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Topology and its Applications 118 (2002) 45–63

TOPOLOGY  
AND ITS  
APPLICATIONS

www.elsevier.com/locate/topol

## The Orlik–Solomon complex and Milnor fibre homology

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Received 12 December 1999; received in revised form 31 August 2000

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### Abstract

Fix a real hyperplane arrangement, and let  $F$  be the Milnor fibre of its complexified defining polynomial. We consider a filtration of the homology of  $F$  that arises from the algebraic monodromy, using integer or  $\mathbb{Z}/p\mathbb{Z}$  coefficients. We compare it with the cohomology of the Orlik–Solomon algebra, over  $\mathbb{Z}$  or  $\mathbb{Z}/p\mathbb{Z}$ , respectively, with respect to a suitable “Koszul” boundary map, and find isomorphisms in certain cases. This continues work by various authors in comparing the cohomology of certain local systems on a hyperplane complement with that of the Orlik–Solomon algebra. © 2002 Elsevier Science B.V. All rights reserved.

*AMS classification:* 32S22; 32S55; 52C35

*Keywords:* Hyperplane arrangement; Milnor fibration; Orlik–Solomon algebra

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### 1. Summary

Let  $\mathcal{A} = \{H_1, \dots, H_m\}$  be a hyperplane arrangement in  $\mathbb{C}^n$  with  $\bigcap_i H_i = 0$ . Choose functionals  $\alpha_H : \mathbb{C}^n \rightarrow \mathbb{C}$  with  $H = \ker \alpha_H$  for each  $H \in \mathcal{A}$ . Given a function  $a : \mathcal{A} \rightarrow \mathbb{Z}_+$  satisfying  $\gcd\{a(H) : H \in \mathcal{A}\} = 1$ , let  $Q = \prod_H \alpha_H^{a(H)}$ . Consider  $Q$  as a map  $Q : \mathbb{C}^n \rightarrow \mathbb{C}$ : let  $M$  be the complement of  $Q^{-1}(0)$  and  $F = Q^{-1}(1)$  the global Milnor fibre. Of the two spaces only  $F$  depends on the choice of  $a$ . Let  $N$  denote the degree of  $Q$ .

The critical locus of the hypersurface defined by  $Q(z) = 0$  has (complex) dimension  $n - 2$ , and so the hypersurface singularity is far from being isolated. An open problem in the theory of hyperplane arrangements is whether or not the (co)homology of the Milnor fibre admits a description in terms of the arrangement’s matroid.

Although  $F$  is smooth only if  $Q$  is square-free, it is a natural object to study from the combinatorial point of view, since it simply reflects the structure of matroids with loops (multiplicities).

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<sup>1</sup> Partially supported by a NSERC Postdoctoral Fellowship.

Let  $U$  be the image of  $M$  under the standard projection  $\mathbb{C}^n - \{0\} \rightarrow \mathbb{P}^{n-1}$ . Cohen and Suciú have shown [3] that  $H_*(F, \mathbb{C})$  can be expressed as a direct sum of local system homology groups  $H_*(U, \mathbb{C}_\zeta)$ , where  $\zeta$  ranges over the  $N$ th roots of unity, and  $\mathbb{C}_\zeta$  is a one-dimensional, complex local system corresponding to a representation  $\pi_1(U) \rightarrow \mathbb{C}^*$  that maps the distinguished generator  $\gamma_H$  for hyperplane  $H$  to  $\zeta$ .

A representation  $\rho: \pi_1(M) \rightarrow \mathbb{C}^*$  factors through a representation of  $\pi_1(U)$  if and only if  $\prod_{H \in \mathcal{A}} \rho(\gamma_H) = 1$ . Given such a representation, let  $\mathbb{C}_\rho$  denote the corresponding one-dimensional local system on  $U$ , and let  $L = L(\mathcal{A})$  denote the intersection lattice of  $\mathcal{A}$ , using the notation of [14]. An element  $X$  of  $L$  is said to be *dense* (following [16]) if  $\beta(X) \neq 0$ , where  $\beta$  is Crapo's beta invariant. By combining the results of Schechtman et al. [16] with those of Yuzvinsky [23], the following is known.

**Theorem 1.1** [16,23]. *If  $\prod_{H \in X} \rho(\gamma_H) \neq 1$  for all dense  $X \in L_p$  with  $p \leq n - 1$ , then*

$$H^p(U, \mathbb{C}_\rho) = \begin{cases} \mathbb{C}^{\beta(\mathcal{A})}, & \text{if } p = n - 1, \\ 0, & \text{otherwise.} \end{cases}$$

However, the theorem's hypotheses and conclusion tend not to apply to the local systems given by the Milnor fibre covering map; conversely, many local systems of interest that do not meet the theorem's "nonresonance" conditions can be expressed in terms of a Milnor fibration. Here, we will consider one-dimensional, complex representations for which the image of each generator is a root of unity. In [2], such representations are called *rational*.

**Proposition 1.2.** *Let  $\mathcal{A}$  be a central arrangement, and  $\rho$  a rational representation of  $\pi_1(U)$ . Then the homology of the local system  $H_*(U, \mathbb{C}_\rho)$  is a monodromy eigenspace of an unreduced Milnor fibre for  $\mathcal{A}$ .*

**Proof.** Lift  $\rho$  to a representation  $\tilde{\rho}$  of  $\pi_1(M)$ . By hypothesis, there exist positive integers  $r$ ,  $k$ , and  $a(H)$ , for all  $H \in \mathcal{A}$ , for which

- $\tilde{\rho}(\gamma_H) = \exp(2\pi i a(H)/r)$ ;
- $\gcd\{a(H)\} = 1$ ; and
- $\sum_{H \in \mathcal{A}} a(H) = kr$ .

