Combinatorics of orbit configuration spaces

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**Orbit configuration spaces** \[ \text{[Xicotencatl `97]} \]

\[ X: \text{"nice" topological space} \quad (\text{eg. } \mathbb{C}, \mathbb{C}^*, \mathbb{S}^1 \times \mathbb{S}^1) \]

\[ G: \text{finite group} \]

\[ G \curvearrowright X \text{ almost freely: } S = \{ x \in X \mid \exists g \in G - e, \ g x = x \} \text{ finite} \]

**Note:** \[ G \curvearrowright (X - S) \text{ freely} \]

\[ \text{Conf}_n^G(X - S) = \left\{ (x_1, \ldots, x_n) \in (X - S)^n \mid G x_i \cap G x_j = \emptyset \right\} \subset (X - S)^n \]

\[ S_n[G] \]

\[ S_n \]

\[ \text{Conf}_n^G(X - S) = \left\{ (x_1, \ldots, x_n) \in X^n \mid G x_i \neq x_j \quad 1 \leq i < j \leq n \quad g \in G \right\} \subset X^n \]

i.e. the complement of the arrangement

\[ A_n(G, X): \quad H_{ij}(g) = \{ (x_1, \ldots, x_n) \in X^n \mid g x_i = x_j \} \quad 1 \leq i < j \leq n \quad g \in G \]

\[ H_k^S = \{ (x_1, \ldots, x_n) \in X^n \mid x_k = s \} \quad 1 \leq k \leq n \quad s \in S \]
Combinatorics of an arrangement

A layer of $A_n = A_n(G,X)$ is a connected component of an intersection $\bigcap_{H \in T} (T \subseteq A)$

The poset of layers $P_n(G,X)$ is the set of layers, partially ordered by reverse inclusion.

**Example**  $X = C$

$G = 1$

$Conf_n(C) = \{(x_1, \ldots, x_n) \in C^n \mid x_i \neq x_j\}$

complement of braid arrangement: hyperplanes $x_i = x_j$  \(1 \leq i < j \leq n\)

$P_n(1,C)$ is the partition lattice $\Pi_n$

An intersection of diagonals in $C^n$ corresponds to a partition of $\{1,2,\ldots,n\}$
Example (Type C toric arrangement)

$X = \mathbb{C}^* \text{ or } S'$

$G = \mathbb{Z}_2$ acting by group inversion $x \mapsto x^{-1}$

$n = 2$

$S = \{ \pm 1 \}$

$x_1 = x_2$

$x_1 = x_2^{-1}$

$x_1 = 1$

$x_2 = 1$

$x_1 = -1$

$x_2 = -1$
<table>
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<th>Topology</th>
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<td>Confₙ(C)</td>
<td>Partition lattice $\Pi_n$</td>
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<tr>
<td>Confₙ⁺(X-S)</td>
<td>??</td>
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**Questions**

- How can we describe layers using partitions?
- How can we use this combinatorics to understand the topology?
**Generalizing partition/Dowling lattices**

$G$: finite group  
$S$: finite $G$-set  
$[n] = \{1, 2, \ldots, n\}$

A partial $G$-partition of $[n]$ consists of a partition

$\beta \rightarrow T$  
where  $T \subseteq [n]$  

along with  $\forall B \in \beta$  an equivalence class of

$G$-colorings  $b: B \rightarrow G$  
($b \sim bg \ \forall g \in G$)

The zero block of $\beta$ is the set  $\emptyset = [n] - T$.  

