Combinatorics of orbit configuration spaces

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Orbit configuration spaces [Xicotencatl '97]

X : "nice" topological space (eg. $\mathbb{C}, \mathbb{C}^\times, S^1\times S^1$)

G: finite group

$G \circlearrowright X$ almost freely : $S=\{x\in X \mid \exists g\in G-e,\ gx=x\}$ finite

Note: $G \circlearrowright (X-S)$ freely

$$\mathrm{Conf}_n^G(X-S) = \{(x_1,\dots,x_n)\in (X-S)^n \mid Gx_i\cap Gx_j=\emptyset\} \subseteq (X-S)^n$$

$\circlearrowleft$

$S_n[G]$

$$= \left\{(x_1,\dots,x_n)\in X^n \;\middle|\; \begin{matrix} gx_i\neq x_j & 1\le i<j\le n & g\in G \\ x_k\neq s & 1\le k\le n & s\in S \end{matrix}\right\} \subseteq X^n$$

i.e. the complement of the arrangement

$\mathcal{A}_n(G,X)$:

$H_{ij}(g) = \{(x_1,\dots,x_n)\in X^n \mid gx_i=x_j\}$ $1\le i<j\le n$ $g\in G$

$H_k^s = \{(x_1,\dots,x_n)\in X^n \mid x_k=s\}$ $1\le k\le n$ $s\in S$

Combinatorics of an arrangement

A layer of $\mathcal{A}_n = \mathcal{A}_n(G, X)$ is a connected component of an intersection $\bigcap_{H \in T} H$ $(T \subseteq \mathcal{A}_n)$

The poset of layers $P_n(G, X)$ is the set of layers, partially ordered by reverse inclusion.

Example $X = \mathbb{C}$, $G = 1$ $\qquad \mathrm{Conf}_n(\mathbb{C}) = \{(x_1, \dots, x_n) \in \mathbb{C}^n \mid x_i \neq x_j\}$

complement of braid arrangement: hyperplanes $x_i = x_j$ $(1 \leq i < j \leq n)$

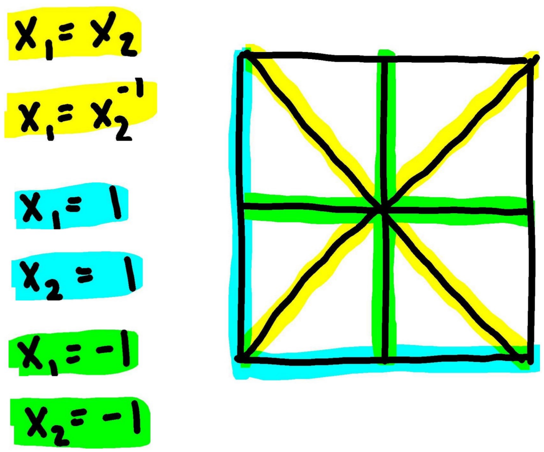
$P_n(1, \mathbb{C})$ is the partition lattice Π_n

