COLORING GRAPHS WITH CROSSINGS

BOGDAN OPOROWSKI* AND DAVID ZHAO†

ABSTRACT. We generalize the Five Color Theorem by showing that it extends to graphs with two crossings. Furthermore, we show that if a graph has three crossings, but does not contain $K_6$ as a subgraph, then it is also 5-colorable. We also consider the question of whether the result can be extended to graphs with more crossings.

1. Introduction

In this paper, $n$ will denote the number of vertices, and $m$ the number of edges, of a graph $G$. A coloring of $G$ is understood to be a proper coloring; that is, one in which adjacent vertices always receive distinct colors.

We will consider drawings of graphs in the plane $\mathbb{R}^2$ for which no three edges have a common crossing. A crossing of two edges $e$ and $f$ is trivial if $e$ and $f$ are adjacent or equal, and it is non-trivial otherwise. A drawing is good if it has no trivial crossings. The following is a well-known easy lemma.

Lemma 1.1. A drawing of a graph can be modified to eliminate all of its trivial crossings, with the number of non-trivial crossings remaining the same.

To avoid complicating the notation, we will use the same symbol for a graph and its drawing in the plane. We will refer to the regions of a drawing of a graph $G$ as the maximal open sets $U$ of $\mathbb{R}^2 - G$ such that for every two points $x, y \in U$, there exists a polygonal $xy$-curve in $U$.

Definition 1.2. The crossing number of a graph $G$, denoted by $\nu(G)$, is the minimum number of crossings in a drawing of $G$. An optimal drawing of $G$ is a drawing of $G$ with exactly $\nu(G)$ crossings.

Definition 1.3. Suppose $G'$ and $G$ are graphs. A function $\alpha$ with domain $V(G') \cup E(G')$ is an immersion of $G'$ into $G$ if the following hold:

(1) the restriction of $\alpha$ to $V(G')$ is an injection into $V(G)$;
(2) for an edge $e$ of $G'$ incident to $u$ and $v$, the image $\alpha(e)$ is a path in $G$ with ends $\alpha(u)$ and $\alpha(v)$; and
(3) for distinct edges $e$ and $f$ of $G'$, their images $\alpha(e)$ and $\alpha(f)$ are edge-disjoint.

The immersion $\alpha$ is essential if additionally $\alpha(e)$ and $\alpha(f)$ are vertex-disjoint whenever $e$ and $f$ are not adjacent, and it is an embedding if $\alpha(e)$ and $\alpha(f)$ are internally vertex-disjoint for all distinct $e$ and $f$. If $v$ is a vertex of $G$, and $\alpha$ is an essential immersion of $G'$ into $G$ such that $v = \alpha(u)$ for some vertex $u$ of $G'$, and $\alpha(e)$ is a single-edge path for each $e$ incident with $u$, then $\alpha$ is called a v-immersion of $G'$ into $G$. We will also say that $\alpha$ is an immersion of $G'$ onto $G$ if the range of $\alpha$ is $V(G) \cup E(G)$. Depending on the

Date: January 19, 2005.
Key words and phrases. chromatic number, clique number, crossing number, immersion.
*Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.
bogdan@math.lsu.edu.
†Department of Computer Science, University of Texas, Austin, TX 78712.
wzhao@cs.utexas.edu.
properties of \( \alpha \), we will say that \( G' \) is \textit{immersed, essentially immersed, embedded,} or \textit{v-immersed into} or \textit{onto} \( G \). An example appears in Figure 1.

It is worth noting that if, for every edge \( e \) of \( G' \), the path \( \alpha(e) \) consists of a single edge, then \( G' \) is a subgraph of \( G \). All immersions considered in the remainder of this paper will be essential.

\textbf{Proposition 1.4.} If \( n \geq 3 \), then \( \nu(G') \geq m - 3n + 6 \).

\textit{Proof.} Since \( m \leq 3n - 6 \) in a planar graph, every edge in excess of this bound introduces at least one additional crossing. \( \square \)

\textbf{Corollary 1.5.} The crossing number of the complete graph \( K_6 \) is three.

\textit{Proof.} It is easy to draw \( K_6 \) with exactly three crossings, while Proposition 1.4 implies that \( \nu(K_6) \geq 3 \). \( \square \)

\section{Immersions and Crossings}

In this section we present several results that relate crossings of a drawing with immersions of a graph.

\textbf{Lemma 2.1.} Suppose \( G \) is a good drawing with exactly \( k \) crossings and there is an essential immersion of \( G' \) onto \( G \). Then \( G' \) has a good drawing with exactly \( k \) crossings.

