

Math 7410 Graph Theory

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Definition of a graph

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Example 1.2

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- ▶ V (or $V(G)$) is a finite set whose elements are called **vertices**;

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- ▶ $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$

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- ▶ $\mathcal{I}$, called the **incidence relation**, is a subset of $V \times E$ in which each edge is in relation with exactly one or two vertices.

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- ▶ $\mathcal{I} = \{(v_1, e_1), (v_1, e_4), (v_1, e_5), (v_1, e_6), (v_2, e_1), (v_2, e_2), (v_3, e_2), (v_3, e_3), (v_3, e_5), (v_3, e_6), (v_4, e_3), (v_4, e_4), (v_4, e_7)\}$

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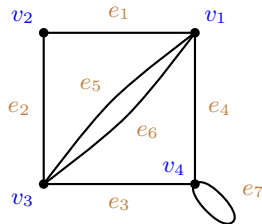
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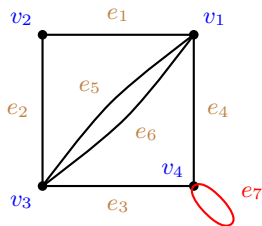
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Simple graphs

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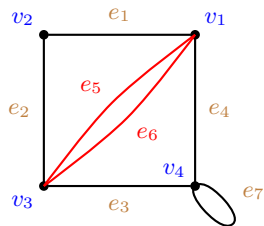
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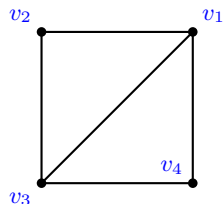
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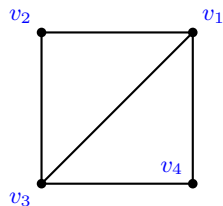


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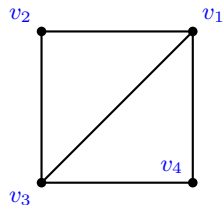
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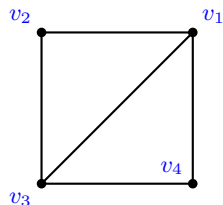
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Note 1.5

*In some books, what we defined as a **graph** is called a **multigraph** and what we defined as a **simple graph** is called a **graph**.*

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- ▶ Similarly, if U is a subset of the vertex set of G , then $N(U)$ is the set of those vertices that are not in U , but are adjacent to a vertex in U .

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The graphs $G_1 = (V_1, E_1, \mathcal{I}_1)$ and $G_2 = (V_2, E_2, \mathcal{I}_2)$ are **isomorphic**, written $G_1 \cong G_2$, if there are bijections $\varphi : V_1 \rightarrow V_2$ and $\psi : E_1 \rightarrow E_2$ such that $(v, e) \in \mathcal{I}_1$ if and only if $(\varphi(v), \psi(e)) \in \mathcal{I}_2$. Such a pair of bijections is an **isomorphism**.

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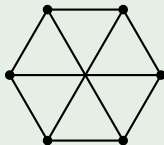
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There is a constant c and an algorithm that can decide whether two graphs on n vertices are isomorphic or not in at most $2^{\mathcal{O}((\log n)^c)}$ steps.*

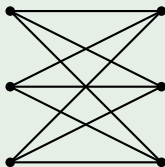
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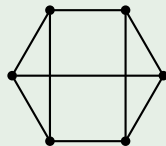
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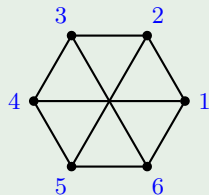


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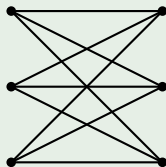
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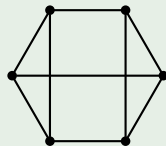
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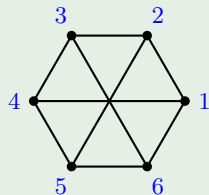


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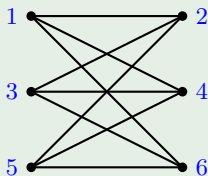
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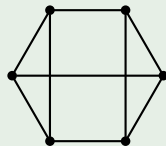
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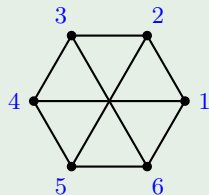


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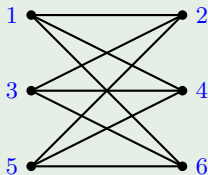
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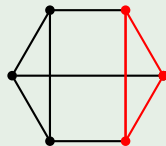
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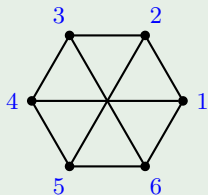


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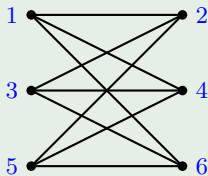
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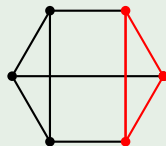
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Problem 1

For every positive integer n , construct a simple graph with exactly n automorphisms.

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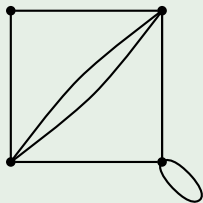
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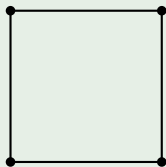
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Subgraph Example

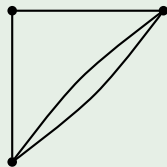
Example 1.16



G_1



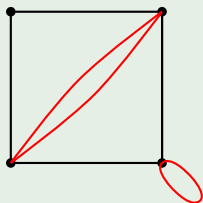
G_2



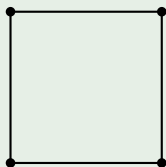
G_3

Subgraph Example

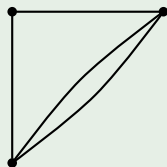
Example 1.16



G_1



G_2

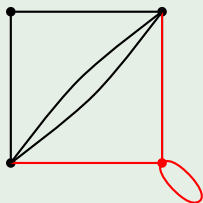


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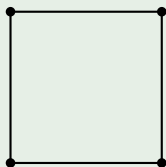
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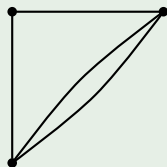
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The **deck** of a graph G is the collection of graphs $G - v$ over all $v \in V(G)$. A graph is **reconstructible** if no other graph (up to isomorphism) has the same deck.

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Conjecture 1.18 (Edge-Reconstruction Conjecture)

Every simple graph on at least four edges is edge-reconstructible.

Walks, Trails, Paths, and Cycles

Definition 1.19

- ▶ A **walk** is a sequence $v_0, e_1, v_1, e_2, v_2, \dots, e_n, v_n$, where each edge e_i is incident with vertices v_{i-1} and v_i .

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- (a) *Show that the order of a self-complementary graph is congruent to 0 or 1 modulo 4.*
- (b) *Construct a self-complementary graph of order n for every positive integer n congruent to 0 or 1 modulo 4.*

Hand-Shaking Lemma

Theorem 1.21 (Hand-Shaking Lemma)

$$\sum_{v \in V(G)} d(v) = 2\|G\|$$

Corollary 1.22

The number of vertices of odd degree is even.

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- ▶ The distance between sets U and W of vertices of G , written $d(U, W)$, is the length of a shortest uw -path where $u \in U$ and $w \in W$, or infinity if no such path exists.

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Theorem 2.2

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But P cannot contain v as $d_T(v) = 1$, and so it also lies in T' . □

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The remainder of the proof is left as an exercise.

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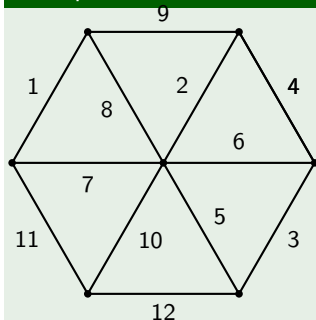
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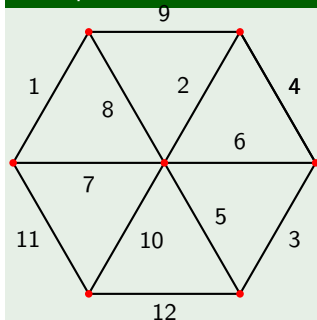
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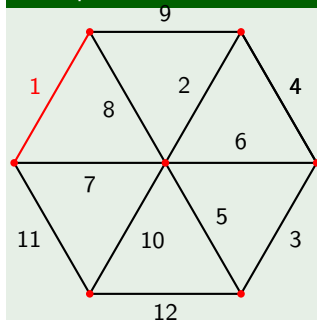
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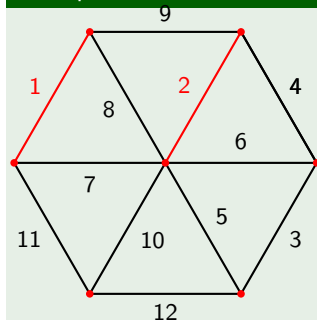
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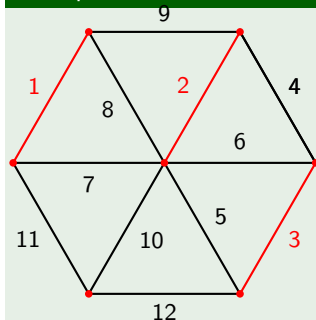
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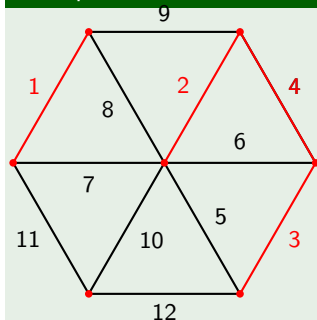
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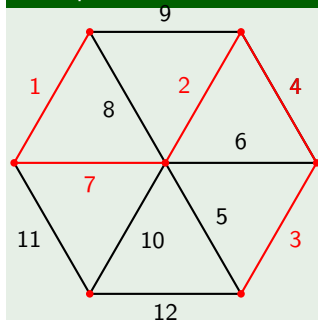
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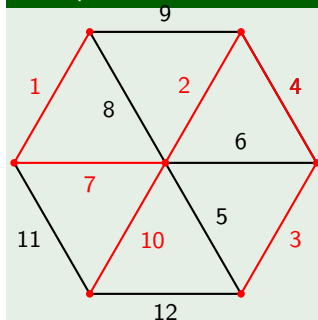
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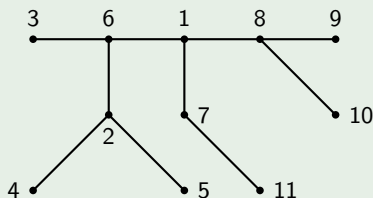
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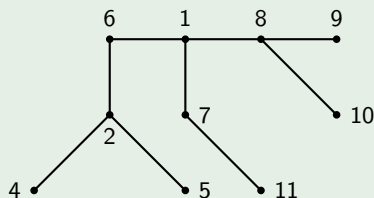
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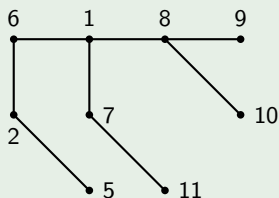
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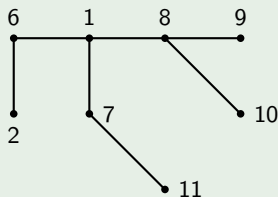
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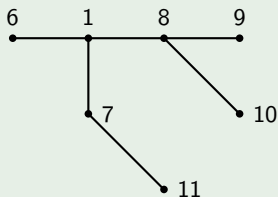
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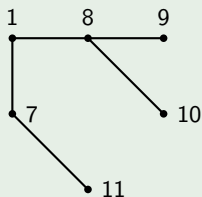
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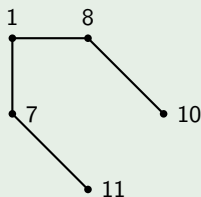
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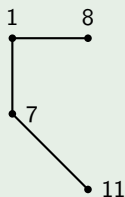
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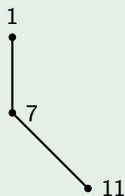
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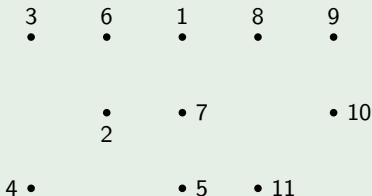
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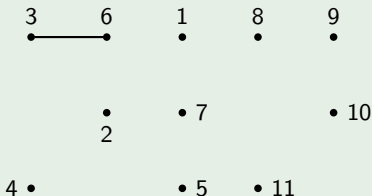
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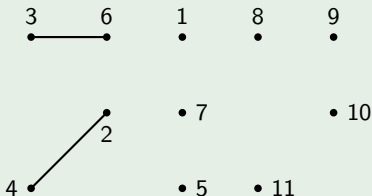
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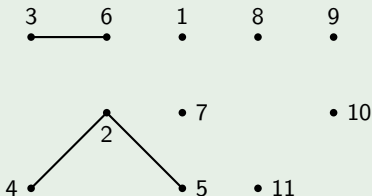
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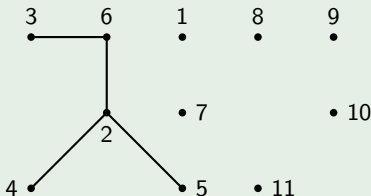
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Finished: 3, 4, 5, 2