Let  $F$  be the unreduced Milnor fibre given by the exponents  $\{a(H)\}$ . Since  $F$  is homotopy-equivalent to an infinite cyclic cover  $F(Q)$  of  $M$ , there is an isomorphism

$$H_*(F, \mathbb{Z}) \cong H_*(M, \mathbb{Z}[t, t^{-1}]), \quad (1.1)$$

where  $\mathbb{Z}[t, t^{-1}]$  is a local system on  $M$ , in which  $\gamma_H$  acts by multiplication by  $t^{a(H)}$ . Under the isomorphism, the algebraic monodromy on the Milnor fibre homology acts by multiplication by  $t$ , making both sides  $\mathbb{Z}[t, t^{-1}]$ -modules [7]. The order of the algebraic monodromy is the degree of the defining polynomial,  $N = kr$ .

If one tensors both sides of (1.1) with  $\mathbb{C}$  using the map that specializes  $t$  to  $\exp(2\pi i/r)$ , one obtains an eigenspace for the monodromy operator on the left, and  $H_*(U, \mathbb{C}_\rho)$  on the right, using [3, Theorem 1.3].  $\square$

In the rest of this paper, then, we consider unreduced Milnor fibrations. For the more general point of view of Alexander invariants, we refer to [9,10].

Let  $\mathbf{d}\mathcal{A}$  denote the deconed arrangement of  $\mathcal{A}$  in  $\mathbb{C}^{n-1}$ . The (integral) Orlik–Solomon rings  $A^\bullet(\mathcal{A}) = H^\bullet(M, \mathbb{Z})$  and  $A^\bullet(\mathbf{d}\mathcal{A}) = H^\bullet(U, \mathbb{Z})$  are torsion-free and are generated as rings by the identity together with the one-forms  $e_H = 1/(2\pi i)d \log \alpha_H$  [14]. Over  $\mathbb{C}$ , the algebra  $A_\mathbb{C}^\bullet$  has been studied in the context of Theorem 1.1 as a cochain complex with the “Koszul” boundary map induced by left multiplication by an element of degree 1,  $\lambda = \sum_H b(H)e_H$ , for weight functions  $b: \mathcal{A} \rightarrow \mathbb{C}$ . In particular, Schechtman et al. show in [16] that, under nonresonance conditions,

$$H^\bullet(U, \mathbb{C}_\rho) \cong H^\bullet(A^\bullet(\mathbf{d}\mathcal{A}), \lambda),$$

where  $\rho(\gamma_H) = \exp(2\pi i b(H))$ .

Here, we relate local systems to the Orlik–Solomon algebra in a slightly different spirit. In place of complex weights  $b(H)$ , we consider a weight function  $a: \mathcal{A} \rightarrow \mathbb{Z}_+$  given by the multiplicities of the hyperplanes in an unreduced arrangement, and  $\omega = \sum_H a(H)e_H$ . In place of the Orlik–Solomon algebra of  $\mathbf{d}\mathcal{A}$ , we consider the central Orlik–Solomon ring  $A^\bullet = A^\bullet(\mathcal{A})$ , over  $\mathbb{Z}$ . From [23, Proposition 2.1], it follows that the cohomology of  $(A^\bullet, \omega)$  is annihilated by multiplication by  $N$ ; that is, the complex is exact over  $\mathbb{C}$ , while its cohomology over  $\mathbb{Z}$  has only torsion. The goal here is to interpret this torsion in terms of the structure of the  $\mathbb{Z}[t, t^{-1}]$ -module  $H_*(F, \mathbb{Z})$ .

In order to state the main theorem precisely, let  $R = \mathbb{Z}[t, t^{-1}]$  and set  $u = 1 - t$ . Define a decreasing filtration of an  $R$ -module  $M$  by  $F^p M = u^{-p} M$ , for  $p \leq 0$ . Let  $\text{Gr}_p M = F^p M / F^{p-1} M$ , and write  $N^\vee$  for the dual of a  $\mathbb{Z}$ -module  $N$ .

**Theorem 1.3.** *If  $\mathcal{A}$  is a complexified real arrangement and  $F$  an unreduced Milnor fibre of  $\mathcal{A}$ , there is an isomorphism of Abelian groups*

$$\text{Gr}_{-1} H_1(F, \mathbb{Z}) \cong H^2(A^\bullet(\mathcal{A}), \omega)^\vee.$$

If  $H_*(F, \mathbb{Z})$  is torsion-free, we find further that, for all  $k < 0$  and all  $p$ ,

$$\text{Gr}_k H_p(F, \mathbb{Z}) \cong H^{p+1}(A^\bullet, \omega)^\vee.$$

More generally, the right side appears as the  $E^2$  term of a spectral sequence converging to the left side.

Rational local systems on  $U$  and their relation to integer torsion in Orlik–Solomon algebras have also been studied in [11,12,2]. In [2], Cohen and Orlik prove the following, where  $U$  is the projective complement of any complex arrangement.