\textit{Proof.} Let \( \alpha \) be an essential immersion of \( G' \) onto \( G \). Draw \( G' \) by placing each vertex \( v \) at \( \alpha(v) \), drawing each edge \( e \) so that it follows \( \alpha(e) \), and then perturbing the edges slightly so that no edge contains a vertex and no three edges cross at the same point. Each crossing of edges \( e \) and \( f \) in \( G' \) arises from the corresponding paths \( \alpha(e) \) and \( \alpha(f) \) either crossing or sharing a vertex. In the latter case, the crossing is trivial as the immersion \( \alpha \) is essential. The conclusion now follows immediately from Lemma 1.1. \( \square \)

Thus we have the following:

\textbf{Corollary 2.2.} If \( G' \) is essentially immersed into \( G \), then \( \nu(G') \leq \nu(G) \).

We may also use essential immersions to extend the Five Color Theorem.

\textbf{Lemma 2.3.} Let \( G \) be a graph and let \( v \) be a vertex in \( G \) of degree at most five such that there is no \textit{v-immersion} of \( K_6 \) into \( G \). If \( G - v \) is 5-colorable, then so is \( G \).

\textit{Proof.} Suppose that \( G \) is not 5-colorable, and let \( c \) be a 5-coloring of \( G - v \). Then \( c \) must assign all five colors to the neighbors of \( v \) and hence \( \deg(v) = 5 \); since otherwise we can extend \( c \) to \( G \). Let the neighbors of \( v \) be \( v_1, v_2, v_3, v_4 \) and \( v_5 \); and denote \( c(v_i) = i \) for each \( i \in \{1, 2, 3, 4, 5\} \).

For each pair of distinct \( i \) and \( j \) in \( \{1, 2, 3, 4, 5\} \), let \( G_{\{i,j\}} \) denote the subgraph of \( G - v \) whose vertices are colored by \( c \) with \( i \) or \( j \). If, for one such pair of \( i \) and \( j \), the graph \( G_{\{i,j\}} \) has \( v_i \) and \( v_j \) in distinct
components, then the colors $i$ and $j$ can be switched in one of the components so that two neighbors of $v$ are colored the same. In this case, the coloring $c$ can be extended to $v$ so that $G$ is 5-colored; a contradiction.

Hence, for each pair of distinct $i$ and $j$, the graph $G - v$ has a path joining $v_i$ and $v_j$ whose vertices are alternately colored $i$ and $j$ by $c$, and thus $G$ contains a $v$-immersion of $K_6$; again, a contradiction. □

**Corollary 2.4** (Generalized Five Color Theorem). *Every graph with crossing number at most two is 5-colorable.*

**Proof.** Suppose not and consider a counterexample $G$ on the minimum number of vertices. Proposition 1.4 implies that $m \leq 3n - 4$, and so $G$ has a vertex $v$ whose degree is at most five. From Corollaries 1.5 and 2.2 we conclude that there is no essential immersion, and hence no $v$-immersion, of $K_6$ into $G$. The minimality of $G$ implies that $G - v$ is 5-colorable, from which Lemma 2.3 provides the required contradiction. □

Lemma 2.3 establishes that a graph $G$ with $\nu(G) \leq 3$ is 5-colorable if there is no $v$-immersion of $K_6$ into $G$. The next lemma addresses the case of graphs with $\nu(G) \leq 3$ for which there is a $v$-immersion of $K_6$ into $G$ for some vertex $v$ in $G$. The following corollary of a result of Kleitman [Kle76] will be used in its proof.

**Proposition 2.5.** *Every good drawing of $K_5$ has odd number of crossings.*

**Lemma 2.6.** *If $G$ is a drawing with exactly three crossings and $\alpha$ is a $v$-immersion of $K_6$ into $G$ for some vertex $v$ in $G$, then $v$ is incident with exactly two crossed edges.*

**Proof.** Let $H$ be the subgraph of $G$ that is the image of $K_6$ under $\alpha$, and let $u$ be the vertex in $K_6$ such that $\alpha(u) = v$. From Lemma 1.1 and Corollaries 1.5 and 2.2, it follows that $H$ is a good drawing containing all three crossings of $G$.

If $v$ were incident with one or three crossed edges in $H$, then $H - v$ would be a good drawing with zero or two crossings with $K_5$ essentially immersed onto it. This, together with Lemma 2.1, would imply that there is a good drawing of $K_5$ with zero or two crossing, which would contradict Proposition 2.5.