Trees from Sequences

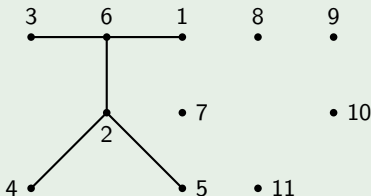
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Example 2.10

Sequence: 6, 2, 2, 6, 1, 8, 8, 1, 7

Finished: 3, 4, 5, 2, 6



Trees from Sequences

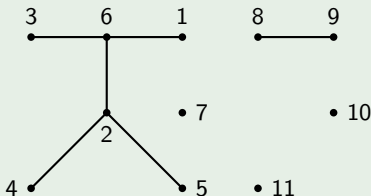
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Example 2.10

Sequence: 6, 2, 2, 6, 1, 8, 8, 1, 7

Finished: 3, 4, 5, 2, 6, 9



Trees from Sequences

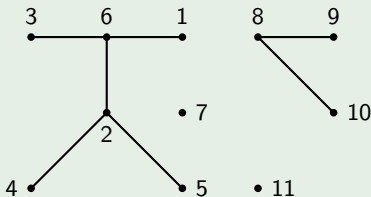
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Finished: 3, 4, 5, 2, 6, 9, 10



Trees from Sequences

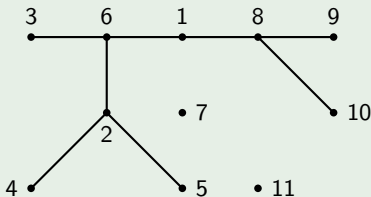
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Example 2.10

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Finished: 3, 4, 5, 2, 6, 9, 10, 8



Trees from Sequences

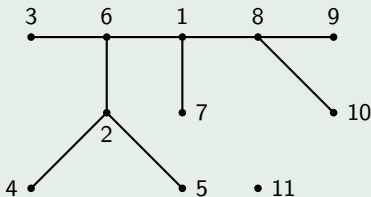
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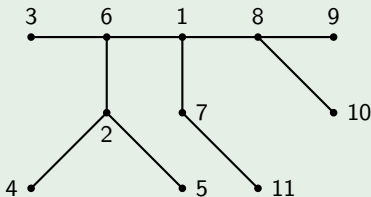
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Counting Trees with Prescribed Degrees

Corollary 2.11

The number of trees with vertex set $\{1, 2, \dots, n\}$ in which vertices $1, 2, \dots, n$ have respective degrees $d_1, d_2, \dots, d_n$ is

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Therefore we count the trees by counting sequences of length $n - 2$ having $d_i - 1$ copies of i , for each i . If we distinguish between various copies of i , then there are $(n - 2)!$ such sequences. Since we really cannot distinguish between the copies, we have over-counted by a factor of $(d_i - 1)!$ for each i . $\square$

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Note 2.13

The order of operations of deleting and contracting to get a minor of a graph is irrelevant.

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Let $\tau(G)$ denote the number of distinct spanning trees of a (labeled) graph G .

Counting Spanning Trees

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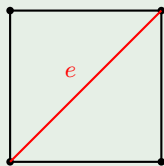
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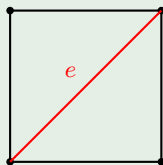
G

Counting Spanning Trees

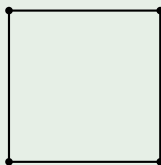
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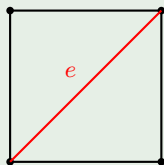
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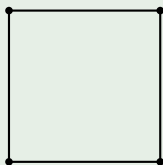
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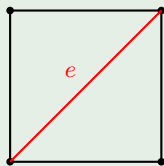
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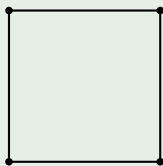
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G/e

$$\tau(G) = \tau(G \setminus e) + \tau(G/e) = 4 + 4 = 8$$

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Using the deletion-contraction formula for calculating the number of spanning trees is inefficient. A much more efficient method is to construct a special matrix, called the **Laplacian** of the graph, and to compute its determinant.

Bipartite Graphs

Definition 3.1

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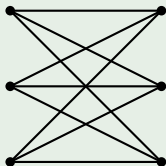
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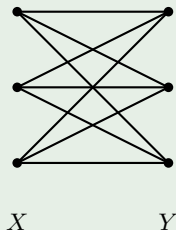


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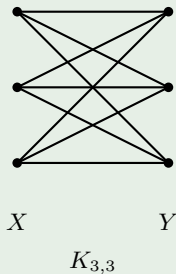


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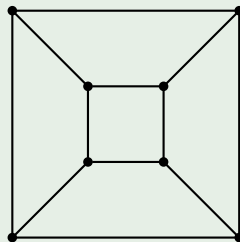
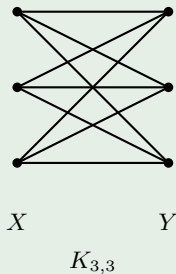


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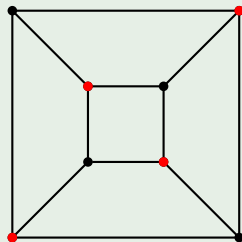
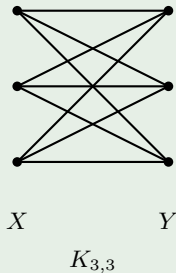


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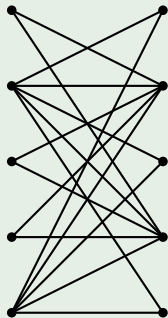
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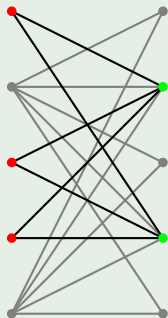
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No! Look at S , which has 3 elements, and $N(S)$, which has only 2 elements.

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Suppose G is a bipartite graph with bipartition $\{X, Y\}$. The graph G has a matching saturating X if and only if $|N(S)| \geq |S|$ for every subset S of X .

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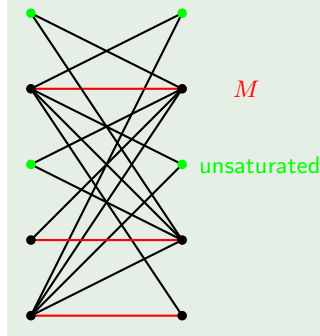
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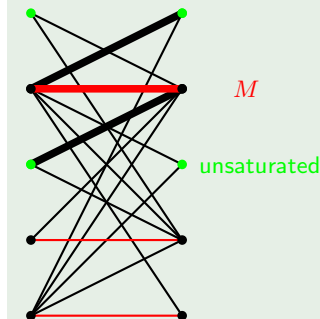
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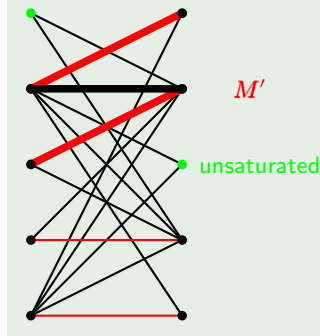
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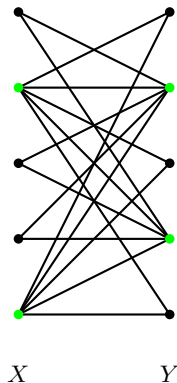
Since distinct vertices must be used to cover the edges of a matching, we have $|U| \geq |M|$ whenever U is a vertex cover and M is a matching. □

Proof of König-Egerváry Theorem, Continued

Given a minimum vertex cover U , we construct a matching of size $|U|$.

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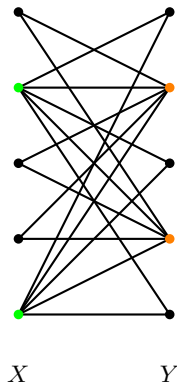
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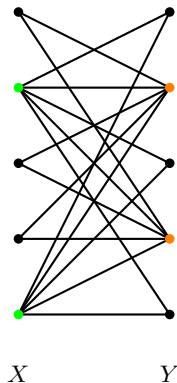
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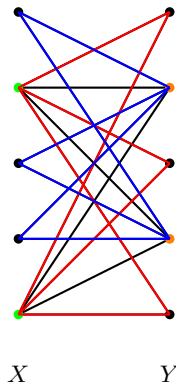
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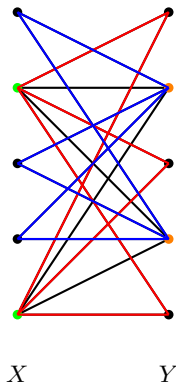
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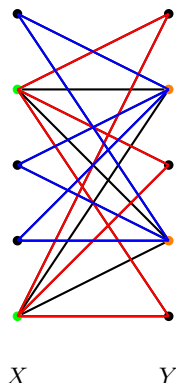
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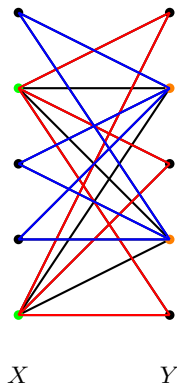
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Matchings in Non-Bipartite Graphs

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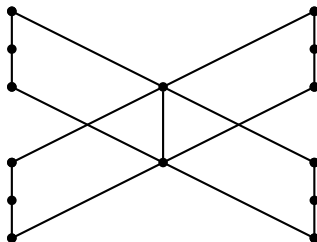
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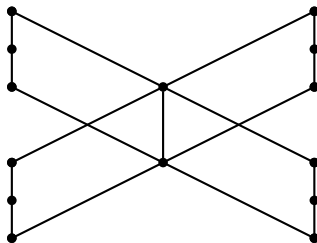


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Does the graph below have a perfect matching?



No, since removing the two vertices in the middle leaves more than two components of odd order.

Tutte's 1-Factor Theorem

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Note that after (1)–(3) are established, the proof is complete.

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Reducing this equation modulo 2, gives $1 \equiv |S| + 1 + q(Q_i - v - S) \pmod{2}$, and thus $q(Q_i - v - S) \equiv |S| \pmod{2}$, and so $q(Q_i - v - S) \geq |S| + 2$.

Now notice that upon deleting $\{v\} \cup S$ from $G - S_0$ the only component of $G - S_0$ that is affected is Q_i , which is lost, and the number of new odd components formed is $q(Q_i - v - S)$. Hence

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Proof of Tutte's 1-Factor Theorem, Continued

Now we turn to (3), which states that G contains a set $s_1v_1, s_2v_2, \dots, s_mv_m$ of edges such that $S_0 = \{s_1, s_2, \dots, s_m\}$ and $v_i \in V(Q_i)$ for all i .

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Homework Set 2

Problem 4

Derive the sufficiency (the non-obvious direction) of the Hall's Marriage Theorem from the Tutte's 1-Factor Theorem.

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Prove that a 3-regular simple graph has a 1-factor if and only if it decomposes into copies of P_4 .

Problem 6

Prove that a tree T has a perfect matching if and only if $q(T - v) = 1$ for every $v \in V(T)$. Do not invoke Tutte's 1-Factor Theorem.

Definition 3.17

A an edge e of G is a **cut-edge** if $G \setminus e$ has more connected components than G .

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We prove that G satisfies the Tutte Condition. Let $S \subseteq V(G)$, and count the edges between S and the odd components of $G - S$. Since G is 3-regular, every vertex in S is incident to at most three such edges.