**Theorem 1.4** [2, Theorem 1.3]. *Let  $\rho$  be a rational representation of  $\pi_1(U)$ , and  $a: \mathcal{A} \rightarrow \mathbb{Z}$  as in Proposition 1.2. Let  $\lambda = \sum_H a(H)e_H$ . Then*

$$\dim_{\mathbb{C}} H^q(U, \mathbb{C}_\rho) \leq \text{rank}_{\mathbb{Z}/N} H^q(A^\bullet(\mathbf{d}\mathcal{A}), \lambda)$$

for all  $q$ .

We find an isomorphism related to this inequality in the case of complexified real arrangements, by using the spectral sequence of the filtration described above over a finite field (Section 5). The right-hand side is shown to count certain nontrivial Jordan blocks of the monodromy operator on  $H_q(F, \mathbb{F}_\ell)$ , where  $\mathbb{F}_\ell$  is the field of (prime) order  $\ell$  (Theorem 5.1).

The arguments are combinatorial in nature. For complexified real arrangements, Salvetti's CW-complex [15] is homotopically equivalent to the complement  $M$ . One can write the differential explicitly for Abelian local systems [21], giving a finite-dimensional complex of  $R$ -modules whose homology is  $H_*(F, \mathbb{Z})$ . We then adapt Varchenko and Gelfand's Heaviside functions [22] to the Salvetti complex, where we expand the differential in terms of powers of  $1 - t$ . It is then possible to write down the first few terms of the spectral sequence of the corresponding filtration, from which Theorem 1.3 follows.

As part of the calculation, we find another combinatorial description of the integral homology of a complexified hyperplane complement. Theorem 3.5 gives an isomorphism between the flag complex of the intersection lattice and certain cycles in Salvetti's CW-complex that are described by the Varchenko–Gelfand ring [22].

## 2. Preliminaries

### 2.1. Flag complex

The material here is based on [17]. By a flag in a graded poset  $P$  we mean tuples of elements  $(X_0, X_1, \dots, X_k)$  of  $P$  where each  $X_i$  has rank  $i$  and  $X_i < X_{i+1}$  for  $0 \leq i < k$ . The flag complex  $Fl_\bullet$  of the intersection lattice of an arrangement is defined in degree  $p$  to be the free Abelian group on flags of length  $p + 1$ , modulo the relations:

$$\sum_{Y: X_{i-1} < Y < X_{i+1}} (X_0, \dots, X_{i-1}, Y, X_{i+1}, \dots, X_p) = 0,$$

where  $i \geq 1$ .

**Proposition 2.1** [17].  *$Fl_\bullet$  is a free Abelian group.*

A set of  $p$  independent hyperplanes  $H_1, H_2, \dots, H_p$  determines a flag with  $X_i = H_1 \cap \dots \cap H_i$  for  $0 \leq i \leq p$ ; denote this flag by  $\lambda(H_1, \dots, H_p)$ . On the other hand, it is easy to check that if a flag  $\Phi = (X_0, \dots, X_p)$  is given and  $H \leq X_p$ , then one can write

$$\Phi = \lambda(H_1, H_2, \dots, H_i, \dots, H_p),$$

for some hyperplanes  $H_1, \dots, H_p$ , where  $H = H_i$ , and where the index  $i$  is unique. Accordingly, set  $\varepsilon(H, \Phi) = (-1)^{i-1}$  for hyperplanes  $H \leq X_p$ .

Recall that there is an isomorphism  $f: A^p \rightarrow Fl_p^\vee$  defined in [17, Lemma 2.3.4] by

$$f(H_1 \wedge \dots \wedge H_p) = \sum_{\sigma \in \mathfrak{S}_p} \operatorname{sgn}(\sigma) \lambda(H_{\sigma(1)}, \dots, H_{\sigma(p)})^\vee,$$

where  $\mathfrak{S}_p$  is the symmetric group and  $\text{sgn}(\sigma)$  denotes the sign of a permutation  $\sigma$ .

Given a function  $a$  assigning weights to each hyperplane, we define a map  $\delta: Fl_p \rightarrow Fl_{p-1}$  as follows. Given a flag  $\Phi$  and  $H \leq X_p$ , let  $D(\Phi, H)$  denote the set of all flags  $\Psi$  of length  $p - 1$  that can be written

$$\Psi = \lambda(H_1, H_2, \dots, H_{i-1}, H_{i+1}, \dots, H_p),$$

for some hyperplanes  $H_1, \dots, H_p$  for which  $\Phi = \lambda(H_1, \dots, H_i = H, \dots, H_p)$ . Now let

$$\delta\Phi = \sum_{H \leq X_p} \sum_{\Psi \in D(\Phi, H)} \varepsilon(H, \Phi) a(H) \Psi. \tag{2.1}$$

It is routine to verify that  $\delta^2 = 0$  and  $f$  induces a quasiisomorphism

$$(A^\bullet, \omega) \rightarrow (Fl_\bullet^\vee, \delta^\vee). \tag{2.2}$$

### 2.2. Salvetti’s complexes

We now assume that our arrangement  $\mathcal{A}$  is defined by real equations. Let  $\mathcal{L}$  denote its face poset, ordered by reverse inclusion. We write covering relations with the symbol “ $\prec$ ”. Using the definitions of [1], Salvetti’s poset  $\text{Sal}$  is the set of pairs  $(F, C)$ , where  $F \in \mathcal{L}$ ,  $C \in \mathcal{L}_0$  (a chamber), and  $C \leq F$ . The order in  $\text{Sal}$  is given by  $(F, C) \leq (G, C')$  iff  $F \leq G$  and  $C = FC'$ , where  $FC'$  is the vector product of faces. The poset inherits the ranking of  $\mathcal{L}$ : if  $F \in \mathcal{L}_p$ ,  $(F, C) \in \text{Sal}_p$ .