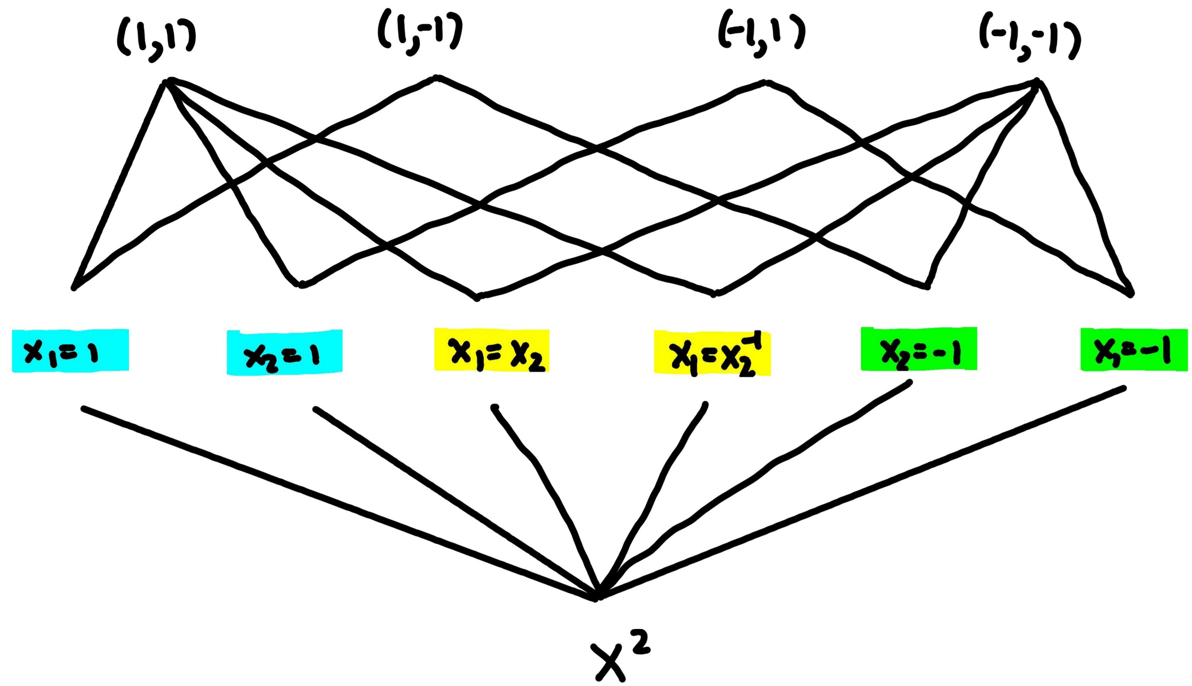
An intersection of diagonals in $\mathbb{C}^n$ corresponds to a partition of $\{1, 2, \dots, n\}$

Example (Type C toric arrangement)

$X = \mathbb{C}^\times$ or S^1 $\qquad$ $G = \mathbb{Z}_2$ acting by group inversion $x \mapsto x^{-1}$ $\qquad$ $n = 2$

$S = \{\pm 1\}$





Topology	Combinatorics
$\mathrm{Conf}_n(\mathbb{C})$	Partition lattice Π_n
$\mathrm{Conf}_n^G(X-S)$	??

Questions

- How can we describe layers using partitions?
- How can we use this combinatorics to understand the topology?

Generalizing partition/Dowling lattices

G: finite group $\qquad$ S: finite G-set $\qquad$ $[n] = \{1, 2, \ldots, n\}$

A partial G-partition of $[n]$ consists of a partition

$\rho \vdash T$ where $T \subseteq [n]$

along with $\forall B \in \rho$ an equivalence class of

G-colorings $\quad b: B \to G \quad (b \sim bg \quad \forall g \in G)$

The zero block of β is the set $Z = [n] - T$.

The Dowling poset $D_n(G,S)$ is the set of pairs

(ρ, z) where ρ is a partial G-partition and

$z: Z \to S$ is an S-coloring of its zero block Z.

covering relations

- merge: $(\{A, B\} \cup \rho, z) \lessdot (\{A \cup B\} \cup \rho, z)$
- color: $(\{B\} \cup \beta, z) \lessdot (\beta, z \cup z_B)$

