COLORING GRAPHS WITH CROSSINGS

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Abstract. We generalize the Five Color Theorem by showing that it extends to graphs with two crossings. Furthermore, we show that if a graph has three crossings, but does not contain K_6 as a subgraph, then it is also 5-colorable. We also consider the question of whether the result can be extended to graphs with more crossings.

1. Introduction

In this paper, n will denote the number of vertices, and m the number of edges, of a graph G. A coloring of G is understood to be a *proper coloring*; that is, one in which adjacent vertices always receive distinct colors.

We will consider *drawings* of graphs in the plane $\mathbb{R}^2$ for which no three edges have a common crossing. A crossing of two edges e and f is *trivial* if e and f are adjacent or equal, and it is *non-trivial* otherwise. A drawing is *good* if it has no trivial crossings. The following is a well-known easy lemma.

Lemma 1.1. *A drawing of a graph can be modified to eliminate all of its trivial crossings, with the number of non-trivial crossings remaining the same.*

To avoid complicating the notation, we will use the same symbol for a graph and its drawing in the plane. We will refer to the *regions* of a drawing of a graph G as the maximal open sets U of $\mathbb{R}^2 - G$ such that for every two points $x, y \in U$, there exists a polygonal xy-curve in U.

Definition 1.2. The *crossing number* of a graph G, denoted by $\nu(G)$, is the minimum number of crossings in a drawing of G. An *optimal drawing* of G is a drawing of G with exactly $\nu(G)$ crossings.

Definition 1.3. Suppose G' and G are graphs. A function α with domain $V(G') \cup E(G')$ is an *immersion* of G' into G if the following hold:

(1) the restriction of α to $V(G')$ is an injection into $V(G)$;
(2) for an edge e of G' incident to u and v, the image $\alpha(e)$ is a path in G with ends $\alpha(u)$ and $\alpha(v)$; and
(3) for distinct edges e and f of G', their images $\alpha(e)$ and $\alpha(f)$ are edge-disjoint.

The immersion α is *essential* if additionally $\alpha(e)$ and $\alpha(f)$ are vertex-disjoint whenever e and f are not adjacent, and it is an *embedding* if $\alpha(e)$ and $\alpha(f)$ are internally vertex-disjoint for all distinct e and f. If v is a vertex of G, and α is an essential immersion of G' into G such that $v = \alpha(u)$ for some vertex u of G', and $\alpha(e)$ is a single-edge path for each e incident with u, then α is called a *v-immersion* of G' into G. We will also say that α is an immersion of G' *onto* G if the range of α is $V(G) \cup E(G)$. Depending on the

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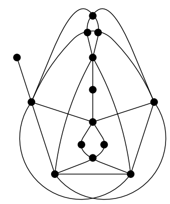

FIGURE 1. A graph with an essential immersion of K_6

properties of α, we will say that G' is *immersed*, *essentially immersed*, *embedded*, or v-*immersed into* or *onto* G. An example appears in Figure 1.

It is worth noting that if, for every edge e of G', the path $\alpha(e)$ consists of a single edge, then G' is a subgraph of G. All immersions considered in the remainder of this paper will be essential.

Proposition 1.4. *If $n \geq 3$, then $\nu(G) \geq m - 3n + 6$.*

Proof. Since $m \leq 3n - 6$ in a planar graph, every edge in excess of this bound introduces at least one additional crossing. □

Corollary 1.5. *The crossing number of the complete graph K_6 is three.*

Proof. It is easy to draw K_6 with exactly three crossings, while Proposition 1.4 implies that $\nu(K_6) \geq 3$. □

2. IMMERSIONS AND CROSSINGS

In this section we present several results that relate crossings of a drawing with immersions of a graph.