In [15], Salvetti constructs a CW-complex, which we call  $\mathbb{S}$  here, whose face poset is  $\text{Sal}$ . He shows that  $\mathbb{S}$  is homotopically equivalent to  $M$ . Suppose that  $\rho: \pi_1(M) \rightarrow R$  is a one-dimensional representation over some commutative domain  $R$ , and  $S$  a domain containing  $R$  in which each element  $\rho(\gamma_H)$  has a fourth root of unity. In [21], Varchenko trivializes the local system on the cells of  $\mathbb{S}$  to find a differential  $\partial$  so that

$$H_p(\mathbb{S}_\bullet \otimes R, \partial) \cong H_p(M, R).$$

In order to write the differential explicitly, let  $E(F, C)$  denote a basis element of  $\mathbb{S}_p$  corresponding to the cell  $(F, C)$ . We fix a coorientation of each subspace in  $L$ , giving coorientations of each face in  $\mathcal{L}$ . For faces  $G \prec F$ , set  $\varepsilon(F, G) = \pm 1$  according to whether or not the coorientations of faces  $F$  and  $G$  agree; see [21]. For a face  $F$  and hyperplane  $H$ ,  $F_H$  denotes the sign of the functional  $\alpha_H$  on the face  $F$ ;  $F_H \in \{0, 1, -1\}$ . Then  $\partial$  is induced by

$$\partial E(F, C) = \sum_{G \prec F} \varepsilon(F, G) b(F, G, C) E(G, GC), \tag{2.3}$$

where one sets

$$b(F, G, C) = \prod_{H: F_H=0} \rho(\gamma_H)^{G_H C_H / 4}. \tag{2.4}$$

The other CW-complex required here,  $\mathcal{Q} = \mathcal{Q}(\mathcal{A})$ , has cells indexed by the elements of the face lattice  $\mathcal{L}$ , and it is contractible [15]. Salvetti’s construction involves a choice of points in each face of the hyperplane arrangement; however, if one prefers the point of view

of oriented matroids, one can define the cells of  $\mathcal{Q}$  abstractly as the geometric realizations of order ideals in the lattice  $\mathcal{L}$ . In either case, the cells of  $\mathcal{Q}$  are labelled  $D(F)$  for  $F \in \mathcal{L}$ , and the boundary map  $\partial_{\mathcal{Q}} : \mathcal{Q}_p \rightarrow \mathcal{Q}_{p-1}$  is given by

$$\partial_{\mathcal{Q}} D(F) = \sum_{G \prec F} \varepsilon(F, G) D(G). \quad (2.5)$$

### 2.3. Heaviside functions

We recall that the Varchenko–Gelfand ring  $VG = VG(\mathcal{A})$  is the ring of integer-valued functions on the chambers of a real arrangement [22] or, more generally, of an oriented matroid [8].

Varchenko and Gelfand identify certain distinguished elements equivalent to the following in its dual, the free Abelian group on the set of chambers  $\mathcal{L}_0$  [6].

**Definition 2.2.** For  $p \geq 0$ , let  $\varphi = (F_0, F_1, \dots, F_p)$  be a flag of faces in  $\mathcal{L}$ ; that is,  $F_i < F_{i+1}$  and  $\text{codim } F_i = i$  for all  $i$ . The *brick* associated with  $\varphi$  is an element  $b(\varphi)$  of  $VG(\mathcal{A})$ . It is defined using the face product by

$$b(\varphi) = \sum_{S \subset [p]} (-1)^{|S|} \varphi(S),$$

where the chamber  $\varphi(S)$  is defined to be

$$\varphi(S) = F_p (\pm F_{p-1}) \cdots (\pm F_1) (\pm F_0), \quad (2.6)$$

where  $F_j$  is chosen to have positive sign iff  $j + 1 \notin S$ .

For  $p \geq 0$ , let  $W^p$  denote the span of the bricks given by flags of length  $p$ . Then  $W^0 = VG^\vee$ , the dual of  $VG$ ,  $W^{n+1} = 0$ , and  $W^p \supseteq W^{p+1}$ . The significance of this filtration comes from setting

$$\text{Gr}_p VG^\vee = \frac{W^p}{W^{p+1}}.$$

For a face  $F$ , let  $|F| \in L$  denote the smallest subspace containing  $F$ . Extending the notation, if  $\varphi$  is a flag in  $\mathcal{L}$ ,  $|\varphi|$  denotes the corresponding flag of subspaces in  $L$ . Define the sign  $\varepsilon(\varphi)$  to be the product of  $\varepsilon(F_i, F_{i-1})$ , for  $0 < i \leq p$ .