The Dowling poset  $D_n(G, S)$ is the set of pairs

$(\beta, \gamma)$  
where  $\beta$ is a partial $G$-partition and

$\gamma: \emptyset \rightarrow S$ is an $S$-coloring of its zero block $\emptyset$.

covering  
merge:  $(\{A \cup B\} \cup \beta, \gamma) < (A \cup B \cup \beta, \gamma)$

relations  
color:  $(B \cup \beta, \gamma) < (\emptyset, \emptyset \cup \gamma)$
Example

Partition lattice $\Pi_n = D_n (1, \emptyset) \cong D_{n-1} (1)$

Dowling lattice $D_n (G) = D_n (G, *)$ [Dowling '73]

$G = \mathbb{Z}_2 = \{ e, i \}$
$
\cap \text{trivial}
$

$s = \{ \pm 1 \}$

$n = 2$
Example

\[ X = \mathbb{C}^x \quad G = \mathbb{Z}_2 \quad S = \{ \pm 1 \} \quad n = 2 \]
Recall:

\[ G : \] finite group \quad \mathcal{X} : \text{G-space} \quad S = \{x \in \mathcal{X} \mid \exists g \in \text{G-e}, gx = x \} \\
\text{An}(G, X) : \text{subspaces} \quad gx_i = x_j, \quad x_k = s \quad (g \in G, s \in S) \\
\text{P}_n(G, X) : \text{poset of layers, connected components of intersections} \\
S_n[G] = G^n \times S_n \cap \text{An}, \text{P}_n \\

Question: How can we describe \( \text{P}_n(G, X) \) using partitions?

Answer: [BG'18]

There is a \( S_n[G] \)-equivariant poset isomorphism

\[
\text{P}_n(G, X) \cong D_n(G, S)
\]

Question: How can we use \( D_n(G, S) \) to understand the topology?
Example

Local Structure

\[ (1,1) \quad (1,-1) \quad (-1,1) \quad (-1,-1) \]

\[ x_1 = 1 \quad x_2 = 1 \quad x_1 = x_2 \quad x_4 = x_2^{-1} \quad x_2 = -1 \quad x_6 = -1 \]

\[ x^2 \]
Local Structure

Example

In general:

layers of a local arrangement

an interval in the poset of layers
Theorem [BG'18] For \((\beta, \gamma) \in D_n(G, S)\),

\[(D_n(G, S))_{\leq (\beta, \gamma)} \equiv \prod_{B \in \beta} \pi_B \times \prod_{G \cdot s \in S/G} D_{sY(Gs)}(Gs)\]

**Note** Closed intervals are geometric lattices and hence homotopy equivalent to a wedge of spheres.

\[(D_n(G, S))_{\geq (\beta, \gamma)} \equiv D_\beta(G, S)\]
Characteristic polynomial \((\text{Assume } S \neq \emptyset)\)

The characteristic polynomial of \(D_n(G,S)\) is

\[
\chi(t) = \sum_{x \in D_n(G,S)} \mu(\delta, x) t^{n - \text{rank}(x)}
\]

\(\mu\) Möbius function

**Theorem** \([BG'18]\)

\[
\chi(t) = \prod_{k=0}^{n-1} (t - |S| - |G|_1 k)
\]

**Corollary** The Euler characteristic of \(\text{Conf}_n^G (X - S)\) is

\[
\prod_{k=0}^{n-1} (\chi_x - |S| - |G|_1 k)
\]

\(\chi_x\) Euler characteristic of \(X\)

And sometimes this tells us the Poincaré polynomial ...

**Example** for \(X = C^* \cup \mathbb{Z}_2 = G\): \(\text{Poin}(t) = \prod_{k=1} (1 + (1+2k) t)\)
Realizability?

**Theorem** [Dowling '73]

\[ D_n(G) \text{ is realized by a complex hyperplane arrangement} \iff G \text{ is cyclic.} \]

*Other fields put a restriction on } |G|

**Question** When is \( D_n(G,S) \) realizable? (by a hypersurface arrangement)

**Note** Need \( \prod_{B \in \beta} \pi_B \times \prod_{G \in S/G} D_{\text{gen}}(G) \) realizable cyclic!

**Consequence**

Smooth complex curve

If \( GCX \) almost freely then the stabilizer of every point is cyclic.
Thank you