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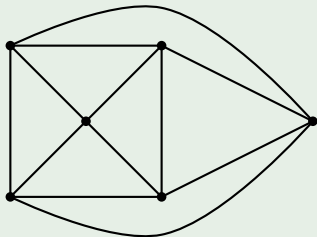
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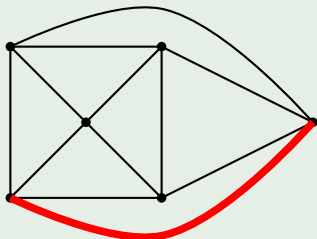
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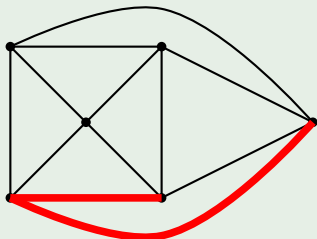
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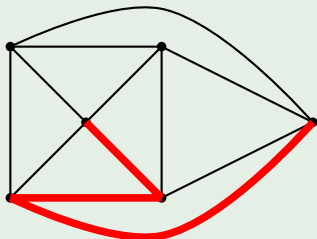
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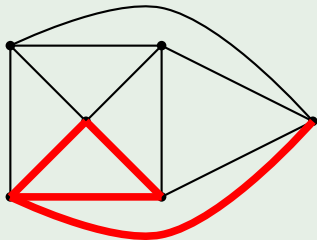
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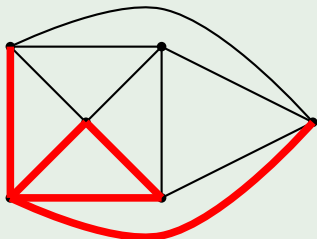
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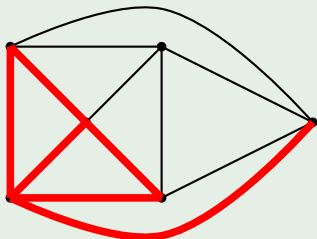
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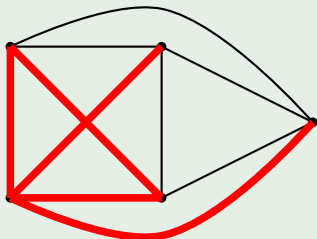
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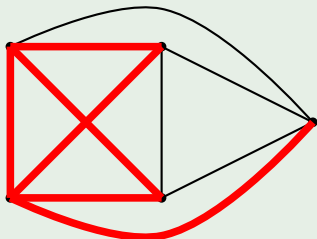
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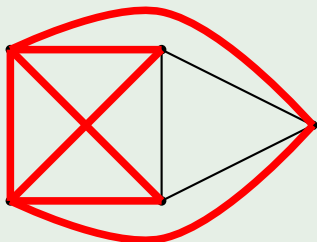
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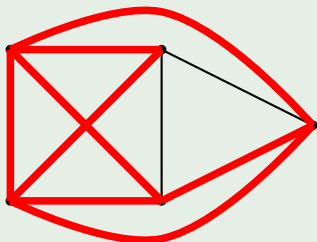
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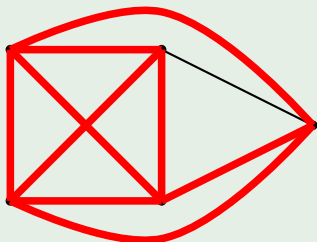
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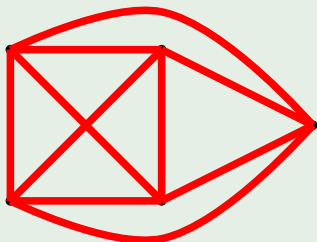
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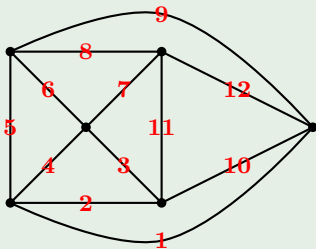
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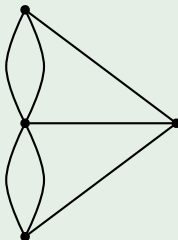
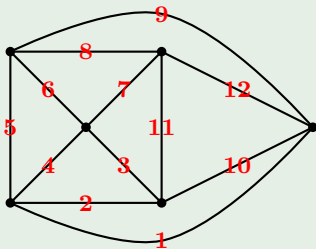
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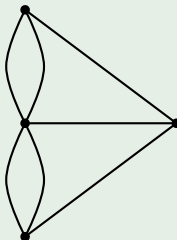
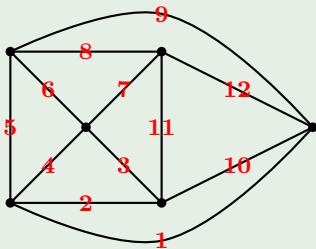
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Let T be a maximal non-trivial trail in some graph G with all degrees even. Since T is maximal, it includes all edges of G incident with its final vertex v . If T is not closed, then the degree of v must be odd, which is impossible. $\square$

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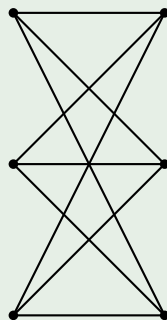
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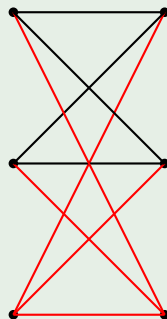
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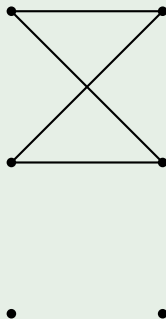
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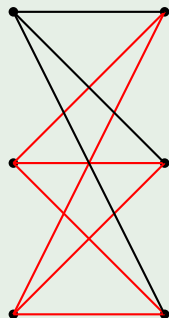
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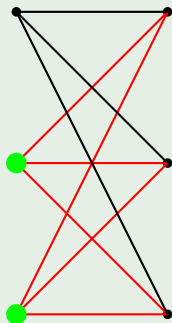
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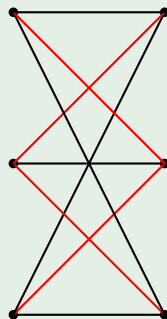
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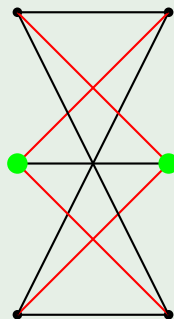
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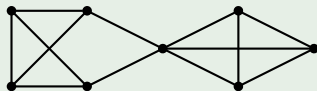
If G is graph with $|G| > 1$, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$.

Proof.

The edges incident to a vertex form a disconnecting set, so $\kappa' \leq \delta$. Clearly, $\kappa(G) \leq |G| - 1$. Suppose $[S, \overline{S}]$ is a minimum edge cut of size $k' = \kappa'(G)$. If every vertex in S is adjacent to every vertex in $\overline{S}$, then $k' = |S||\overline{S}| \geq |G| - 1$, and the inequality follows. Hence we may assume that there are vertices $s \in S$ and $\overline{s} \in \overline{S}$ that are non-adjacent. Let T be the vertex set consisting of all neighbors of s in $\overline{S}$ and all vertices in $S - s$ that have neighbors in $\overline{S}$. Then T is a vertex cut consisting of one endpoint of each edge in $[S, \overline{S}]$. Hence $\kappa \leq \kappa'$. □

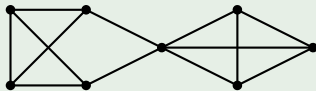
Connectivity Example

Example 5.8



Connectivity Example

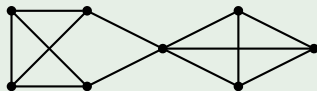
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$$\kappa = 1 < \kappa' = 2 < \delta = 3$$

Connectivity Example

Example 5.8



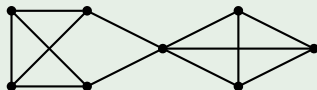
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Theorem 5.9

If G is a connected graph and S is a non-empty proper subset of $V(G)$, then $F = [S, \overline{S}]$ is a bond if and only if $G \setminus F$ has two components.

Connectivity Example

Example 5.8



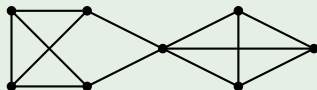
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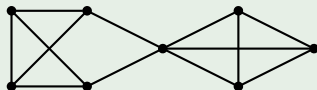
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If $G \setminus F$ has two components, then F is a bond, since $G \setminus F'$ is connected for every proper subset F' of F .



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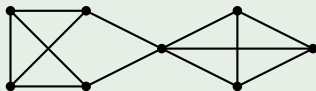
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If $G \setminus F$ has more than two components, then we may assume $S = A \cup B$ with no edges between A and B . Then $[A, \overline{A}]$ is an edge cut which is a proper subset of F ; a contradiction. □

Definition 5.10

- ▶ A **k -separation** of a graph G is a pair of subgraphs $\{A, B\}$ of G such that each of A and B has size at least k , $A \neq G$, $B \neq G$, $A \cup B = G$, and $A \cap B$ is trivial of order at most k .

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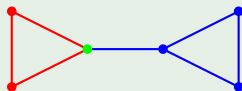
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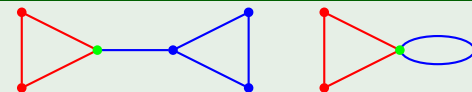
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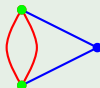
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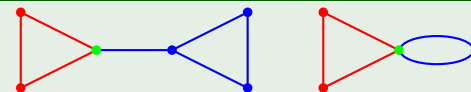
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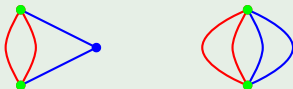
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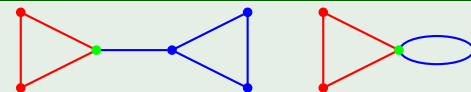
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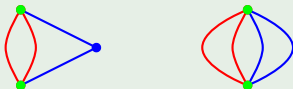
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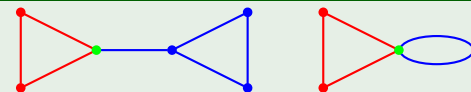
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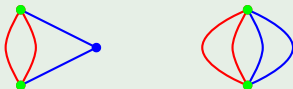
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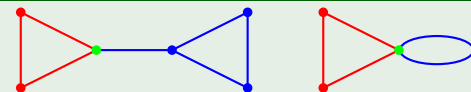
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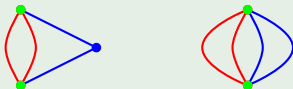
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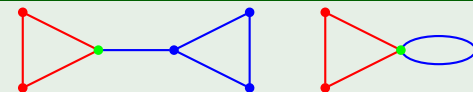
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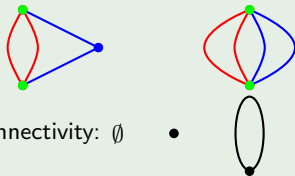
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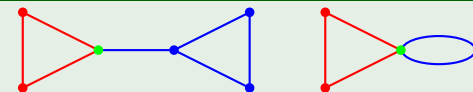
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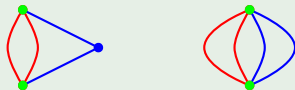
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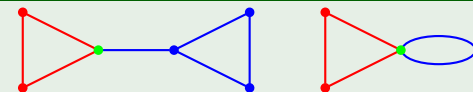
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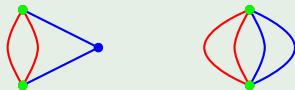
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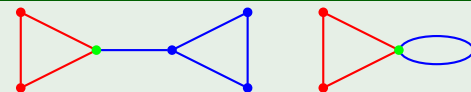
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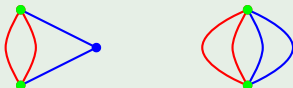
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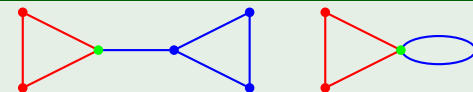
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Tutte Connectivity vs. Vertex Connectivity

Theorem 5.12

*If G is a graph on at least 3 vertices and $G \not\cong K_3$, then the Tutte connectivity of G is $\min(\kappa(G), g(G))$, where $g(G)$ is the **girth** of G , that is, the length of a shortest cycle in G .*

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Note 5.14

A block of a non-empty graph is an isolated vertex, a loop-graph, a graph on two vertices with a positive number of edges between those vertices, or is vertex-2-connected.

Note 5.15

Two distinct blocks in a graph share at most one vertex since otherwise their union would be Tutte-2-connected.

Block Tree

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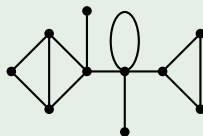
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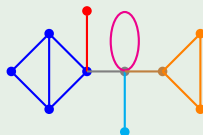
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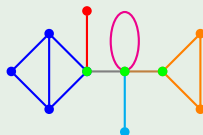
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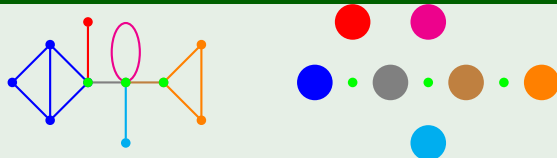
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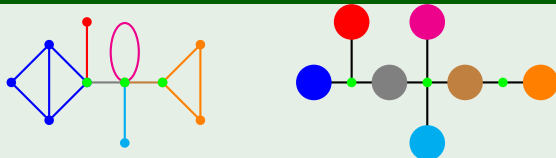
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Whitney's Characterization of 2-Connected Graphs

Definition 5.18

Two paths are **internally-disjoint** if neither contains a non-endpoint of the other.