**Theorem 2.3** [20]. *There is an isomorphism of Abelian groups  $Fl_p \cong \text{Gr}_p VG^\vee$ , induced by the map  $\Phi \mapsto \varepsilon(\varphi)b(\varphi)$ , where  $\varphi$  is any flag of faces with  $\Phi = |\varphi|$ .*

One can extend the brick filtration to a filtration of  $\mathbb{S}_\bullet$  as follows. For a fixed  $F \in \mathcal{L}$ , we identify chambers of  $\mathcal{A}_{|F|}$  with chambers  $C$  of  $\mathcal{A}$  satisfying  $C \leq F$ . Then the natural map identifies a chain in  $W^p VG^\vee(\mathcal{A}_{|F|})$  with an element of  $\mathbb{S}_q$  supported on cells  $\{E(F, C) : C \in \mathcal{L}_0, C \leq F\}$ . With this identification, we set

$$W^p \mathbb{S}_q = \bigoplus_{F \in \mathcal{L}_q} W^p VG^\vee(\mathcal{A}_{|F|}). \quad (2.7)$$

**Definition 2.4.** Let  $\varphi$  be a flag of faces. Then, for faces  $F \geq F_p$  of rank  $q$  in  $\mathcal{L}$ , the chain  $E(F, \varphi) \in \mathbb{S}_q$  is defined by

$$E(F, \varphi) = \sum_{S \subset [r]} (-1)^{|S|} E(F, \varphi(S)),$$

where  $\varphi(S)$  is defined in (2.6). We shall call such elements  $q$ -bricks. By construction, their span is  $W^p \mathbb{S}_q$ .

### 3. Filtration of the Salvetti complex

In view of the isomorphism (1.1), one can compute  $H_*(F, \mathbb{Z})$  for a complexified real arrangement using the Salvetti complex  $\mathbb{S}$ . Specifically, (2.4) becomes

$$b(F, G, C) = \prod_{H: F_H=0} t^{a(H)G_H C_H/4}.$$

We show (Section 3.1) that the two filtrations of  $(\mathbb{S}, \partial)$  considered so far are closely related. That is,  $\partial$  preserves the  $q$ -brick filtration, modulo filtration by powers of  $1 - t$ . Consequently, the first two terms of the spectral sequence of the latter filtration have combinatorial descriptions (Section 3.2). In particular, the  $E^2$  term is essentially the homology of the flag complex.

The  $E^1$  term is just the homology of the Salvetti complex with constant (integer) coefficients. As a byproduct of the calculations here, we show that the  $q$ -bricks provide a basis in a natural way (Theorem 3.5). That is, the homology cycles of a complexified real arrangement can be described in terms of the Varchenko–Gelfand ring.

#### 3.1. Boundary map in terms of Heaviside functions

We begin by reinterpreting the definition of the  $q$ -bricks. Let  $E(F, \varphi)$  be a  $q$ -brick, where  $\varphi = (F_0, F_1, \dots, F_p)$ , and  $F$  is a face of codimension  $q$  satisfying  $F \geq F_p$ . For  $1 \leq i \leq p$ , set

$$\Pi(i) = \{H \in \mathcal{A}: (F_i)_H = 0, (F_{i-1})_H \neq 0\}. \tag{3.1}$$

**Lemma 3.1.** *The set  $\{\Pi(i): i \in [p]\}$  partitions  $\{H: (F_p)_H = 0\}$ , and*

$$\varphi(S)_H = \begin{cases} (F_{i-1})_H & \text{if } H \in \Pi(i) \text{ and } i \in S, \\ -(F_{i-1})_H & \text{if } H \in \Pi(i) \text{ and } i \notin S, \\ (F_p)_H & \text{if } H \notin \Pi(i) \text{ for any } i. \end{cases}$$

**Proof.** For some unique  $i > 0$ ,  $(F_i)_H = 0$  and  $(F_{i-1})_H \neq 0$  if  $(F_p)_H = 0$ . From (2.6),

$$\varphi(S) = F_p (\pm F_{p-1}) (\pm F_{p-2}) \cdots (\pm F_0),$$

with the sign of  $F_{i-1}$  chosen positively if  $i \notin S$ . The claim follows by the definition of the vector product.  $\square$

A routine argument shows that one may extend the vector product to flags of faces as follows.

**Lemma 3.2.** *Let  $\varphi = (F_0, F_1, \dots, F_p)$  be a flag, and  $G \in \mathcal{L}$  a face. Let  $0 < k \leq p + 1$  be the largest integer for which  $|G| \geq |F_{k-1}|$ . If  $k \leq p$ , then  $GF_k = GF_{k-1}$ , and  $(GF_0, GF_1, \dots, GF_{k-1} = GF_k, \dots, GF_p)$  is a flag of faces. Furthermore,  $G_H = \pm(F_{k-1})_H$  for all  $H \in \Pi(k)$ , where the choice of sign does not depend on  $H$ . In particular,  $G_H \neq 0$  for  $H \in \Pi(k)$ .*

Accordingly, we let  $G\varphi$  denote the flag of faces obtained by applying  $G$  to each element of  $\varphi$ . The flag has the same length as  $\varphi$  if  $|G| \geq |F_p|$ ; otherwise, it is shorter by one. If  $F$ ,  $G$ , and  $\varphi$  are defined as in the lemma, set  $G_\varphi = 1$  if  $G_H = (F_{k-1})_H$  for  $h \in \Pi(k)$ , and  $-1$ , otherwise. Note that  $(F_{k-1})_H = (F_j)_H$ , for  $j \leq k - 1$ .

Now, for notational convenience, we define a product of a face with a  $q$ -brick in the following circumstances. Let  $E(F, \varphi)$  be a  $q$ -brick. Then  $G \cdot E(F, \varphi)$  is a linear combination of  $q - 1$ -bricks: if  $q = p + 1$ , set

$$G \cdot E(F, \varphi) = b(F, G, F_p)E(G, G\varphi).$$

Otherwise,

$$G \cdot E(F, \varphi) = \left(1 - \prod_{H \in \Pi(k)} t^{-G_\varphi a(H)/2}\right) b(F, G, F_{k-1})E(G, G\varphi).$$

The next statement should be compared with equation (2.3). In particular, for a flag  $\varphi = (C)$  of length 1, we have  $E(F, \varphi) = E(F, C)$ , and the expression below simplifies to (2.3).

