MATH 1552-02 Answers to Sample Test 3
July 14, 2004

1. The center of the hyperbola is half-way between it foci and so has coordinates $(4, 2)$. The foci are 2 units away from the center and so $c = 2$. Since the slopes of the asymptotes and 1 and -1, it follows that $a = b$. Since $a^2 + b^2 = c^2 = 4$, it follows that $a = b = \sqrt{2}$. Finally, since the foci are located on a horizontal line, the equation of the hyperbola is
$$\frac{(x-4)^2}{2} - \frac{(y-2)^2}{2} = 1.$$

2. Since $\arctan x \to \pi/2$ as $x \to \infty$, it follows that the sequence $\{\arctan 2n\}$ converges to $\pi/2$.

3. Use partial fractions to determine that
$$s_n = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$$
and conclude that the series sums up to $5/6$.

4. This series has positive terms, so the integral estimate of the remainder must be used. We get
$$\int_n^\infty 2x^{-4}\,dx = \frac{2}{3n^3} < 0.01,$$
and the first integer n for which the above holds is $n = 5$.

5. Use the comparison test, we get
$$\sum_{n=1}^{\infty} \frac{n+2}{(n+1)^3} < \sum_{n=1}^{\infty} \frac{2(n+1)}{(n+1)^3} = 2\sum_{n=2}^{\infty} \frac{1}{n^2}.$$
The last series converges, as it is a p-series with $p = 2$.

6. Since this is an alternating series, it converges whenever $\frac{(\ln n)^p}{n}$ is eventually decreasing and has limit zero as $n \to \infty$. Consider the function $f_p(x) = \frac{(\ln x)^p}{x}$. To show that each f_p is eventually decreasing, compute its derivative and show that it is negative for sufficiently large x. To show that the function has limit zero as $x \to \infty$, use l'Hospital Rule k times where k is the first integer that is equal to or larger than p. Conclude that the series converges for all p.

7. Since the sequence $\{n/(n^2+1)\}$ is decreasing and approaches zero, the series converges. However, the series does not converge absolutely since $n/(n^2+1)$ can be bounded from below by $n/(n^2+n^2) = 1/(2n)$ which is forms the harmonic series when summed up.

8. Use the Ratio Test to conclude that the series converges.

9. Use the Ratio Test together with l'Hospital Rule to conclude that the radius of convergence is 4. Since the series is alternating when x is positive, it converges when $x = 4$, but when $x = -4$, the series diverges, as may be seen, for example, by comparing it to the Harmonic Series. Thus the interval of convergence is $(-4, 4]$.

10. Consider $b_1 = 1$ and $b_{n+1} = 3b_n/\sqrt{n}$. Then $a_n \le b_n$ for all n. To show that $\sum b_n$ converges, note that $b_{n+1} = 3^n/\sqrt{n!}$ and apply the Ratio Test. The convergence of $\sum a_n$ follows from the Comparison Test.