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If G has two internally-disjoint uv -paths, then deletion of one vertex cannot separate u from v . Hence G has no one-element vertex-cuts and so is 2-connected. □

Proof of Sufficiency

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Whitney's Theorem 5.19 establishes the equivalence of (A) and (B). Clearly, (B) and (C) are equivalent. To see that (D) implies (C), apply (D) to edges incident to the desired x and y . □

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Subdivisions and 2-Connectedness

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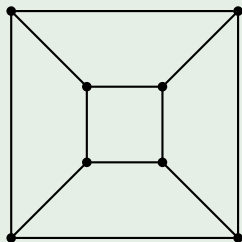
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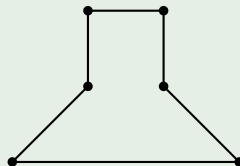
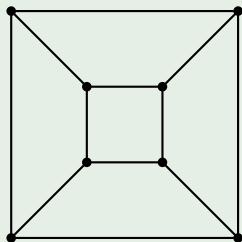
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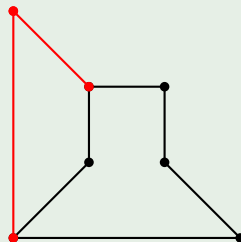
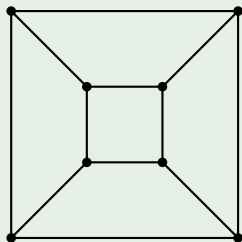
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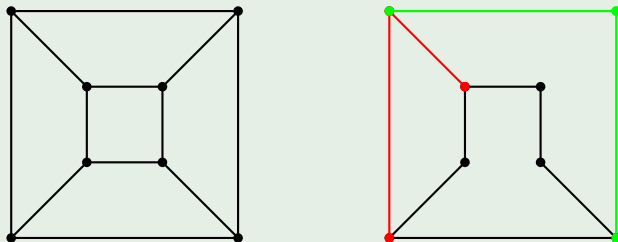
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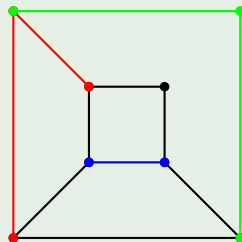
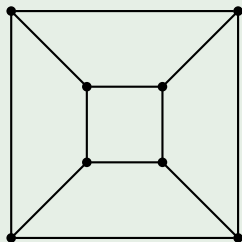
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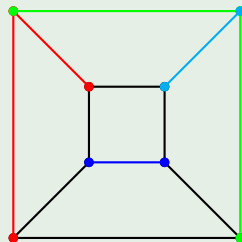
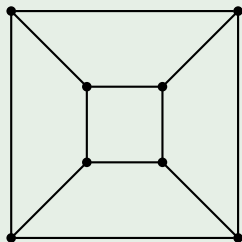
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Whitney's Ear Decomposition

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Closed-Ear Decomposition

Definition 5.27

A **closed-ear decomposition** of a graph G is a partition of $E(G)$ into sets $R_0, R_1, \dots, R_k$ such that R_0 is a cycle and R_i for $i > 0$ is either a path addition

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Theorem 5.28

A simple graph is 2-edge-connected if and only if it has a closed-ear decomposition. Moreover, every cycle in a 2-edge-connected graph is the initial cycle in some closed-ear decomposition.

Proof omitted.

The Menger Theorem

Theorem 5.29 (Menger 1927)

If x and y are non-adjacent distinct vertices of a graph G , then the minimum size of a vertex-cut separating x from y equals the maximum number of pairwise internally-disjoint xy -paths.

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Let $\kappa(x, y)$ denote the minimum size of a vertex-cut separating x from y . Let $\lambda(x, y)$ denote the maximum number of pairwise internally-disjoint xy -paths. An vertex-cut separating x from y must contain an internal vertex from every xy -path, and so $\kappa(x, y) \geq \lambda(x, y)$.



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To prove the opposite inequality, we use induction on $|G|$. If $|G| = 2$, then $\kappa(x, y) = \lambda(x, y) = 0$. For the induction step, suppose $|G| > 2$ and let $k = \kappa(x, y)$; we construct k pairwise internally-disjoint xy -paths. □

Proof of the Menger Theorem, Case 1

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Case 2: Every minimum xy -vertex-cut contains $N(x)$ or $N(y)$.

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If G has a vertex v outside $\{x, y\} \cup N(x) \cup N(y)$, then v is in no minimum xy -vertex-cut; hence $\kappa_{G-v}(x, y) = k$ and applying the induction hypothesis to $G - v$ yields the desired xy -paths in G .

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If $v \in N(x) \cap N(y)$, then v appears in **every** xy -vertex-cut, and so $k_{G-v}(x, y) = k - 1$.

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If $v \in N(x) \cap N(y)$, then v appears in **every** xy -vertex-cut, and so $k_{G-v}(x, y) = k - 1$. Now, applying the induction hypothesis to $G - v$ yields $k - 1$ xy -paths, which combine with the path xvy to get the desired k paths.

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If G has a vertex v outside $\{x, y\} \cup N(x) \cup N(y)$, then v is in no minimum xy -vertex-cut; hence $\kappa_{G-v}(x, y) = k$ and applying the induction hypothesis to $G - v$ yields the desired xy -paths in G .

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The Edge Version of Menger's Theorem

Theorem 5.30 (Edge Version of Menger's Theorem)

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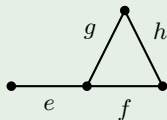
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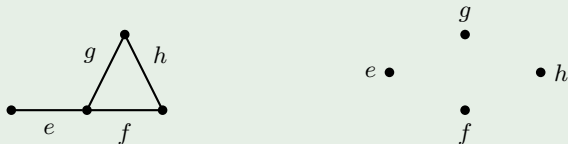
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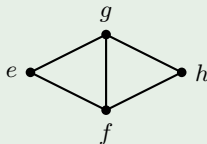
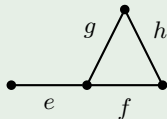
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$$\lambda_G(x, y) = 1 + \lambda_{G \setminus xy}(x, y) = 1 + \kappa_{G \setminus xy}(x, y) \geq 1 + \kappa(G \setminus xy) \geq \kappa(G).$$



Tutte's Wheel Theorem

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If G is a Tutte-3-connected graph on at least four vertices that is not a wheel, then there is an edge e of G such that at least one of G/e and $G \setminus e$ is also Tutte-3-connected.

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Lemma 5.35 (Thomassen 1980)

Every 3-connected graph G on at least five vertices has an edge e such that G/e is 3-connected.

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- ▶ F consists of a path on vertices (listed in order) $v_0, v_1, \dots, v_k$, and a vertex c adjacent to all vertices of the path;
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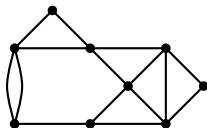
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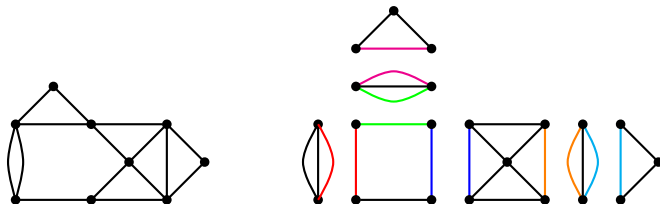
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Homework Set 3

Problem 7

Suppose G is a graph that is non-trivial, connected, and such that every edge e is in some two cycles that meet only at e . What is the highest edge-connectivity of G that can be inferred from these properties?

Problem 8

*Find all non-negative integers k for which the following statement is true:
For every simple k -regular graph G on at least two vertices, $\kappa(G) = \kappa'(G)$.*

Problem 9

Suppose G is a simple r -connected graph of even order with no $K_{1,r+1}$ as an induced subgraph for a positive integer r . Prove that G has a perfect matching.

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Definition 6.2

A **polygonal curve** in the plane is the union of finitely many line segments such that each segment starts at the end of the previous one and no point lies in more than one segment, except the end of one segment and the beginning of the next one coincide.

A **simple open** polygonal curve is one homeomorphic to a closed interval.

A **simple closed** polygonal curve is one homeomorphic to a unit circle.

Definition 6.3

- ▶ A **drawing** of a graph G is a function that maps each vertex $v \in V(G)$ to a point $f(v)$ in the plane, and each uv -edge to a simple polygonal $f(u)f(v)$ -curve in the plane.

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Note 6.4

*A plane embedding corresponds to an embedding of the graph in the sphere through a **stereographic projection**.*

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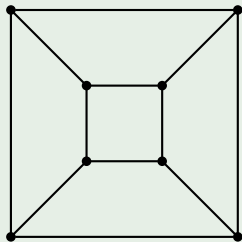
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 - ▶ the edges of G^* are the edges of G ;
 - ▶ a vertex and an edge of G^* are incident if and only if the edge is the boundary of the corresponding face of G .

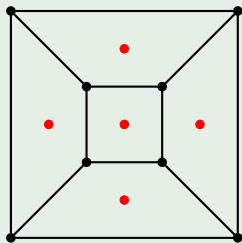
Example of a Dual Graph

Example 6.7



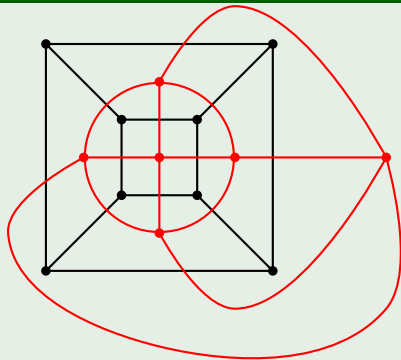
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- ▶ *Contracting an edge in a plane graph can be visualized as sliding the two endvertices towards each other until they meet, pulling all incident edges along.*
- ▶ *Thus the class of planar graphs is **minor-closed**, that is, all minors of planar graphs are also planar.*

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Theorem 6.13 (Euler's Formula)

If a connected non-empty plane graph has v vertices, e edges, and f faces, then $v - e + f = 2$.

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Suppose $v > 1$. Since G is connected, it has a non-loop edge. Contract such an edge to obtain a plane graph with $v' = v - 1$ vertices, $e' = e - 1$ edges, and $f' = f$ faces.



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Is K_5 planar?

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Then $-kl + 2l + 2k > 0$, and so $-kl + 2l + 2k - 4 > -4$, And so $(k - 2)(l - 2) < 4$, and so $k, l \geq 3$ and $k, l \leq 5$.

The Zoo of Platonic Solids

$$e = \frac{2kl}{2k + 2l - kl} \quad v = \frac{2e}{k} \quad f = \frac{2e}{l}$$

k	l	$(k - 2)(l - 2)$	e	v	f	name

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3					

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$$e = \frac{2kl}{2k + 2l - kl} \quad v = \frac{2e}{k} \quad f = \frac{2e}{l}$$

k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1				

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6			

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4		

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	

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$$e = \frac{2kl}{2k + 2l - kl} \quad v = \frac{2e}{k} \quad f = \frac{2e}{l}$$

k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4					

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2				

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12			

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3	3	1	6	4	4	tetrahedron
3	4	2	12	8		

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	

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3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube

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3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3					

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2				

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5					

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$$e = \frac{2kl}{2k + 2l - kl} \quad v = \frac{2e}{k} \quad f = \frac{2e}{l}$$

k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3				

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k	l	$(k-2)(l-2)$	e	v	f	name
3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30			

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3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20		

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3	5	3	30	20	12	

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3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron

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3	3	1	6	4	4	tetrahedron
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3	5	3	30	20	12	dodecahedron
5	3					

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3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
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3	5	3	30	20	12	dodecahedron
5	3	3				

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3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30			

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3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30	12		

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3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30	12	20	

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3	3	1	6	4	4	tetrahedron
3	4	2	12	8	6	cube
4	3	2	12	6	8	octahedron
3	5	3	30	20	12	dodecahedron
5	3	3	30	12	20	icosahedron

Statement of the Kuratowski Theorem

Theorem 6.17 (Kuratowski 1930)

A graph is planar if and only if it has neither K_5 nor $K_{3,3}$ as a topological minor.

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Proof.

Apply stereographic projection twice. □

Minimally No-Planar Graphs

Definition 6.20

A graph is **minimally non-planar** if it is non-planar, but every proper subgraph of it is planar.

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Every minimal non-planar graph is 2-connected.

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If G is disconnected, we can embed one component of G inside one face of the rest of G .



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If G is disconnected, we can embed one component of G inside one face of the rest of G . Similarly, if G has a cut-vertex v , let $G_1, G_2, \dots, G_k$ be the subgraphs of G induced by v together with the components of $G - v$.



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Lemma 6.22

Suppose $G = H_1 \oplus_2 H_2$ is non-planar.

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Kuratowski Graphs and Topological Minors

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Let e be the common edge of H_1 and H_2 .



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Let e be the common edge of H_1 and H_2 . Suppose both H_1 and H_2 are planar. By Lemma 6.19, each of H_1 and H_2 can be embedded in the plane with e in the boundary of the infinite face.



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Kuratowski Graphs and Topological Minors

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Definition 6.23

- ▶ A **Kuratowski subgraph** is a subgraph isomorphic to a subdivision of K_5 or of $K_{3,3}$.

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Definition 6.23

- ▶ A **Kuratowski subgraph** is a subgraph isomorphic to a subdivision of K_5 or of $K_{3,3}$.
- ▶ A vertex of a graph G is a **branch vertex** of a Kuratowski subgraph H of G , if its degree in H exceeds two.

Lemma 6.24

If G/e has a Kuratowski subgraph, then so does G .

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Proof.