**Proposition 3.3.** *Given a flag of faces  $\varphi = (F_0, F_1, \dots, F_p)$  and  $F \geq F_p$ , the boundary map  $\partial : \mathbb{S}_q \rightarrow \mathbb{S}_{q-1}$  can be written*

$$\partial E(F, \varphi) = \sum_{G \prec F} \varepsilon(F, G)G \cdot E(F, \varphi) + (1 - t)^2 v,$$

where  $v$  is some element of  $C_{q-1}$ .

**Proof.** Recall that the  $q$ -brick  $E(F, \varphi)$  is a linear combination of basis elements  $E(F, C)$  for some chambers  $C$ , with  $F$  fixed. The image under the boundary map (2.3) is supported on basis elements  $E(G, C)$  for (different) chambers  $C$  and  $G \prec F$ . To prove the formulas above, then, it is enough to consider each  $G \prec F$  separately. More precisely, let  $\pi_G$  denote projection to the span of basis elements  $\{E(G, C) : C \leq G, C \in \mathcal{L}_0\}$ . We show that  $\pi_G \partial(E(F, \varphi)) = \varepsilon(F, G)G \cdot E(F, \varphi)$  using the definitions given above.

First, suppose that  $|G| \geq |C|$ . From (2.3),

$$\pi_G \partial(E(F, \varphi)) = \varepsilon(F, G) \sum_{S \subseteq [p]} (-1)^{|S|} b(F, G, \varphi(S))E(G, G\varphi(S)).$$

By assumption,  $(F_p)_H = 0$  implies  $G_H = 0$ , so  $G_H = 0$  for  $H \in \Pi(i)$  for all  $1 \leq i \leq p$ . By Lemma 3.1,  $\varphi(S)_H = (F_p)_H$  for  $H \notin \Pi(i)$ , so  $b(F, G, \varphi(S)) = b(F, G, F_p)$  in the expression above. Furthermore, from the definition of the vector product,

$$G\varphi(S) = GF_p(\pm F_{p-1}) \cdots (\pm F_0) = F'_p(\pm F'_{p-1}) \cdots (\pm F'_0),$$

where  $F'_i = GF_i$ . By the definition of  $E(G, G\varphi)$ , then, we have

$$\pi_G \partial(E(F, \varphi)) = \varepsilon(F, G)E(G, G\varphi),$$

as claimed.

Otherwise, choose  $k$  so that  $|G| \geq |F_{k-1}|$  but  $|G| \not\geq |F_k|$ . By Lemma 3.2,  $G_H \neq 0$  for  $H \in \Pi(k)$ . It follows that  $G\varphi(S) = G\varphi(S \Delta \{k\})$ , where  $\Delta$  denotes symmetric difference. If  $k \notin S$ ,

$$\frac{b(F, G, \varphi(S))}{b(F, G, \varphi(S \cup \{k\}))} = \prod_{H \in \Pi(k)} t^{a(H)G_H(F_{k-1})_H/2} = \prod_{H \in \Pi(k)} t^{G_\varphi a(H)/2},$$

where Lemma 3.2 shows that the sign  $G_\varphi$  is independent of  $H$ .

By separating terms according to whether or not  $k \in S$ , we find

$$\begin{aligned} \pi_G \partial(E(F, \varphi)) &= \varepsilon(F, G) \sum_{S \subseteq [p]} (-1)^{|S|} b(F, G, \varphi(S)) E(G, G\varphi(S)) \\ &= \left(1 - \prod_{H \in \Pi(k)} t^{-G_\varphi a(H)/2}\right) \sum_{S \subseteq [p] - \{k\}} (-1)^{|S|} b(F, G, \varphi(S)) E(G, G\varphi(S)). \end{aligned}$$

For  $H \notin \Pi(k)$  consider whether or not  $G_H = 0$ . If  $G_H = 0$ , then  $b(F, G, \varphi(S))$  has no factor depending on  $a(H)$ , for any choice of  $S$ .

Otherwise,  $G_H \neq 0$ , and the factor of  $b(F, G, \varphi(S))$  that is indexed by  $H$  is  $t^{a(H)G_H \varphi(S)_H/4}$ . If  $(F_p)_H \neq 0$ , then  $\varphi(S)_H = (F_p)_H$ , so we can reexpress this as  $t^{a(H)G_H (F_p)_H/4}$ . Trivially, for any  $H \in \mathcal{A}$ ,

$$t^{a(H)G_H \varphi(S)_H/4} \equiv t^{a(H)G_H (F_{k-1})_H/4} \text{ modulo } 1 - t,$$

so we have

$$\begin{aligned} \pi_G \partial(E(F, \varphi)) &= \varepsilon(F, G) b(F, G, F_{k-1}) \left(1 - \prod_{H \in \Pi(k)} t^{-G_\varphi a(H)/2}\right) \\ &\quad \times \left(\sum_{S \subseteq [p] - \{k\}} (-1)^{|S|} E(G, G\varphi(S)) + (1 - t)v\right), \end{aligned}$$

where  $v$  is some  $R$ -linear combination of basis elements of the form  $E(G, C)$ ,  $C \in \mathcal{L}_0$ .