__Example__ Partition lattice $\Pi_n = D_n(1,\emptyset) \cong D_{n-1}(1)$

__Example__ Dowling lattice $D_n(G) = D_n(G,*)$ [Dowling '73]

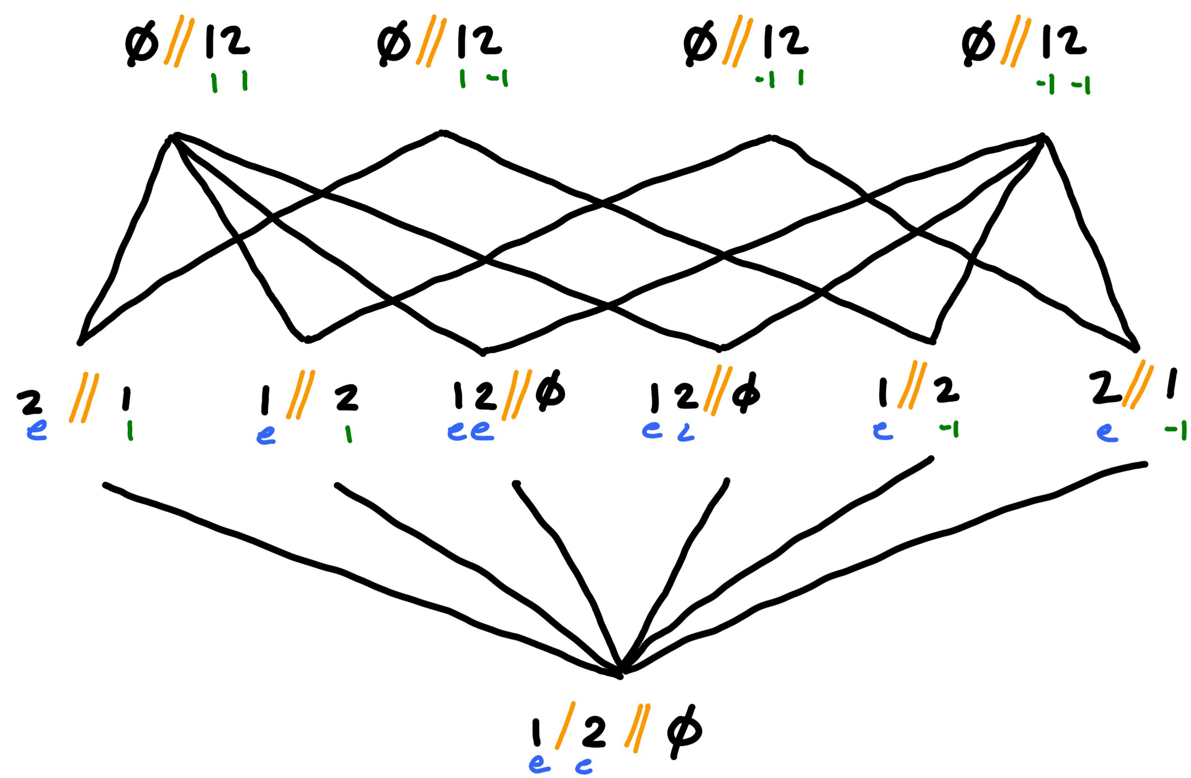
__Example__

$G = \mathbb{Z}_2 = \{e, \iota\}$

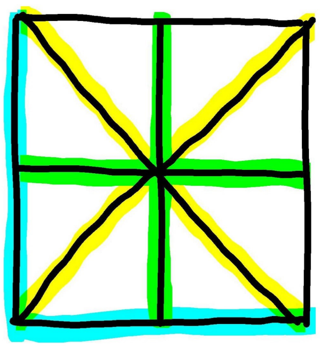
↻ trivial

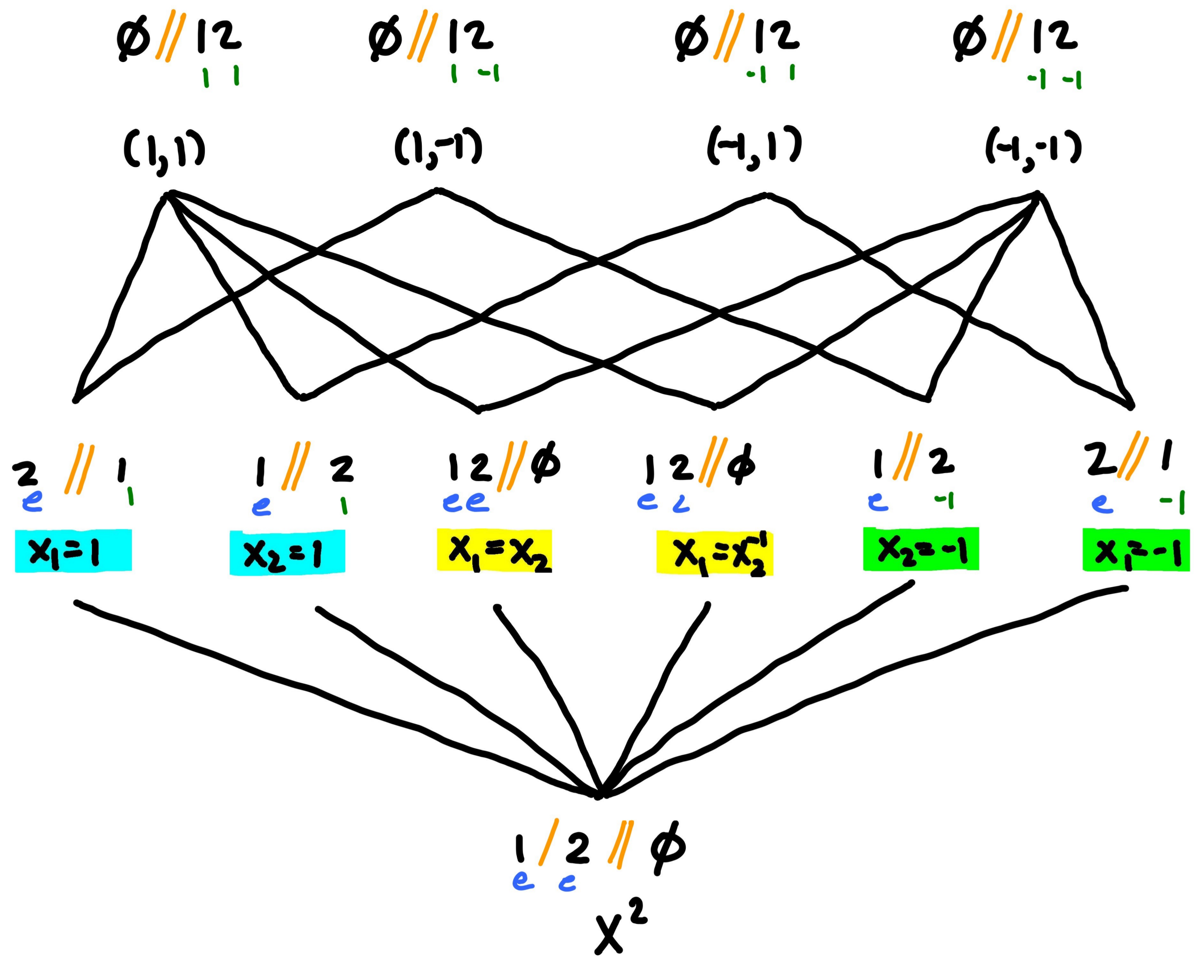
$S = \{\pm 1\}$

$n = 2$



Example $X = \mathbb{C}^\times$ $G = \mathbb{Z}_2$ $S = \{\pm 1\}$ $n = 2$





Recall:

