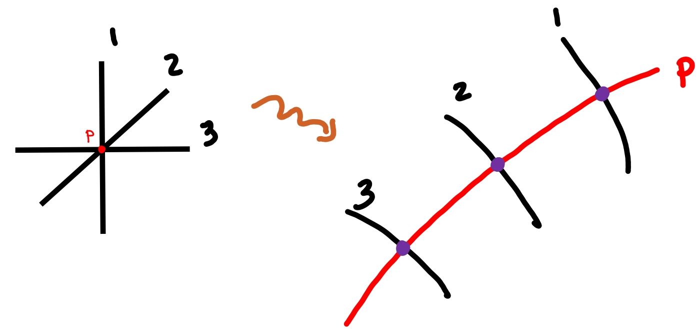
$Bl_p\mathcal{L}$ is a locally geometric semilattice via: $y > x$ in $\mathcal{L}$ $\Rightarrow$ $y > x$, $(p,y) > (p,x)$, $(p,y) > x$

Example

$U(\mathcal{A})$ in $\mathbb{P}^2$:

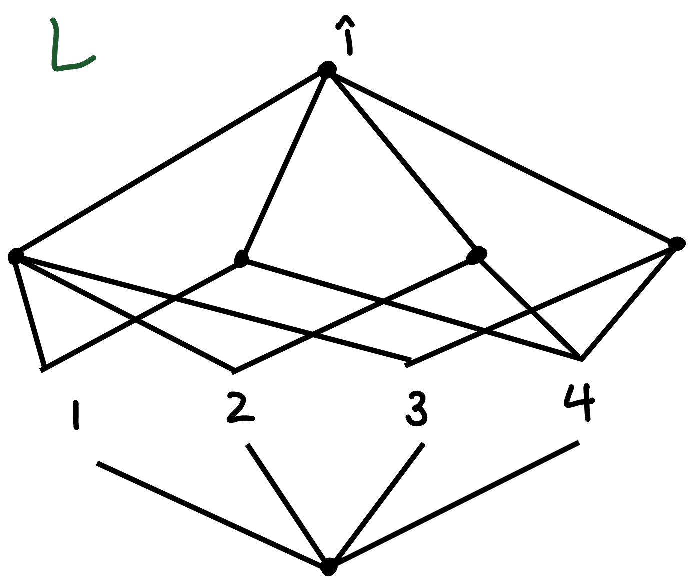


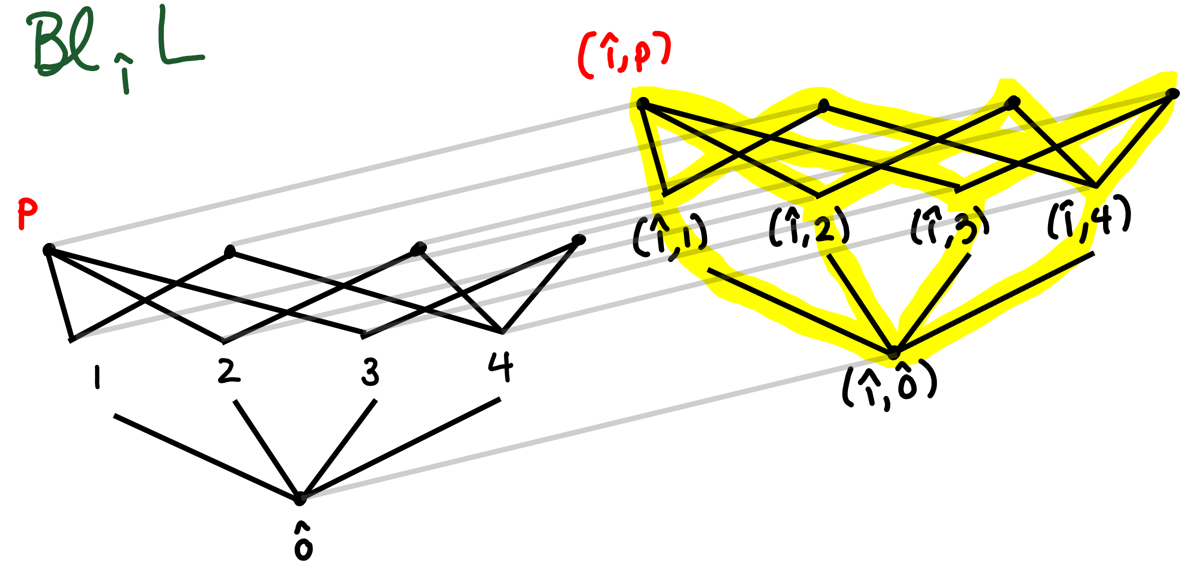
$Y(\mathcal{A},\mathcal{G}) = Bl_p\mathbb{P}^2$