To complete the proof, we note that

$$\sum_{S \subseteq [p] - \{k\}} (-1)^{|S|} E(G, G\varphi(S)) = E(G, \varphi'),$$

where  $\varphi' = (GF_0, \dots, GF_{k-1} = GF_k, \dots, GF_p)$ , using (2.6).  $\square$

3.2. *Filtration by powers of 1 – t*

Using the explicit formula provided by Proposition 3.3, one can expand the boundary map  $\partial$  of  $\mathbb{S}_\bullet$  in terms of powers of  $u = 1 - t$ ,  $\partial = \partial^0 + u\partial^1 + u^2\partial^2 + \dots$ . Once again,  $\mathbb{S}_\bullet$  is a complex of  $R = \mathbb{Z}[t, t^{-1}]$ -modules, filtered by setting  $F^p\mathbb{S}_q = u^{-p}\mathbb{S}_q$  for  $p \leq 0$  and  $q \geq 0$ . The filtration is decreasing, bounded above by  $F^0\mathbb{S}_q = \mathbb{S}_q$ , and exhaustive; accordingly the spectral sequence that starts with

$$E_{pq}^0 = \text{Gr}_p\mathbb{S}_{p+q} \tag{3.2}$$

weakly converges to  $H_\bullet(F, \mathbb{Z})$ . The use of integer coefficients is important; rationally, this filtration is trivial. Both of the terms  $E^1$  and  $E^2$  are combinatorially significant, and we describe them here in Theorems 3.5 and 3.6, respectively.

To begin, each column of the  $E^0$  term is isomorphic to the Salvetti complex with trivial coefficients, and the boundary map can be obtained by specializing Proposition 3.3 with  $t = 1$ . Then:

**Lemma 3.4.** *In  $(\mathbb{S}, \partial^0)$ , the boundary map is given by*

$$\partial^0 E(F, \varphi) = \sum_{\substack{G: G < F \\ |G| \geq |F_p|}} \varepsilon(F, G) E(G, G\varphi),$$

where  $\varphi = (F_0, F_1, \dots, F_p)$  is a flag of faces and  $F \geq F_p$ . In particular, the boundary map  $\partial^0$  preserves the filtration  $W^\bullet(\mathbb{S}_\bullet)$ .

**Theorem 3.5.** *The homology classes of p-bricks  $E(F_p, \varphi)$  generate  $H_p(\mathbb{S}, \partial^0)$ . There is an isomorphism of Abelian groups  $g: Fl_p \rightarrow H_p(\mathbb{S}, \partial^0)$  defined on flags  $\Phi$  by*

$$\Phi \mapsto \varepsilon(\varphi) E(F_p, \varphi).$$

**Proof.** We shall consider the spectral sequence of the filtration  $\mathbb{S}_p = W^0\mathbb{S}_p \supseteq W^1\mathbb{S}_p \supseteq \dots \supseteq W^p\mathbb{S}_p \supset 0$ , defined in (2.7). The spectral sequence begins with  $e_{pq}^0 = W^p\mathbb{S}_{p+q} / W^{p+1}\mathbb{S}_{p+q}$ . Note that  $e_{pq}^0 = 0$  unless  $0 \leq p \leq n$  and  $0 \leq q \leq n - p$ .

For  $p$  and  $q$  in this range, we define a map

$$f: e_{pq}^0 \rightarrow \bigoplus_{X \in L_p(\mathcal{A})} Fl_p(\mathcal{A}_X) \otimes \mathcal{Q}_q(\mathcal{A}^X) \tag{3.3}$$

as follows. First, we abuse notation and identify the cells of  $\mathcal{Q}(\mathcal{A}^X)$  with faces  $F \in \mathcal{L}$  having  $|F| \geq X$ . Then, for  $F \in \mathcal{L}$  of rank  $p + q$  and  $\varphi = (F_0, \dots, F_p)$ , set  $f(E(F, \varphi)) = \varepsilon\varphi|\varphi| \otimes D(F)$ , in the component indexed by  $X = X_p = |F_p|$ , where the sign is chosen according to whether or not the orientations of  $\varphi$  and  $|\varphi|$  agree.

For a fixed face  $F$ ,  $f$  restricts to a map

$$\begin{aligned} \text{Gr}_p V G^\vee(\mathcal{A}_{|F|}) &\rightarrow \bigoplus_{X_p \leq |F|} Fl_p(\mathcal{A}_{X_p}) \\ &\cong Fl_p(\mathcal{A}_{|F|}). \end{aligned}$$

From Theorem 2.3,  $f$  is an isomorphism, and it becomes an isomorphism of chain complexes by defining the boundary map on the right side as  $\text{Id} \otimes \partial_Q$ . (Compare the formula (2.5) for  $\partial_Q$  with that of Lemma 3.4.)

We continue to the next term of the spectral sequence. Since the group  $Fl_p$  is free abelian (Proposition 2.1), homology commutes with the tensor products as well as direct sums. We obtain

$$e_{pq}^1 = \bigoplus_{X \in L_p(\mathcal{A})} Fl_p(\mathcal{A}_X) \otimes H_q(\mathcal{Q}_\bullet(\mathcal{A}^X)).$$

Since the chain complexes  $\mathcal{Q}_\bullet(\mathcal{A}^X)$  are acyclic, our spectral sequence has only one nonzero row,

$$\begin{aligned} e_{p0}^1 &\cong \bigoplus_{X \in L_p(\mathcal{A})} Fl_p(\mathcal{A}_X) \\ &= Fl_p(\mathcal{A}). \end{aligned}$$

The proof is completed by observing that the boundary map here is zero, and the isomorphism  $g : Fl_p \cong H_p(\mathbb{S}, \partial^0)$  is induced by the inverse of the map  $f$  defined in (3.3).  $\square$

Restating the theorem in terms of the spectral sequence (3.2), we have  $E_{pq}^1 \cong Fl_{p+q}$  for all  $p \leq 0$  and  $q \geq 0$ . We can now pass to the  $E^2$  term.