Moreover, if $v$ were incident with no crossed edges in $H$, then $H - v$ would be a drawing with a region $R$ that is incident with all vertices in the set $S = \{\alpha(w) : w \in V(K_6 - u)\}$. The boundary of $R$ then induces a cyclic order on the set $S$, and hence also on $V(K_6 - u)$. If $e$ and $f$ are distinct non-adjacent edges of $K_6 - u$ and each joins a pair of non-consecutive vertices, then $\alpha(e)$ and $\alpha(f)$ must cross. It follows that $H$ would have at least five crossings; a contradiction. □

3. Colorings and Crossings

Lemmas 2.3 and 2.6, respectively, characterize a graph $G$ when it does not and does contain a $v$-immersion of $K_6$. With these, we now proceed to the main theorem. We will use $\omega(G)$ to denote the *clique number* of $G$, that is, the largest $n$ for which $K_n$ is a subgraph of $G$.

**Main Theorem 3.1.** *If $\nu(G) \leq 3$ and $\omega(G) \leq 5$, then $G$ is 5-colorable.*

**Proof.** Let $G$ denote the class of all graphs with crossing number at most three that are not 5-colorable, and let $G$ be a member of $G$ with the minimum number of vertices. Suppose that $\omega(G) \leq 5$ and that $G$ is drawn optimally in the plane.

If $G$ contains a vertex $v$ of degree less than five, then $G$ is not a minimal member of $G$, since a 5-coloring of $G - v$ extends to a 5-coloring of $G$. Hence, the minimum degree of $G$ is five. By Proposition 1.4, the graph $G$ has at most $3n - 3$ edges, and thus has at least six vertices of degree five.
Let $v$ be a vertex of degree five. Lemma 2.3 implies that there is a $v$-immersion of $K_6$ into $G$, and Corollary 2.2 implies that the image of $K_6$ in $G$ contains three crossed edges. Then Lemma 2.6 implies that two crossed edges of $G$ are incident with $v$. Since $G$ is not $K_6$, it contains a vertex $w$ of degree five not adjacent to $v$. However, Lemma 2.3 implies that there is also a $w$-immersion of $K_6$ into $G$, and so $w$ is also incident with two crossed edges. Since $v$ and $w$ are not adjacent, these two crossed edges are different from the crossed edges incident with $v$, which implies that $G$ contains four crossings; a contradiction.

We also show that when Theorem 3.1 is applied to a 4-connected graph $G$ other than $K_6$, then the assumption $\omega(G) \leq 5$ may be discarded. More precisely, we have:

**Corollary 3.2.** If $G$ is 4-connected, $\nu(G) \leq 3$ and $G \neq K_6$, then $G$ is 5-colorable.

**Proof.** Let $G$ be a drawing with at most three crossings of a 4-connected graph not isomorphic to $K_6$. We show that $\omega(G) \leq 5$, from which the conclusion follows immediately by Theorem 3.1.

Suppose, to the contrary, that $G$ has a complete subgraph $K$ on six vertices. Let $v$ be a vertex of $G$ that is not in $K$, and let $K'$ be the plane drawing obtained from $K$ by replacing each crossing with a new vertex. By Corollary 1.5, all three crossings of $G$ are in $K$, and so $|V(K')| = 9$ and $|E(K')| = 21$. Thus $K'$ is a triangulation and so every region of $K$ contains at most three vertices in its boundary. But this is impossible, as $G$, being 4-connected, has four paths from $v$ to vertices of $K$, with each pair of paths having only $v$ in common.

Lastly, note that $C_3 \vee C_5$, the graph in which every vertex of $C_3$ is adjacent to every vertex of $C_5$, contains no $K_6$ subgraph and is not 5-colorable.

![Figure 2. $C_3 \vee C_5$ drawn with the minimum number of crossings](image)

**Proposition 3.3.** The crossing number of $C_3 \vee C_5$ is six.

**Proof.** Let $G$ be an optimal drawing of $K \vee L$, where $K$ and $L$ are cycles on, respectively, three and five vertices. Suppose that $G$ has fewer than six crossings. Note that $G \setminus (E(K) \cup E(L))$ is isomorphic to $K_{3,5}$, which has crossing number four [Kle70]. This implies that the edges of $K \cup L$ are involved in at most one crossing, and thus $L$ has at most three regions, one of which contains $K$. Thus at least one region of $L$ avoids $K$ and has two non-adjacent vertices of $L$ in its boundary. These two vertices of $L$ can be joined by a new edge that crosses no edges of $G$ thereby creating a graph with 8 vertices, 24 edges, and 5 crossings; a contradiction to Proposition 1.4. Hence, $G$ has six crossings. Figure 2 shows a drawing which achieves this bound, proving that $\nu(C_3 \vee C_5) = 6$.

We do not currently know whether the Main Theorem 3.1 extends to graphs with four or five crossings, and hence conclude with the following question:

**Question 3.4.** Does a graph $G$ have a 5-coloring if $\nu(G) \leq 5$ and $\omega(G) \leq 5$?
References