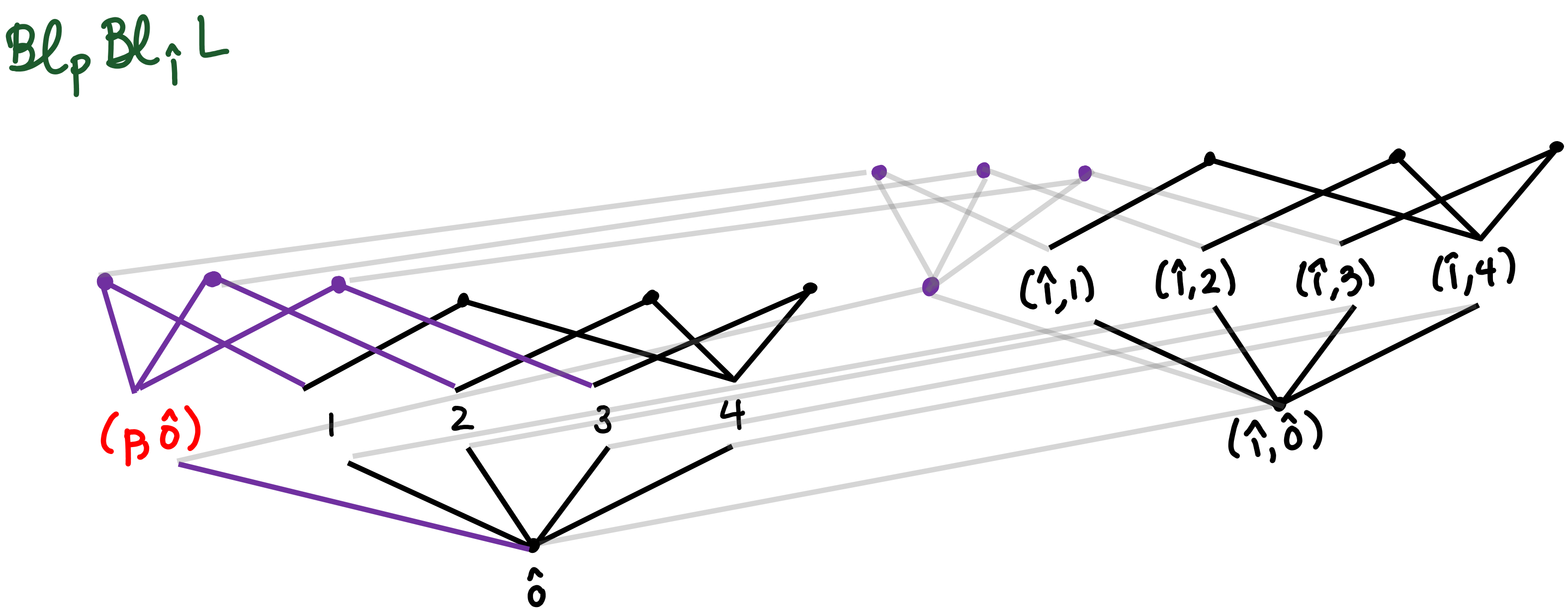


$\mathcal{G} \subseteq \mathcal{L}$ $\qquad$ $Bl_p\mathcal{L}$

Example (continued)







For a partial building set $\mathcal{H} = \text{atoms}(L) \cup \{p_1, \ldots, p_m\}$ in a geometric lattice L, define

$$\mathcal{L}(L, \mathcal{H}) = Bl_{p_m} \cdots Bl_{p_2} Bl_{p_1} L$$

This is a locally geometric semilattice with atoms $\equiv \mathcal{H}$

Define $B(L, \mathcal{H})$ as the graded-commutative $\mathbb{Q}$-algebra with

generators: e_g x_g $g \in \mathcal{H}$

in degrees: $(0,1)$ $(2,0)$

and relations:

(i) $e_S x_T$ if $\vee(S \cup T) \notin \mathcal{L}(L, \mathcal{H})$ — Stanley-Reisner relations from an associated simplicial complex

(ii) ∂e_C if C is dependent in some $[0, x] \subseteq \mathcal{L}(L, \mathcal{H})$ — From Orlik-Solomon algebra

(iii) $\sum_{g \geq h} x_g$ if $h \in \text{atoms}(L)$ — linear relations

and differential: $d(e_g) = x_g$ $d(x_g) = 0$

Theorem [B - Denham - Feichtner] If $\hat{1} \in \mathcal{H}$ there are isomorphisms

$$OS(L) \cong H^*(B(L,\mathcal{H}), d) \qquad \overline{OS}(L) \cong H^*\left(B(L,\mathcal{H})/e_{\hat{1}}, d \right)$$

induced by an injection

$$e_h \longmapsto \sum_{g \geq h} e_g \qquad h \in \text{atoms}(L)$$

Remarks

① If L is realizable over $\mathbb{C}$, then $(B(L,\mathcal{G}), d) \cong (E_2, d_2)$

[De Concini - Procesi '95]

② The subalgebra of $B(L,\mathcal{G})$ generated by x_g's is $DP(L,\mathcal{G})$

③ The subalgebra of $B(L,\mathcal{H})$ generated by e_g's is $OS(\mathcal{L}(L,\mathcal{H}))$

④ $B(L, \text{atoms}(L)) \cong OS(L)$

Idea of the Proof.

① one step at a time:

$$B(L,\mathcal{H}) \xrightarrow{\varphi} B(L,\mathcal{H}\cup\{p\})$$

$$e_g \longmapsto \begin{cases} e_g & \text{if } g \not\leq p \\ e_g + e_p & \text{if } g \leq p \end{cases}$$

$$x_g \longmapsto \begin{cases} x_g & \text{if } g \not\leq p \\ x_g + x_p & \text{if } g \leq p \end{cases}$$

② Upgrade relations to a Gröbner basis to obtain a monomial basis and prove that φ is injective.

3. Dualize B, using Poincaré duality of $DP(L,\mathcal{H})$ – the subalgebra generated by x_g's – and local versions modeling cohomology of strata.

4. Upgrade the dual to a sheaf of algebras on the poset $\{(x,y) \mid x,y \in \mathcal{L}(L,\mathcal{H}),\ x \leq y\}$, whose global sections are $B(L,\mathcal{H})$, use this to establish the quasi-isomorphism

Thank you!

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