G : finite group X : G-space $S = \{x \in X \mid \exists g \in G - e,\ gx = x\}$

$A_n(G,X)$: subspaces $gx_i = x_j$, $x_k = s$ $(g \in G,\ s \in S)$

$P_n(G,X)$: poset of layers, connected components of intersections

$S_n[G] = G^n \rtimes S_n \circlearrowright A_n, P_n$

Question: How can we describe $P_n(G,X)$ using partitions?

Answer: [BG'18]

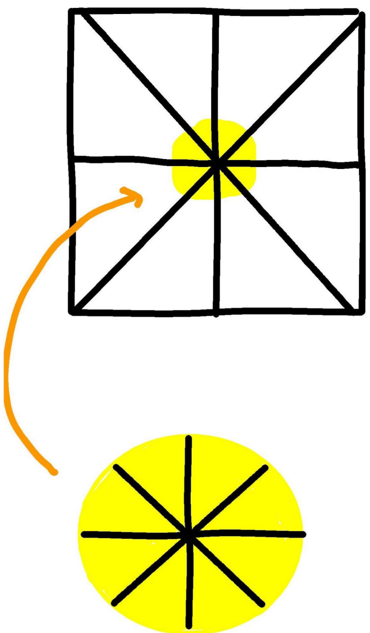
There is a $S_n[G]$-equivariant poset isomorphism