Lemma 2.1. *Suppose G is a good drawing with exactly k crossings and there is an essential immersion of G' onto G. Then G' has a good drawing with exactly k crossings.*

Proof. Let α be an essential immersion of G' onto G. Draw G' by placing each vertex v at $\alpha(v)$, drawing each edge e so that it follows $\alpha(e)$, and then perturbing the edges slightly so that no edge contains a vertex and no three edges cross at the same point. Each crossing of edges e and f in G' arises from the corresponding paths $\alpha(e)$ and $\alpha(f)$ either crossing or sharing a vertex. In the latter case, the crossing is trivial as the immersion α is essential. The conclusion now follows immediately from Lemma 1.1. □

Thus we have the following:

Corollary 2.2. *If G' is essentially immersed into G, then $\nu(G') \leq \nu(G)$.*

We may also use essential immersions to extend the Five Color Theorem.

Lemma 2.3. *Let G be a graph and let v be a vertex in G of degree at most five such that there is no v-immersion of K_6 into G. If $G - v$ is 5-colorable, then so is G.*

Proof. Suppose that G is not 5-colorable, and let c be a 5-coloring of $G - v$. Then c must assign all five colors to the neighbors of v and hence $\deg(v) = 5$; since otherwise we can extend c to G. Let the neighbors of v be v_1, v_2, v_3, v_4 and v_5; and denote $c(v_i) = i$ for each $i \in \{1, 2, 3, 4, 5\}$.

For each pair of distinct i and j in $\{1, 2, 3, 4, 5\}$, let $G_{\{i,j\}}$ denote the subgraph of $G - v$ whose vertices are colored by c with i or j. If, for one such pair of i and j, the graph $G_{\{i,j\}}$ has v_i and v_j in distinct components, then the colors i and j can be switched in one of the components so that two neighbors of v are colored the same. In this case, the coloring c can be extended to v so that G is 5-colored; a contradiction.

Hence, for each pair of distinct i and j, the graph $G - v$ has a path joining v_i and v_j whose vertices are alternately colored i and j by c, and thus G contains a v-immersion of K_6; again, a contradiction. $\square$

Corollary 2.4 (Generalized Five Color Theorem)**.** *Every graph with crossing number at most two is* 5*-colorable.*

Proof. Suppose not and consider a counterexample G on the minimum number of vertices. Proposition 1.4 implies that $m \leq 3n-4$, and so G has a vertex v whose degree is at most five. From Corollaries 1.5 and 2.2 we conclude that there is no essential immersion, and hence no v-immersion, of K_6 into G. The minimality of G implies that $G - v$ is 5-colorable, from which Lemma 2.3 provides the required contradiction. $\square$

Lemma 2.3 establishes that a graph G with $\nu(G) \leq 3$ is 5-colorable if there is no v-immersion of K_6 into G. The next lemma addresses the case of graphs with $\nu(G) \leq 3$ for which there is a v-immersion of K_6 into G for some vertex v in G. The following corollary of a result of Kleitman [Kle76] will be used in its proof.

Proposition 2.5. *Every good drawing of K_5 has odd number of crossings.*

Lemma 2.6. *If G is a drawing with exactly three crossings and α is a v-immersion of K_6 into G for some vertex v in G, then v is incident with exactly two crossed edges.*

Proof. Let H be the subgraph of G that is the image of K_6 under α, and let u be the vertex in K_6 such that $\alpha(u) = v$. From Lemma 1.1 and Corollaries 1.5 and 2.2, it follows that H is a good drawing containing all three crossings of G.

If v were incident with one or three crossed edges in H, then $H - v$ would be a good drawing with zero or two crossings with K_5 essentially immersed onto it. This, together with Lemma 2.1, would imply that there is a good drawing of K_5 with zero or two crossing, which would contradict Proposition 2.5.

Moreover, if v were incident with no crossed edges in H, then $H - v$ would be a drawing with a region R that is incident with all vertices in the set $S = \{\alpha(w) \colon w \in V(K_6 - u)\}$. The boundary of R then induces a cyclic order on the set S, and hence also on $V(K_6 - u)$. If e and f are distinct non-adjacent edges of $K_6 - u$ and each joins a pair of non-consecutive vertices, then $\alpha(e)$ and $\alpha(f)$ must cross. It follows that H would have at least five crossings; a contradiction. $\square$

3. Colorings and Crossings

Lemmas 2.3 and 2.6, respectively, characterize a graph G when it does not and does contain a v-immersion of K_6. With these, we now proceed to the main theorem. We will use $\omega(G)$ to denote the *clique number* of G, that is, the largest n for which K_n is a subgraph of G.