Let H be a Kuratowski subgraph of $G' = G/e$, and let z be the vertex of G' obtained by contracting $e = xy$. If z is not a branch vertex of H , then G also has a Kuratowski subgraph obtained from H by lengthening a path through z if necessary.



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Let H be a Kuratowski subgraph of $G' = G/e$, and let z be the vertex of G' obtained by contracting $e = xy$. If z is not a branch vertex of H , then G also has a Kuratowski subgraph obtained from H by lengthening a path through z if necessary. If z is a branch vertex of H and at most one of the edges incident to z in H is incident to x in G ,



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The only remaining case to consider is when H is a subdivision of K_5 , z is a branch vertex of H , and each of x and y is incident in G to two of the four edges incident to z in H .



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The only remaining case to consider is when H is a subdivision of K_5 , z is a branch vertex of H , and each of x and y is incident in G to two of the four edges incident to z in H . Let u_1, u_2 be the branch vertices of H that are at the other ends of paths leaving z on the edges incident with x , and let v_1, v_2 be the other branch vertices of H . By deleting the edges of the u_1u_2 -path and the v_1v_2 -path, we obtain a subdivision of $K_{3,3}$. □

Tutte's Version of Kuratowski's Theorem

Definition 6.25

A plane embedding is **convex** if every face except the infinite one is a convex polygon.

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Theorem 6.26 (Tutte 1960–63)

If G is a simple 3-connected graph with neither K_5 nor $K_{3,3}$ as the topological minor, then G has a convex embedding in the plane with no three vertices on a line.

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We proceed by induction on $|G|$.



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We proceed by induction on $|G|$. The only 3-connected simple graph on at most 4 vertices is K_4 and it has such an embedding.



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Proof.

We proceed by induction on $|G|$. The only 3-connected simple graph on at most 4 vertices is K_4 and it has such an embedding. Let G be a graph on $n \geq 5$ vertices and suppose the theorem holds for all graphs on fewer than n vertices. □

Proof, Continued

By Lemma 5.35, G has an edge $e = xy$ such that G/e is also 3-connected;

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By Lemma 5.35, G has an edge $e = xy$ such that G/e is also 3-connected; let z denote the vertex resulting from contracting e . By Lemma 6.24, G/e has no Kuratowski subgraph, and so by the induction hypothesis, $H = G/e$ has a convex embedding with no 3 vertices on a line.

Proof, Continued

By Lemma 5.35, G has an edge $e = xy$ such that G/e is also 3-connected; let z denote the vertex resulting from contracting e . By Lemma 6.24, G/e has no Kuratowski subgraph, and so by the induction hypothesis, $H = G/e$ has a convex embedding with no 3 vertices on a line. Let C denote the cycle that is the boundary of the face $H - z$ that is not a face of H .

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2. y has neighbors u, v in C that are in different components of $C - \{x_i x_{i+1}\}$ for some i , in which case we obtain a subdivision of $K_{3,3}$.

Proof of Kuratowski Theorem

Recall: G is planar if and only if neither K_5 nor $K_{3,3}$ is a topological minor of G .

Proof.

Without loss of generality, we may assume that G is simple.



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Problem 12

*A plane graph is **outerplane** if it has a face incident with all the vertices. A graph is **outerplanar** if it is isomorphic to an outerplane graph. Prove that a graph is outerplanar if and only if it has neither K_4 nor $K_{2,3}$ as a topological minor.*

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Suppose G is connected, not a clique, not an odd cycle, and let $k = \Delta(G)$. If G is not regular, then choose v_n so that $d(v_n) < k$. Since G is connected, we can grow a spanning tree of G from v_n , assigning indices in the decreasing order as we reach vertices. Then each vertex v_i , other than v_n , has at least one neighbor among $v_{i+1}, v_{i+2}, \dots, v_n$,



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5-Color Theorem

Theorem 7.3 (Heawood 1890)

Every loopless planar graph has a proper 5-coloring.

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Proof.

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Suppose G is a plane graph that is a minimal counter-example. Then G is simple, and so $\|G\| \leq 3|G| - 6$ by Corollary 6.15.



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Proof, Continued

Let $P_{i,j}$ be a path in $G_{i,j}$ joining the neighbors of v colored i and j .

Proof, Continued

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Chromatic Polynomial

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Theorem 7.6 (Four-Color Theorem, restated)

If G is a planar loopless graph, then $P_G(4) > 0$.

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Definition 7.7

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Theorem 7.9 (Strong Graph Theorem (formerly Berge's Strong Graph Conjecture), Chudnovsky, Robertson, Seymour, Thomas 2002)

A graph is perfect if and only if it has no induced subgraph that is an odd cycle of length at least five or its complement.

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$$\Delta(G) \leq \chi'(G).$$

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Theorem 7.12 (König 1916)

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Hence we may assume that a_0 appears at v_l , but a_k does not. Let P be the (unique) maximal path of edges colored a_0 or a_k that begins at v_l . **Switching on P** means interchanging the colors a_0 and a_k on the edges of P .

Proof of Vizing's Theorem, Continued

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But there are graphs for which $\text{ch}(G) \neq \chi(G)$. Consider $K_{3,3}$ where each side of the bipartition has lists $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$. The list-chromatic number of this graph is 3, while the chromatic number is 2.

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Theorem 7.14 (Thomassen 1994)

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We prove this by induction. The claim is obvious for the smallest graph for which it makes sense, that is, a triangle. Suppose the claim is true for every graph on fewer than n vertices, and suppose that G is like described above, and $|G| = n$.

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Case 2

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Theorem 7.15

There are simple planar graphs that are not 4-choosable.

List Colorings

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Conjecture 7.16

$$ch'(G) = \chi'(G).$$

Flows and Circulations

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- ▶ If f is a circulation and $e = xy$ is a cut-edge, then $f(e, x, y) = 0$.

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Corollary 7.20

If $\mathbb{H}$ and $\mathbb{H}'$ are two finite abelian groups of equal order, then G has an $\mathbb{H}$ -flow if and only if it has an $\mathbb{H}'$ -flow.

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Proof of $\Rightarrow$ only.

Use the natural map $i \mapsto \bar{i}$ from $\mathbb{Z}$ to $\mathbb{Z}_k$.



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Conversely, let G be bipartite with bipartition (X, Y) . Since G is cubic, the map $\vec{E} \rightarrow \mathbb{Z}_3$ defined by $f(e, x, y) = 1$ and $(e, y, x) = 2$ for all edges xy with $x \in X$ and $y \in Y$ is a $\mathbb{Z}_3$ -flow. □

Theorem 7.25

$$\varphi(K_n) = \begin{cases} 2 & \text{if } n \text{ is odd;} \\ 4 & \text{if } n = 4; \text{ and} \\ 3 & \text{if } n \text{ is even and exceeds 4.} \end{cases}$$

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The case for n odd follows from Theorem 7.23, and $n = 4$ can be checked directly.



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The case for n odd follows from Theorem 7.23, and $n = 4$ can be checked directly. We handle the remaining cases by induction.



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Let $\mathbb{H} = \mathbb{Z}_2 \times \mathbb{Z}_2$. By Theorems 7.20 and 7.22, a graph has a 4-flow if and only if it has an $\mathbb{H}$ -flow.



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Assume a cubic graph G has an $\mathbb{H}$ -flow f . It is easy to check that f gives a 3-edge-coloring. Conversely, since the non-zero elements of $\mathbb{H}$ sum up to 0, every proper 3-edge-coloring of G using colors $\mathbb{H} \setminus 0$ defines an $\mathbb{H}$ -flow on G .



Theorem 7.27 (Tait 1878)

A simple 2-edge-connected 3-regular plane graph G is 3-edge-colorable if and only if its dual is 4-colorable.

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Problem Set 5

Problem 13

Prove that every loopless planar graph on fewer than thirteen vertices admits a proper 4-coloring. In other words, prove the Four-Color Theorem for graphs on at most twelve vertices.

Problem 14

Show that every graph without a cut-edge admits a flow.

Problem 15

Show that if a graph has a spanning cycle, then it admits a 4-flow.

Tutte's Flow Conjectures

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Theorem 7.31 (Grötzsch 1959)

Every planar graph with no edge-cuts of size 1 and 3 has a 3-flow.

Miscellany of Coloring Results

Theorem 7.32 (Tutte 1954)

$$\chi(G) = \varphi(G^*)$$

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Hamilton Cycles

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Sufficient Condition for Hamiltonicity

Theorem 8.2 (Dirac 1952)

Every graph of order $n \geq 3$ and $\delta \geq n/2$ is Hamiltonian.

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Let $P = x_0x_1 \dots x_k$ be a longest path in G . By the maximality of P , all neighbors of x_0 and all neighbors of x_k lie on P . Hence at least $n/2$ of the vertices $x_0, x_1, \dots, x_{k-1}$ are adjacent to x_k , and at least $n/2$ of the same $k < n$ vertices x_i are such that $x_0x_{i+1} \in E(G)$. By the Pigeon-Hole Principle, there is a vertex x_i that has both properties, that is, $x_0x_{i+1} \in E(G)$ and $x_ix_k \in E(G)$. Let C be the cycle obtained from P by deleting the edge x_ix_{i+1} and adding edges x_0x_{i+1} and x_ix_k . If C is not Hamiltonian, then, since G is connected, C would have a neighbor in $G - C$,



Sufficient Condition for Hamiltonicity

Theorem 8.2 (Dirac 1952)

Every graph of order $n \geq 3$ and $\delta \geq n/2$ is Hamiltonian.

Proof.

Let G be a graph as described. Note that G is connected; otherwise a vertex in a smallest component would have degree less than $n/2$.

Let $P = x_0x_1 \dots x_k$ be a longest path in G . By the maximality of P , all neighbors of x_0 and all neighbors of x_k lie on P . Hence at least $n/2$ of the vertices $x_0, x_1, \dots, x_{k-1}$ are adjacent to x_k , and at least $n/2$ of the same $k < n$ vertices x_i are such that $x_0x_{i+1} \in E(G)$. By the Pigeon-Hole Principle, there is a vertex x_i that has both properties, that is, $x_0x_{i+1} \in E(G)$ and $x_ix_k \in E(G)$. Let C be the cycle obtained from P by deleting the edge x_ix_{i+1} and adding edges x_0x_{i+1} and x_ix_k . If C is not Hamiltonian, then, since G is connected, C would have a neighbor in $G - C$, which would yield a path longer than P ; a contradiction. □

Note on Dirac's Theorem

Note 8.3

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Note that $n/2$ in Dirac's Theorem 8.2 is the best possible. We cannot replace it with $\lfloor n/2 \rfloor$ if n is odd, since then G which is a 1-sum of two copies of $K^{\lceil n/2 \rceil}$ would have $\delta = \lfloor n/2 \rfloor$, but no Hamilton cycle.

Another Sufficient Condition

Theorem 8.4

Every graph G with $|G| \geq 3$ and $\kappa(G) \geq \alpha(G)$ is Hamiltonian.

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Let $k = \kappa(G)$ and let C be a longest cycle in G .



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Let $k = \kappa(G)$ and let C be a longest cycle in G . Enumerate the vertices of C cyclically so that $V(C) = \{v_i : i \in \mathbb{Z}_n\}$ with $v_i v_{i+1} \in E(C)$ for all $i \in \mathbb{Z}_n$.



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A Necessary Condition

Theorem 8.5

If G is a Hamiltonian graph, then for every set $\emptyset \neq S \subseteq V(G)$, the graph $G - S$ has at most $|S|$ components.

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When leaving a component of $G - S$, a Hamilton cycle can go only to S and the arrivals in S must occur at different vertices of S .



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Proof.

When leaving a component of $G - S$, a Hamilton cycle can go only to S and the arrivals in S must occur at different vertices of S . Hence S must have at least as many vertices as $G - S$ has components. $\square$

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Note that if we managed to prove that every 3-connected cubic plane graph is Hamiltonian, then we would have proved that every such graph has a 4-flow, and so is 3-edge-colorable, and so is 4-face-colorable. Unfortunately, there are 3-connected cubic plane graphs that are not Hamiltonian.

Grinberg's Theorem

Theorem 8.6 (Grinberg 1968)

If G is a loopless plane graph with a Hamilton cycle C , and G has f'_i faces of length i inside C and f''_i faces of length i outside C , then

$$\sum_i (i - 2)(f'_i - f''_i) = 0.$$

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Corollary 8.7

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Theorem 8.10 (Thomas, Yu 1997)

Every 5-connected toroidal graph is Hamiltonian.

Suppose we have a finite family $\mathcal{F}$ of pairwise disjoint convex polygons (with interiors) in the plane with all sides of length 1.

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Theorem 9.1

Every surface is homeomorphic to a triangulated surface.

Proof omitted.

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Consider the sphere $\mathbb{S}_0$ (tetrahedron). If we add h handles to $\mathbb{S}_0$, then we obtain $\mathbb{S}_h$, which we call the **orientable surface of genus h** . If we add h crosscaps to $\mathbb{S}_0$, we obtain $\mathbb{N}_h$, the **non-orientable surface of genus h** .