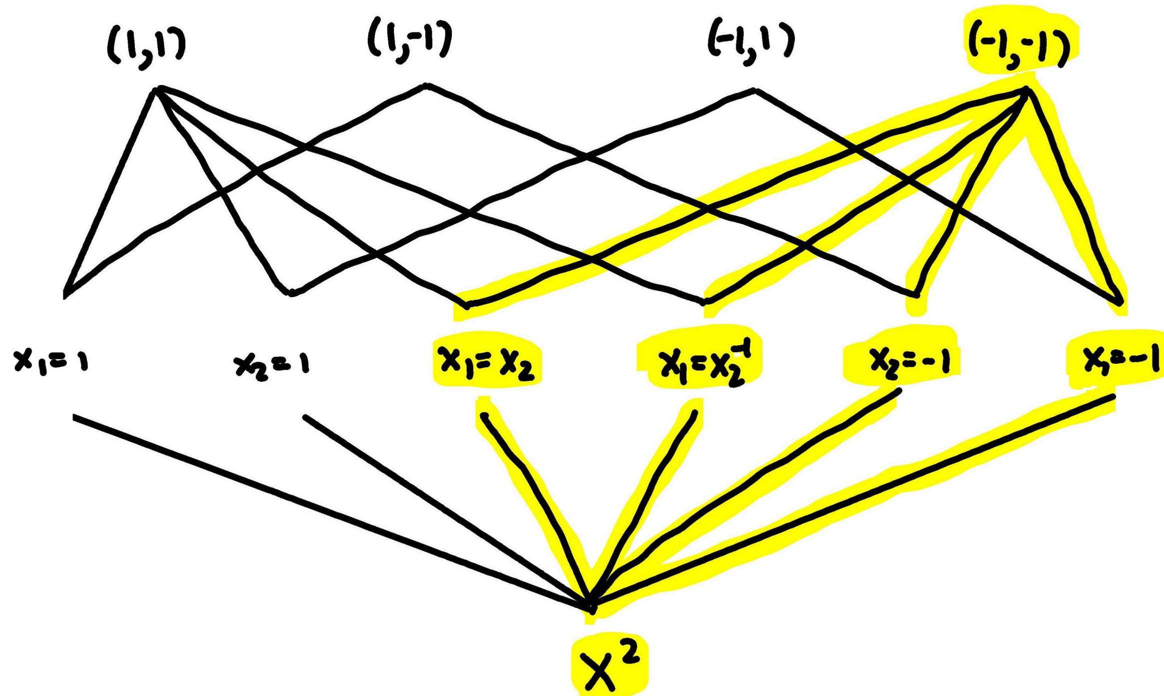
$$P_n(G,X) \cong D_n(G,S)$$

Question: How can we use $D_n(G,S)$ to understand the topology?