Main Theorem 3.1. *If $\nu(G) \leq 3$ and $\omega(G) \leq 5$, then G is* 5*-colorable.*

Proof. Let $\mathcal{G}$ denote the class of all graphs with crossing number at most three that are not 5-colorable, and let G be a member of $\mathcal{G}$ with the minimum number of vertices. Suppose that $\omega(G) \leq 5$ and that G is drawn optimally in the plane.

If G contains a vertex v of degree less than five, then G is not a minimal member of $\mathcal{G}$, since a 5-coloring of $G - v$ extends to a 5-coloring of G. Hence, the minimum degree of G is five. By Proposition 1.4, the graph G has at most $3n - 3$ edges, and thus has at least six vertices of degree five.

Let v be a vertex of degree five. Lemma 2.3 implies that there is a v-immersion of K_6 into G, and Corollary 2.2 implies that the image of K_6 in G contains three crossed edges. Then Lemma 2.6 implies that two crossed edges of G are incident with v. Since G is not K_6, it contains a vertex w of degree five not adjacent to v. However, Lemma 2.3 implies that there is also a w-immersion of K_6 into G, and so w is also incident with two crossed edges. Since v and w are not adjacent, these two crossed edges are different from the crossed edges incident with v, which implies that G contains four crossings; a contradiction. $\square$

We also show that when Theorem 3.1 is applied to a 4-connected graph G other than K_6, then the assumption $\omega(G) \leq 5$ may be discarded. More precisely, we have:

Corollary 3.2. *If G is 4-connected, $\nu(G) \leq 3$ and $G \neq K_6$, then G is 5-colorable.*

Proof. Let G be a drawing with at most three crossings of a 4-connected graph not isomorphic to K_6. We show that $\omega(G) \leq 5$, from which the conclusion follows immediately by Theorem 3.1.

Suppose, to the contrary, that G has a complete subgraph K on six vertices. Let v be a vertex of G that is not in K, and let K' be the plane drawing obtained from K by replacing each crossing with a new vertex. By Corollary 1.5, all three crossings of G are in K, and so $|V(K')| = 9$ and $|E(K')| = 21$. Thus K' is a triangulation and so every region of K contains at most three vertices in its boundary. But this is impossible, as G, being 4-connected, has four paths from v to vertices of K, with each pair of paths having only v in common. $\square$

Lastly, note that $C_3 \vee C_5$, the graph in which every vertex of C_3 is adjacent to every vertex of C_5, contains no K_6 subgraph and is not 5-colorable.

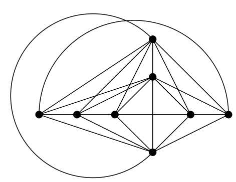

FIGURE 2. $C_3 \vee C_5$ drawn with the minimum number of crossings

Proposition 3.3. *The crossing number of $C_3 \vee C_5$ is six.*

Proof. Let G be an optimal drawing of $K \vee L$, where K and L are cycles on, respectively, three and five vertices. Suppose that G has fewer than six crossings. Note that $G \backslash (E(K) \cup E(L))$ is isomorphic to $K_{3,5}$, which has crossing number four [Kle70]. This implies that the edges of $K \cup L$ are involved in at most one crossing, and thus L has at most three regions, one of which contains K. Thus at least one region of L avoids K and has two non-adjacent vertices of L in its boundary. These two vertices of L can be joined by a new edge that crosses no edges of G thereby creating a graph with 8 vertices, 24 edges, and 5 crossings; a contradiction to Proposition 1.4. Hence, G has six crossings. Figure 2 shows a drawing which achieves this bound, proving that $\nu(C_3 \vee C_5) = 6$. $\square$

We do not currently know whether the Main Theorem 3.1 extends to graphs with four or five crossings, and hence conclude with the following question:

Question 3.4. Does a graph G have a 5-coloring if $\nu(G) \leq 5$ and $\omega(G) \leq 5$?

References

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