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Theorem 9.2

Let S be the surface obtained from the sphere by adding h handles, t twisted handles, and c crosscaps. If $t = c = 0$, then $S = \mathbb{S}_h$. Otherwise, $S = \mathbb{N}_{2h+2t+c}$.

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Theorem 9.3

Let S be a surface and let G be a graph that is 2-cell embedded in S with v vertices, e edges and f faces. Then S is homeomorphic to either $\mathbb{S}_h$ or $\mathbb{N}_k$, where $v - e + f = 2 - 2h = 2 - k$.

Euler's Characteristic

Definition 9.4

The **Euler characteristic** $\chi(S)$ of a surface S is defined as

$$\chi(S) = \begin{cases} 2 - 2h, & \text{if } S = \mathbb{S}_h; \\ 2 - k, & \text{if } S = \mathbb{N}_k. \end{cases}$$

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Theorem 9.5

*Every **cellular** embedding (an embedding where each face is homeomorphic to an open disk) into an orientable surface is determined by its rotation system.*

Embedding Schemes

An **embedding scheme** is a pair $\Pi = (\pi, \lambda)$ where π is a rotation system, and $\lambda : E(G) \rightarrow \{-1, 1\}$ is a **signature**.

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Theorem 9.6

Every cellular embedding of a graph in some surface is uniquely determined, up to homeomorphism, by its embedding scheme.

Genus of a Graph

Definition 9.7

The **genus** $\gamma(G)$ and the **non-orientable genus** $\tilde{\gamma}(G)$ of a graph G are the minimum h and the minimum k , respectively, such that G has an embedding into the surface $\mathbb{S}_h$, respectively into $\mathbb{N}_k$.

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Let G be a connected graph. If $\tilde{\gamma}(G) < 2\gamma(G) + 1$, then every non-orientable minimum genus embedding of G is cellular. If $\tilde{\gamma}(G) = 2\gamma(G) + 1$ and G is not a tree, then G has both a cellular and a non-cellular embedding in $\mathbb{N}_{\tilde{\gamma}(G)}$.

Cycle Double-Cover Conjecture

Conjecture 9.10 (Cycle Double-Cover Conjecture)

Every 2-edge-connected graph G can be expressed as a union of cycles so that every edge of G appears in exactly two cycles.

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Every 2-edge-connected graph has an embedding in some surface so that every face with the boundary is homeomorphic to the closed unit disk. Holds for 4-connected graphs.

Bounds on Genus

Theorem 9.12

Let G be a simple connected graph with v vertices ($v \geq 3$) and e edges. Then

$$\gamma \geq \left\lceil \frac{e}{6} - \frac{v}{2} + 1 \right\rceil \quad \text{and} \quad \tilde{\gamma} \geq \left\lceil \frac{e}{3} - v + 2 \right\rceil.$$

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Proof.

Let f denote the number of facial walks.



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$$\gamma \geq 1 - \frac{v - e/3}{2} = \frac{e}{6} - \frac{v}{2} + 1.$$



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$$\gamma \geq \left\lceil \frac{e}{6} - \frac{v}{2} + 1 \right\rceil \quad \text{and} \quad \tilde{\gamma} \geq \left\lceil \frac{e}{3} - v + 2 \right\rceil.$$

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Let f denote the number of facial walks. The sum of lengths of those walks is $2e$. Since the graph is simple, $2e \geq 3f$. By Euler's Formula, $3\chi = 3v - 3e + 3f \leq 3v - e$, and so $\chi \leq v - e/3$. Note that $\gamma = 1 - \chi/2$ and $\tilde{\gamma} = 2 - \chi$. In the orientable case, we have

$$\gamma \geq 1 - \frac{v - e/3}{2} = \frac{e}{6} - \frac{v}{2} + 1.$$

Similarly, in the nonorientable case, we have

$$\tilde{\gamma} \geq 2 - (v - \frac{e}{3}) = \frac{e}{3} - v + 2.$$



Bounds on Genus

Theorem 9.12

Let G be a simple connected graph with v vertices ($v \geq 3$) and e edges. Then

$$\gamma \geq \left\lceil \frac{e}{6} - \frac{v}{2} + 1 \right\rceil \quad \text{and} \quad \tilde{\gamma} \geq \left\lceil \frac{e}{3} - v + 2 \right\rceil.$$

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The theorem follows now from the fact that both γ and $\tilde{\gamma}$ are integres. □

Genera of Complete Graphs

Corollary 9.13

$$\gamma(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \quad \text{and} \quad \tilde{\gamma}(K_n) \geq \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$$

Heawood's Theorem

Theorem 9.14 (Ringel, Youngs)

If $n \geq 3$ and $n \neq 7$, then

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \quad \text{and} \quad \tilde{\gamma}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$$

Prof of Heawood's Formula

Theorem 9.15 (Heawood's Formula)

Let S be a surface with Euler genus $g = 2 - (v - e + f) > 0$ and let G be a loopless graph embedded in S . Then

$$\chi(G) \leq \left\lfloor \frac{7 + \sqrt{1 + 24g}}{2} \right\rfloor.$$

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Theorem 9.16 (Ringel–Youngs)

The bound in Heawood's Formula is the best possible, except that maximum chromatic number of graphs embedded in the Klein bottle is 6.

Problem 16

Prove that for every number n there is a bipartite graph whose choosability number is greater than n .

Problem 17

Find the (orientable) genus of the Petersen graph.

Problem 18

Does K_5 have cellular embeddings into two different orientable surfaces? Into two different non-orientable surfaces?

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- ▶ Then (x_i, x_j) is a **good pair** for the sequence.
- ▶ An infinite sequence containing a good pair is **good**; otherwise it is **bad**.

Antichains and Decreasing Sequences

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A quasi ordering $\preceq$ on X is a wqo if and only if X contains neither an infinite antichain nor an infinite strictly descending chain $x_0 \succ x_1 \succ \dots$.

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The forward implication is obvious.



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We start with $\mathbf{y}$ as the empty sequence, A and B as empty sets, and $\mathbf{x}_0 = \mathbf{x}$. Suppose that for some $i = 0, 1, \dots$, the first i elements of $\mathbf{y}$ and the sequence $\mathbf{x}_i$ have been defined, and each of the first i elements of $\mathbf{y}$ has been placed in exactly one of A and B .



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- If infinitely many elements of $\mathbf{x}_i$ are incomparable with x , make $\mathbf{x}_{i+1}$ be an infinite subsequence of $\mathbf{x}_i$ consisting of the elements incomparable with x , and put x into A .



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- ▶ If infinitely many elements of $\mathbf{x}_i$ are incomparable with x , make $\mathbf{x}_{i+1}$ be an infinite subsequence of $\mathbf{x}_i$ consisting of the elements incomparable with x , and put x into A .
- ▶ If this doesn't happen, there are infinitely many elements of $\mathbf{x}_i$ smaller than x . In that case, let $\mathbf{x}_{i+1}$ be an infinite subsequence of $\mathbf{x}_i$ consisting of the elements x' such that $x \succ x'$ and put x into B .



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Proof: Exercise.

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- ▶ The relation $\preccurlyeq$ on X is extended to $X^{<\omega}$ as follows: If $\mathbf{x} = (x_1, x_2, \dots, x_m)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$, we write $\mathbf{x} \preccurlyeq \mathbf{y}$ whenever there is a strictly increasing function $f : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, n\}$ such that $x_i \preccurlyeq y_{f(i)}$ for all $i \in \{1, 2, \dots, m\}$.

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- ▶ A quasi-order X is **well-founded** if it has no infinite strictly descending chains.

Higman's Theorem

Theorem 10.5 (Higman)

If X is well-quasi-ordered by $\preccurlyeq$, then so is $X^{<\omega}$.

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Proof of Higman's Theorem, Continued

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Proof of Higman's Theorem, Continued

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Corollary 10.6

If X is well-quasi-ordered by $\preceq$, then so is $[X]^{<\omega}$.

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Note 10.7 (Rado)

Higman's Theorem does not extend to infinite sequences.

Ordering Trees

Consider two trees T and T' with roots, respectively r and r' .

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Theorem 10.8 (Kruskal 1960)

Trees are well-quasi-ordered by the topological minors relation.

Proof of Kruskal's Theorem

We show that rooted trees are well-quasi-ordered by $\preceq$.

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Let G be a graph, T be a tree, and let $\mathcal{V} = \{V_t\}_{t \in V(T)}$ be a family of vertex sets $V_t \subseteq V(G)$ (called **bags**).

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The **width of the decomposition** $(T, \mathcal{V})$ is the maximum of $|V_t| - 1$ taken over all $v \in V(T)$. The **tree-width** of G , denoted by $\text{tw}(G)$ is the minimum width over all possible tree-decompositions.

Properties of Tree-Decompositions

Theorem 10.10

If H is a subgraph of G , and $(T, \{V_t\}_{t \in V(T)})$ is a tree-decomposition of G , then $(T, \{V_t \cap V(H)\}_{t \in V(T)})$ is a tree-decomposition of H .

Proof is very easy.

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Let t_1t_2 be an edge of T , and let T_1 and T_2 be the components of $T \setminus t_1t_2$, with $t_1 \in V(T_1)$ and $t_2 \in V(T_2)$.

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Both t_1 and t_2 lie on every s_1s_2 -path in T with $s_1 \in V(T_1)$ and $s_2 \in V(T_2)$.



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Let t be the last vertex of a maximal directed path in T ; we claim that $W \subseteq V_t$.



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We orient the edges of T as follows. For each edge t_1t_2 of T , define U_1 and U_2 as in Lemma 10.11; then $V_{t_1} \cap V_{t_2}$ separates U_1 from U_2 . If $V_{t_1} \cap V_{t_2}$ does not separate any two vertices of W , then $W \subseteq U_i$ for some $i \in \{1, 2\}$; we orient t_1t_2 towards that t_i .

Let t be the last vertex of a maximal directed path in T ; we claim that $W \subseteq V_t$. Suppose $w \in W$ and let $t' \in V(T)$ be such that $w \in V_{t'}$. If $t' \neq t$, then the edge e at t that separates t from t' is directed towards t , so w also lies in $V_{t''}$ for some t'' in the component of $T \setminus e$ containing t .



Lemma 10.12

For every $W \subseteq V(G)$, one of the following holds:

- (i) there is a $t \in V(T)$ such that $W \subseteq V_t$;*
- (ii) there are vertices w_1 and w_2 in W and an edge t_1t_2 of T such that w_1 and w_2 lie outside of $V_{t_1} \cap V_{t_2}$ and are separated by it in G .*

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We orient the edges of T as follows. For each edge t_1t_2 of T , define U_1 and U_2 as in Lemma 10.11; then $V_{t_1} \cap V_{t_2}$ separates U_1 from U_2 . If $V_{t_1} \cap V_{t_2}$ does not separate any two vertices of W , then $W \subseteq U_i$ for some $i \in \{1, 2\}$; we orient t_1t_2 towards that t_i .

Let t be the last vertex of a maximal directed path in T ; we claim that $W \subseteq V_t$. Suppose $w \in W$ and let $t' \in V(T)$ be such that $w \in V_{t'}$. If $t' \neq t$, then the edge e at t that separates t from t' is directed towards t , so w also lies in $V_{t''}$ for some t'' in the component of $T \setminus e$ containing t . Therefore $w \in V_t$ by (T3).



Corollary 10.13

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If G is obtained from H by deleting an edge, then a tree-decomposition of H is also a tree-decomposition of G . If G is obtained from H by deleting a vertex, then a tree-decomposition of H may be modified by removing the vertex from all bags to form a tree-decomposition of G .



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Theorem 10.16 (Robertson-Seymour 1990)

For every positive integer k , the graphs of tree-width less than k are well-quasi-ordered by the minor relation.

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- ▶ $\text{tw}(G) < 2$ if and only if K_3 is not a minor of G .

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- ▶ $\text{tw}(G) < 2$ if and only if K_3 is not a minor of G .
- ▶ $\text{tw}(G) < 3$ if and only if K_4 is not a minor of G .

Definition 10.18

- ▶ Two subsets U and W of $V(G)$ **touch** if a vertex of U is in W or is a neighbor of a vertex in W .

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Any set of vertices separating two covers of a bramble also covers that bramble.

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Since each set in a bramble is connected and meets both of the covers, it also meets any set separating these covers. □

Tree-width vs. Bramble

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Tree-width vs. Bramble

Theorem 10.20 (Seymour-Thomas 1993)

Let k be a non-negative integer. $\text{tw}(G) \geq k$ if and only if G contains a bramble of order greater than k .

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Theorem 10.21

The tree-width of an $n \times n$ grid ($n > 1$) is n .

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Every planar graph is a minor of a sufficiently large grid.

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Theorem 10.24 (Robertson-Seymour)

Planar graphs are well-quasi-ordered by the minor relation.