Local Structure

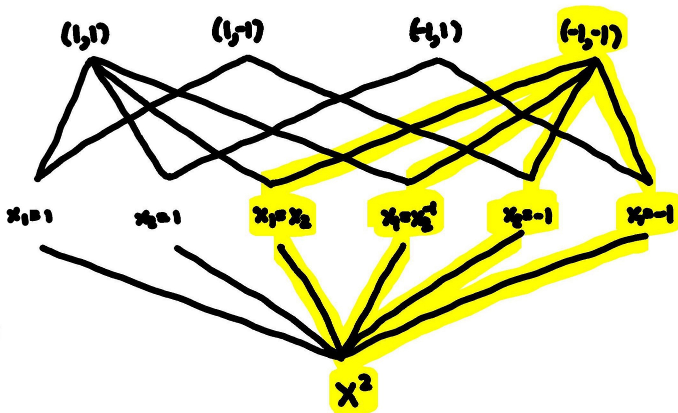
Example

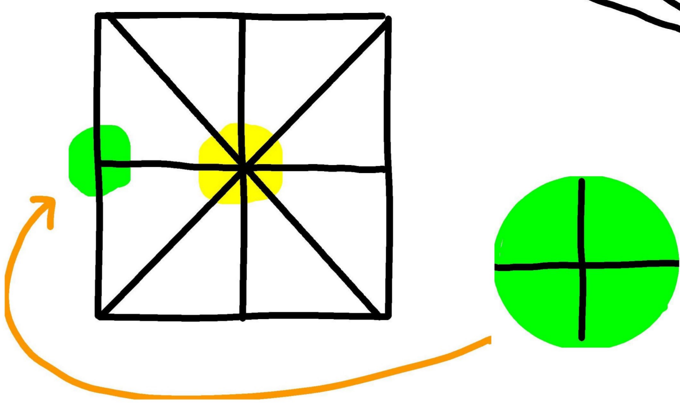


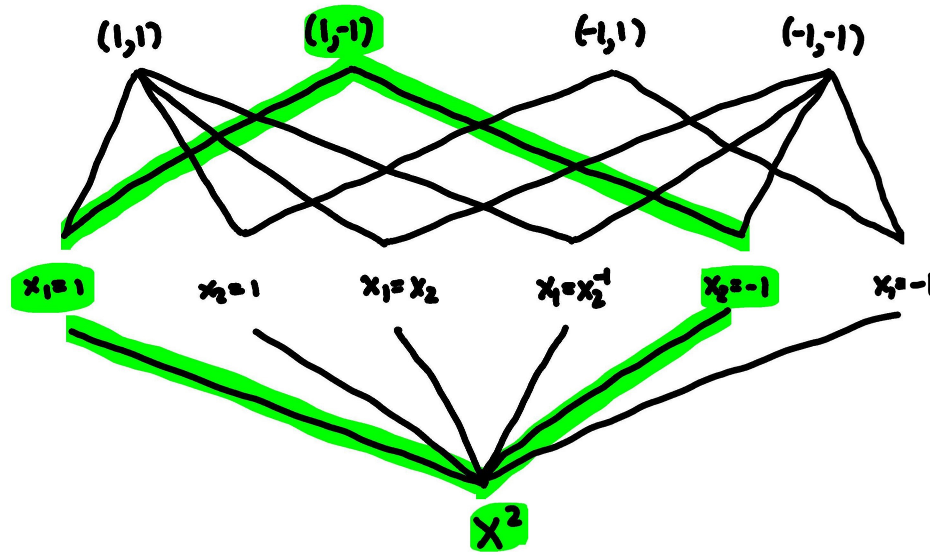


Local Structure

Example







In general:

layers of a local arrangement $\longleftrightarrow$ an interval in the poset of layers

Theorem [BG '18] For $(\beta, z) \in D_n(G,S)$,

stabilizer subgroup

(a) $$(D_n(G,S))_{\leq(\beta,z)} \cong \prod_{B \in \beta} \Pi_B \times \prod_{G\cdot s \in S/G} D_{z^{-1}(G\cdot s)}(G_s)$$

partition lattice

G-orbits of S

Dowling lattice

Note Closed intervals are geometric lattices and hence homotopy equivalent to a wedge of spheres

(b) $$(D_n(G,S))_{\geq(\beta,z)} \cong D_\beta(G,S)$$

Characteristic polynomial (Assume $S \neq \emptyset$)

The characteristic polynomial of $D_n(G,S)$ is

$$\chi(t) = \sum_{x \in D_n(G,S)} \mu(\hat{0}, x)\, t^{n-\operatorname{rank}(x)}$$

möbius function

Theorem [BG'18] $\chi(t) = \prod_{k=0}^{n-1} (t - |S| - |G|k)$

Corollary The Euler characteristic of $\operatorname{Conf}_n^G(X-S)$ is

$$\prod_{k=0}^{n-1} (\chi_X - |S| - |G|k)$$

Euler characteristic of X

And sometimes this tells us the Poincaré polynomial ...

Example for $X = \mathbb{C}^* \circlearrowleft \mathbb{Z}_2 = G$: $\operatorname{Poin}(t) = \prod_{k=1}^{n} (1 + (1+2k)t)$

Realizability?

Theorem [Dowling '73]

$D_n(G)$ is realized by a complex hyperplane arrangement $\iff$ G is cyclic.

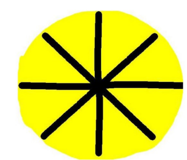
other fields put a restriction on $|G|$

Question When is $D_n(G,S)$ realizable?

(by a hypersurface arrangement)

Note Need $\prod_{B \in \beta} \pi_B \times \prod_{G_s \in S/G} D_{S(G_s)}(G_s)$ realizable

cyclic!



Consequence

smooth complex curve

If $G \curvearrowright X$ almost freely then the stabilizer of every point is cyclic.

Thank you