Definition 10.25

Suppose G is a graph embedded in a surface S . The **representativity** of G is the smallest number of points that a homotopically non-trivial curve in S intersects the graph.

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Theorem 10.26 (Robertson-Seymour)

Every graph embeddable on a surface S is a minor of a graph of sufficiently high S -representativity.

Theorem 10.27

For every surface S (orientable or not), the graphs embeddable in S are well-quasi-ordered by the minors relation.

Graphs Almost Embedded on Surfaces

Let r , s , t , and u be non-negative integers.

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Theorem 10.28 (Robertson-Seymour)

The class $\mathcal{G}(r, s, t, u)$ is well-quasi-ordered by the minor relation.

Graph Minors Theorem

Theorem 10.29 (Robertson-Seymour)

For every integer k there are integers r , s , t , and u such that every graph without K_k -minor belongs to $\mathcal{G}(r, s, t, u)$.

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Every minor-closed class of graphs other than the class of all graphs is a subclass of some $\mathcal{G}(r, s, t, u)$.

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Every minor-closed class of graphs other than the class of all graphs is a subclass of some $\mathcal{G}(r, s, t, u)$.

Corollary 10.31

The class of all (finite) graphs is well-quasi-ordered by the minor relation.

Problem 19

For each integer n exceeding one, find a bramble of order $n + 1$ in the $n \times n$ grid.

Problem 20

*A tree T is a **caterpillar** if T contains a path P such that every vertex of T either lies on P or is adjacent to a vertex of P . A **caterpillar forest** is a disjoint union of caterpillars. Find the minor-minimal graphs that are not caterpillar forests.*

Problem 21

What is the tree-width of the graph obtained from the Petersen graph by deleting one edge?

The Turán Graph

Question: Given a graph H , what is the greatest possible number of edges in a simple graph of order n that does not have H as a subgraph? We will answer this question when H is a complete graph.

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Theorem 11.2 (Turán 1941)

Given integers r and n exceeding 1, the unique simple graph of order n without K_r as a subgraph of maximum possible size is $T^{r-1}(n)$.

Proof of Turán's Theorem

First, observe that among all simple k -partite ($k < r$) graphs on n vertices, $T^{r-1}(n)$ has the largest size.

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If not, then non-adjacency is not an equivalence relation on $V(G)$, that is, there are vertices y_1 , x , and y_2 such that y_1x and xy_2 do not form edges of G , but y_1y_2 does. If $d(y_1) > d(x)$, then deleting x and duplicating y_1 yields another K_r -free graph with more edges than G . So $d(y_1) \leq d(x)$ and $d(y_2) \leq d(x)$. But then deleting y_1 and y_2 and duplicating x twice gives a K_r -free graph with more edges than G ; a contradiction.

Erdős-Stone Theorem

Corollary 11.3

If G is a simple graph of order n and size more than $t_{r-1}(n)$, then G contains K_r as a subgraph.

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Theorem 11.4 (Erdős-Stone, 1946)

For all integers $r \geq 2$ and $s \geq 1$, and every $\epsilon > 0$, there is an integer n_0 such that every simple graph of order $n \geq n_0$ and size at least $t_{r-1}(n) + \epsilon n^2$ contains the complete r -partite graph with each part of cardinality s .

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Definition 11.5

Given a simple graph H and an integer n , let $h_n(H)$ denote the maximum edge density that a simple H -free graph of order n can have;

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Lemma 11.6

$$\lim_{n \rightarrow \infty} h_n(K_r) = \frac{r-2}{r-1}.$$

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Corollary 11.3

If G is a simple graph of order n and size more than $t_{r-1}(n)$, then G contains K_r as a subgraph.

Theorem 11.4 (Erdős-Stone, 1946)

For all integers $r \geq 2$ and $s \geq 1$, and every $\epsilon > 0$, there is an integer n_0 such that every simple graph of order $n \geq n_0$ and size at least $t_{r-1}(n) + \epsilon n^2$ contains the complete r -partite graph with each part of cardinality s .

Proof omitted.

Definition 11.5

Given a simple graph H and an integer n , let $h_n(H)$ denote the maximum edge density that a simple H -free graph of order n can have; that is, the maximum number of edges that a simple H -free graph of order n can have divided by $\binom{n}{2}$.

Lemma 11.6

$$\lim_{n \rightarrow \infty} h_n(K_r) = \frac{r-2}{r-1}.$$

Corollary of Erdős-Stone Theorem

Corollary 11.7

For every simple, non-trivial graph H ,

$$\lim_{n \rightarrow \infty} h_n(H) = \frac{\chi(H) - 2}{\chi(H) - 1}.$$

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Proof.

Let $r = \chi(H)$.



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On the other hand, if K_r^s denotes the complete r -partite graph on rs vertices with every part of cardinality s , then $h_n(H) \leq h_n(K_r^s)$ for all sufficiently large s .



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On the other hand, if K_r^s denotes the complete r -partite graph on rs vertices with every part of cardinality s , then $h_n(H) \leq h_n(K_r^s)$ for all sufficiently large s . Moreover, Erdős-Stone Theorem 11.4 implies that for every $\epsilon > 0$ and n large enough

$$h_n(K_r^s) < h_n(K_r) + \frac{\epsilon n^2}{\binom{n}{2}}.$$



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Hence for large n , we have

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Then Lemma 11.6 finishes the proof.



Ramsey Theorem

Theorem 11.8 (Ramsey 1930)

For every natural number r there is a natural number n such that every simple graph of order at least n contains either K_r or $\overline{K_r}$ as an induced subgraph.

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Let V_1 be any set of 2^{2r-3} vertices of and pick $v_1 \in V_1$ arbitrarily. Then (i)–(iii) hold trivially.



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Inductively, $|V_{i-1} \setminus \{v_{i-1}\}| = 2^{2^{r-1-i}} - 1$, and $V_{i-1} \setminus \{v_{i-1}\}$ contains a subset V_i satisfying (i)–(iii); pick v_i arbitrarily in V_i .



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Inductively, $|V_{i-1} \setminus \{v_{i-1}\}| = 2^{2r-1-i} - 1$, and $V_{i-1} \setminus \{v_{i-1}\}$ contains a subset V_i satisfying (i)–(iii); pick v_i arbitrarily in V_i . Among the vertices $v_1, v_2, \dots, v_{2r-3}$, at least $r-1$ show the same behavior described in (iii). Those $r-1$ vertices together with v_{2r-2} induce either K_r or $\overline{K_r}$. □

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Theorem 11.9

Let k and c be positive integers, and let X be an infinite set. If $[X]^k$ is c -colored, then X has an infinite monochromatic subset.

Proof

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Start with $X_0 = X$ and pick $x_0 \in X_0$ arbitrarily. Having chosen X_i and $x_i \in X_i$, we c -color $[X_i \setminus \{x_i\}]^{k-1}$ by giving each set Z the color of $\{x_i\} \cup Z$ in our c -coloring of $[X]^k$.

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Since c is finite, one of the colors is associated with infinitely many x_i —they form an infinite monochromatic subset of X .

Theorem 11.10 (König Infinity Lemma)

*Let $V_0, V_1, \dots$ be an infinite sequence of disjoint non-empty finite sets, and let G be an infinite graph on their union. Assume that every vertex v in V_n , for $n \geq 1$, has a neighbor $f(v)$ in V_{n-1} . Then G contains a **ray**, that is a one-way-infinite path, $v_0 v_1 \dots$ with $v_n \in V_n$ for all n .*

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For all positive integers k , c , and r there is an integer $n \geq k$ such that every n -element set X has a monochromatic r -element subset with respect to any c -coloring of $[X]^k$.

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Suppose the theorem fails for some k , c , and r . Then for every $n \geq k$ there is a c -coloring of $[n]^k$ such that n contains no monochromatic r -element subset.

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For every $n \geq k$, let V_n be the (nonempty) set of bad colorings of $[n]^k$. For $n > k$, the restriction $f(g)$ of any $g \in V_n$ to $[n-1]^k$ is still bad, and so lies in V_{n-1} .

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For every $n \geq k$, let V_n be the (nonempty) set of bad colorings of $[n]^k$. For $n > k$, the restriction $f(g)$ of any $g \in V_n$ to $[n-1]^k$ is still bad, and so lies in V_{n-1} . By König Infinity Lemma 11.10, there is an infinite sequence $g_k, g_{k+1}, \dots$ of bad colorings $g_n \in V_n$ such that $f(g_n) = g_{n-1}$ for all $n > k$.

Theorem 11.11

For all positive integers k , c , and r there is an integer $n \geq k$ such that every n -element set X has a monochromatic r -element subset with respect to any c -coloring of $[X]^k$.

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Then g is a bad coloring of $[\mathbb{N}]^k$ since every r -element subset S of $\mathbb{N}$ is contained in some sufficiently large $[n]$, and so S cannot be monochromatic since g coincides on $[n]^k$ with the bad coloring g_n .

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Then g is a bad coloring of $[\mathbb{N}]^k$ since every r -element subset S of $\mathbb{N}$ is contained in some sufficiently large $[n]$, and so S cannot be monochromatic since g coincides on $[n]^k$ with the bad coloring g_n . This contradicts 11.9.

Ramsey numbers

Definition 11.12

The least integer n associated with k , c , and r as in Theorem 11.11 is the **Ramsey number** for k , c and r , and is denoted by $R(k, c, r)$.

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We proved before that $R(2, 2, 3) = 6$, and that $R(K_3, K_3) = 6$. In most cases the exact Ramsey numbers are not known. Most known values and bounds are listed at <http://mathworld.wolfram.com/RamseyNumber.html>

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Let s and t be positive integers, and let T be a tree of order t . Then $R(T, K_s) = (s - 1)(t - 1) + 1$.

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Theorem 11.14 (Chvatál, Rödl, Szemerédi and Trotter 1983)

For every positive integer Δ there is a constant c such that $R(H) \leq c|H|$ for all graphs H with $\Delta(H) \leq \Delta$.

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Proof omitted—uses the Regularity Lemma.

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Theorem 11.15 (Deuber; Erdős, Hajnal, Pósa; Rödl 1973)

Every graph has a Ramsey graph. For every graph H there is a graph G such that, for every partition $\{E_1, E_2\}$ of $E(G)$, has an induced subgraph H with $E(H) \subseteq E_1$ or $E(H) \subseteq E_2$.

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Given two graphs $G = (V, E)$ and H , and $U \subseteq V$, we write $G[U \rightarrow H]$ to denote the graph obtained from G by replacing each vertex u in U by a copy $H(u)$ of H ,

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(*) For any two graphs H_1 and H_2 , there is a graph $G = G(H_1, H_2)$ such that every edge-coloring of G with colors 1 and 2 yields either an induced $H_1 \subseteq G$ with all edges colored 1, or an induced $H_2 \subseteq G$ with all edges colored 2.

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We proceed by induction on $|H_1| + |H_2|$. If either H_1 or H_2 has no edges, in particular, $|H_1| + |H_2| \leq 1$, then (*) holds with $G = \overline{K_n}$ for sufficiently large n . For the induction step, assume that both H_1 and H_2 have at least one edge, and that (*) holds for all pairs (H'_1, H'_2) with smaller $|H'_1| + |H'_2|$.

For each $i \in \{1, 2\}$, pick a vertex $x_i \in H_i$ that is incident with an edge, let $H'_i = H_i - x_i$, and let H''_i be the subgraph of H_i induced by the neighbors of x_i .

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Let F be the family of these graphs $H'_1(u)$, one for each $u \in U^{i-1}$ and let $x = x(F)$.

If, for some $u \in U^{i-1}$, all the $x - H_1''(u)$ edges in G^i are also colored 1, then we have an induced copy of H_1 in G^i and again the conclusion holds.

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If, for some $u \in U^{i-1}$, all the $x - H_1''(u)$ edges in G^i are also colored 1, then we have an induced copy of H_1 in G^i and again the conclusion holds. So we may assume that each $H_1''(u)$ has a vertex y_u for which the edge xy_u is colored 2. The restriction $y_u \mapsto u$ of f to $\hat{U}^{i-1} = \{y_u | u \in U^{i-1}\} \subseteq V^i$ extends by $(v, \emptyset) \mapsto v$ to an isomorphism from

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to G^{i-1} , and so our edge-coloring of G^i induces an edge-coloring of G^{i-1} . If this coloring gives an induced $H_1 \subseteq G^{i-1}$ colored 1, or an induced $H_2 \subseteq G^{i-1}$ colored 2, we have these also in $\hat{G}^{i-1} \subseteq G^i$ and again the conclusion holds.

By (**) for $i-1$ we may then assume that G^{i-1} has an induced H' colored 2 with $V(H') \subseteq V^{i-1}$ and such that the restriction of f^{i-1} to $V(H')$ is an isomorphism from H' to $G^0[W'_k] \cong H'_2$ for some $k \in \{i-1, \dots, n-1\}$. Let $\hat{H}'$ be the corresponding induced subgraph of $\hat{G}^{i-1} \subseteq G^i$ (also colored 2). then $V(\hat{H}') \subseteq V_i$,

$$f^i(V(\hat{H}')) = f^{i-1}(V(H')) = W'_k,$$

and $F^i : \hat{H}' \rightarrow G^0[W'_k]$ is an isomorphism.

If $k \geq i$, then the proof of (**) is complete with $H = \hat{H}'$.

We thus assume that $k < i$, and so $k = i - 1$.

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Problem 22

Prove that for every positive integer k there is an integer N such that if G is a 2-connected graph of order at least N , then G has a subdivision of C_k or $K_{2,k}$. Find an upper bound on N in terms of k .

Problem 23

Find a Ramsey graph for C_4 , that is, find a graph G such that if the edges of G are partitioned into $\{E_1, E_2\}$, then G has a induced subgraph isomorphic to C_4 all of whose edges belong to one of E_1 or E_2 .

Ramsey Theorem for Connected Graphs

Theorem 11.16

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Parallel Minors

Recall that a graph is a topological minor of another if it can be obtained by

- ▶ deleting edges or isolated vertices
- ▶ contracting edges in series that are in series with another edge

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A graph is a **parallel minor** of another if it can be obtained by

- ▶ contracting edges
- ▶ deleting edges that are in parallel with other edges (simplifying)

Unavoidable Parallel Minors

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Conjecture 11.24 (Chartrand, Geller, Hedetniemi)

*Every **planar** graph is a union of two **outerplanar** graphs.*

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“Proof”:

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 - $\Rightarrow$ Hamilton cycle C
 - $\Rightarrow (C + \text{edges inside } C) \cup (C + \text{edges outside } C)$

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 - $\Rightarrow G$ is 0-, 1-, 2-, or 3-sum of A and B
 - $\Rightarrow$ decompose each A and B and make parts “fit together” **FAIL!**

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 - $\Rightarrow$ decompose each A and B and make parts “fit together” **FAIL!**
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Partitioning Planar Graphs

Conjecture 11.24 (Chartrand, Geller, Hedetniemi)

Every *planar* graph is a union of two *outerplanar* graphs.

~~“Proof”~~:

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Theorem 11.25 (Ding, O., Sanders, Vertigan; Kedlaya)

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Partitioning Graphs on Surfaces

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Every *projective* graph has a vertex-partition into two graphs of $tw \leq 2$.

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Minor-Closed Classes

Conjecture 11.30 (Thomas)

For every G there is an integer k such that every graph with no G -minor has a vertex-partition and edge-partition into two graphs of $tw \leq k$.

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Partitions and Contractions

Question 11.34 (Oxley)

Can every co-graphic matroid be partitioned into two series-parallel matroids?

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The edges of every **toroidal** graph can be partitioned into E_1 and E_2 such that **tw** $(G/E_1) \leq 3$ and **tw** $(G/E_2) \leq 4$.

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Theorem 11.37 (Demaine, Hajiaghayi, Mohar)

The edges of a graph of **genus** g can be partitioned into E_1 and E_2 such that each of G/E_1 and G/E_2 has **tw** $\leq O(g^2)$.

Definition of $T(k, l, r)$

- ▶ Start with K_k , and assign all of its vertices level 0

$T(2, 4, 1)$

level 0



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- ▶ Start with K_k , and assign all of its vertices level 0
- ▶ Inductively, for each K_k subgraph H of level $n - 1$, add r new vertices, join each of them to all vertices of H and declare all newly created vertices and K_k subgraphs to have level n .

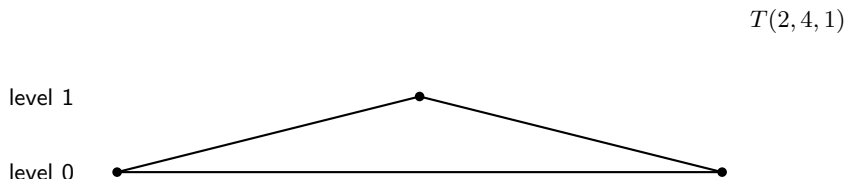
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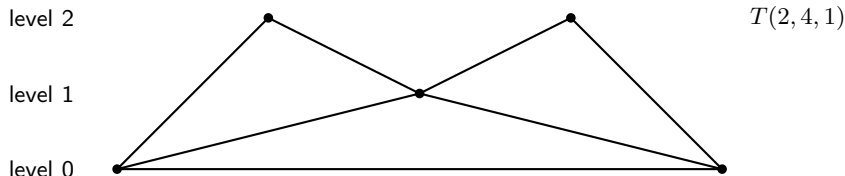
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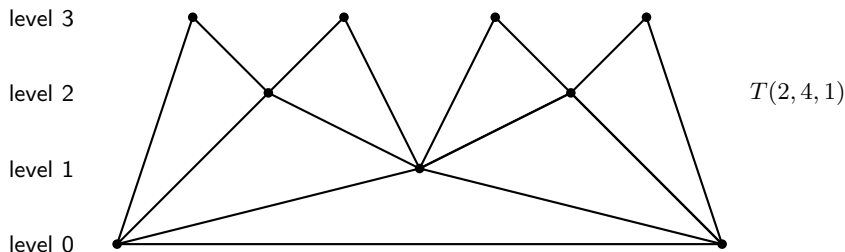
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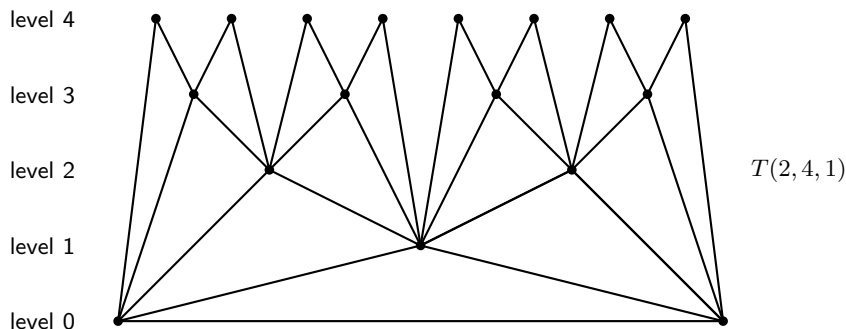
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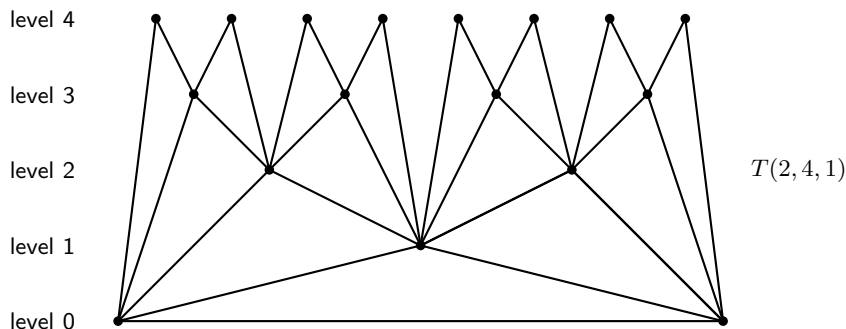
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- ▶ Stop after having created all level- l subgraphs.



Partitioning k -Trees

Definition 11.38

k -tree: $T(k, l, r)$ where l is arbitrary, and r can vary arbitrarily at every stage.

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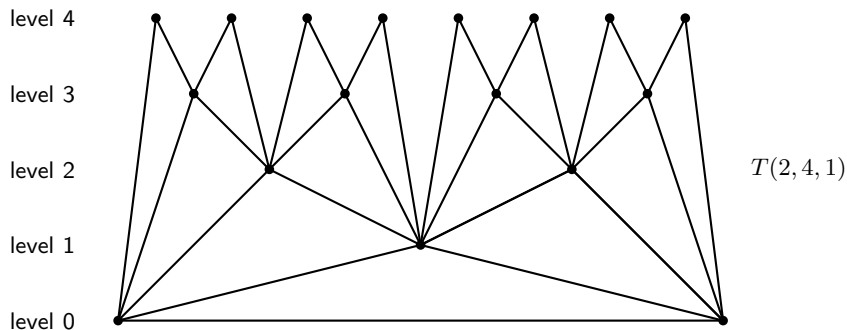
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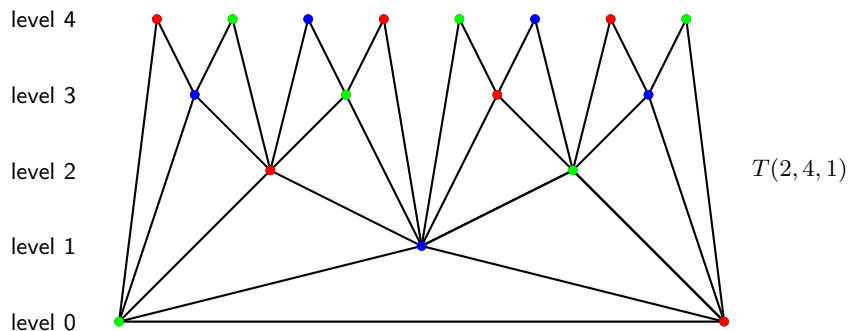
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For every k, l , and r there is L such that
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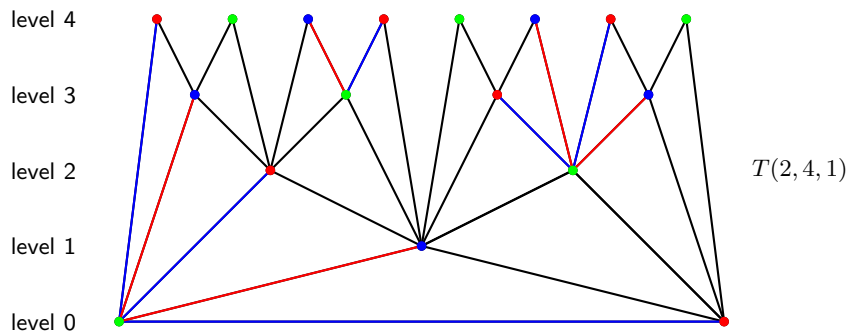
Large k -Trees and Edge-Partitions



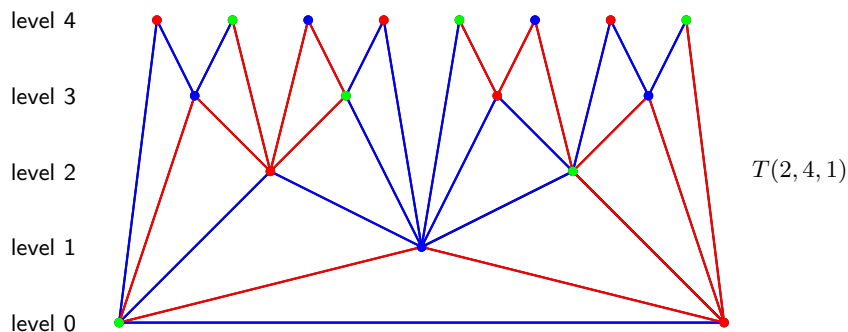
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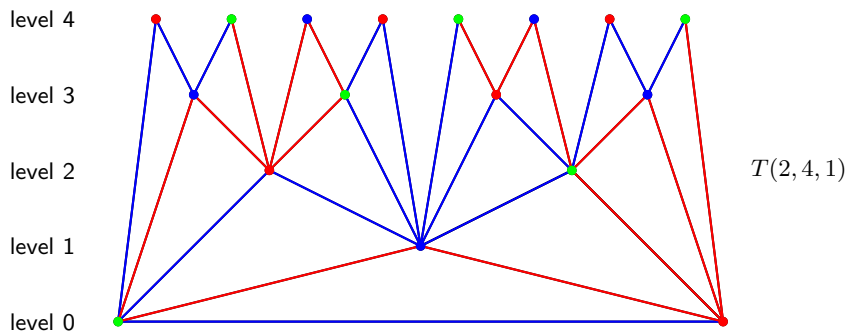
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Large k -Trees and Edge-Partitions



Theorem 11.44 (DOSV)

For every l and r there are L and R such that if $T(2, L, R)$ has its edges colored red and blue, then it contains a **red** $T(1, l, r)$ or a **blue** subdivision of $T(1, l, r)$.

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- ▶ for vertex-partitions, consider line graphs of those graphs

$\Delta \leq 4$ and Vertex-Partitions

Theorem 11.46 (ADOV)

If $\Delta(G) \leq 4$, then G has a vertex-partition into two graphs on components with at most

$\Delta \leq 4$ and Vertex-Partitions

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Four-Color Theorem for Minor-Closed Classes of Graphs

Theorem 11.51 (ADOV+S)

Let $\mathcal{G}$ be a minor-closed class of graphs other than the class of all graphs, and pick Δ .

There is a number $c(\mathcal{G}, \Delta)$ such that every member of $\mathcal{G}$ whose max degree is $\leq \Delta$ can be vertex 4-colored so that all monochromatic components have at most c vertices.

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- ▶ This gives a 4-coloring of G with components on at most c vertices.