**Theorem 3.6.** *The isomorphism  $g$  of Theorem 3.5 makes the following diagram commute, where  $\delta$  is given by (2.1):*

$$\begin{array}{ccc} Fl_q & \xrightarrow{\delta} & Fl_{q-1} \\ \downarrow g & & \downarrow g \\ E_{p,q-p}^1 & \xrightarrow{d^1} & E_{p-1,q-p}^1 \end{array}$$

**Proof.** Let  $E(F_q, \varphi)$  be the image of a flag  $\Phi \in Fl_q$ . Then  $E(F_q, \varphi) \in W^q \mathbb{S}_q$ , and we use the boundary map calculation of Proposition 3.3 to find those terms in  $\partial E(F_q, \varphi)$  that are multiples of  $1 - t$ , but of no higher powers. The expression simplifies, since there are no faces  $G < F_q$  with  $|G| \geq |F_q|$ , and since the remainder term is divisible by  $(1 - t)^2$ . We find

$$d^1 E(F_q, \varphi) = \frac{1}{1 - t} \sum_{G < F_q} \varepsilon(F_q, G) G \cdot E(F_q, \varphi),$$

where

$$G \cdot E(F_q, \varphi) = r(G, \varphi, t) b(F_q, G, F_{k-1}) E(G, G\varphi),$$

setting

$$r(G, \varphi, t) = 1 - \prod_{H \in \Pi(k)} t^{-G_\varphi a(H)/2}.$$

For any flat  $X \prec |F_q|$ , there are exactly two faces  $G \prec F_q$  contained in the flat  $X$ ; choose one by requiring that  $\varepsilon(F_q, G) = 1$ . Then

$$d^1 E(F_q, \varphi) = \frac{1}{1-t} \sum_{X \prec |F_q|} (G \cdot E(F_q, \varphi) - F_q(-G) \cdot E(F_q, \varphi)).$$

Now observe that  $b(F_q, G, F_{k-1}) = b(F_q, F_q(-G), F_{k-1})$  and that, by Theorem 3.5,  $E(G, G\varphi)$  and  $E(F_q(-G), F_q(-G)\varphi)$  are equal in  $E^1$ . Then, in  $E^1$ ,

$$d^1 E(F_q, \varphi) = \sum_{X \prec |F_q|} \frac{r(G, \varphi, t) - r(G, \varphi, t^{-1})}{1-t} b(F_q, G, F_{k-1}) E(GF_q, G\varphi).$$

Simplifying with  $t = 1$ , we see that

$$d^1 E(F_q, \varphi) = \sum_{X \prec |F_q|} \sum_{H \in \Pi(k)} G_\varphi a(H) E(GF_q, G\varphi). \tag{3.4}$$

On the other hand, using (2.1),

$$\begin{aligned} \delta\Phi &= \sum_{H \leq X_q} \sum_{\Psi \in D(\Phi, H)} \varepsilon(H, \Phi) a(H) \Psi \\ &= \sum_{X \prec X_q} (-1)^{k-1} \left( \sum_{H_k \in \Pi(k)} a(H_k) \right) \lambda(H_1, \dots, \widehat{H}_k, \dots, H_q), \end{aligned}$$

since the sets  $\Pi(k)$  partition the set of hyperplanes containing  $X_q$ . Now apply the map  $g$  to obtain an expression for  $g \circ \delta(\Phi)$ . From (3.4), this agrees with  $d^1 \circ g(\Phi)$  in  $E^1$ .  $\square$

Let  $Z_p$  denote the kernel of the map  $\delta : Fl_p \rightarrow Fl_{p-1}$ . Under the isomorphism  $g$  of Theorem 3.5, we find:

**Corollary 3.7.**

$$E_{pq}^2 \cong \begin{cases} H_{p+q}(Fl_\bullet, \delta), & \text{if } p < 0, \\ Z_q, & \text{if } p = 0, \\ 0, & \text{otherwise.} \end{cases}$$

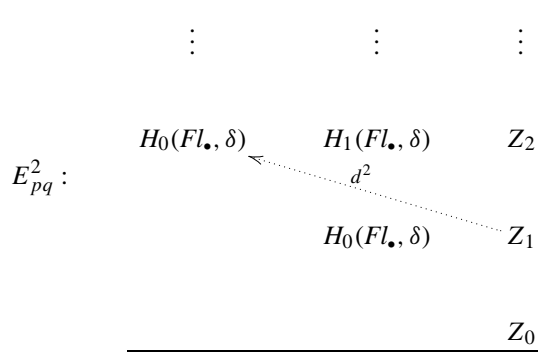
The zero column here is extraneous in the following sense. We recall that the polynomial  $\pi(\mathcal{A}, x) = \sum_{q=0}^n (\text{rank } Fl_q) x^q$  is called the Poincaré polynomial of  $\mathcal{A}$ . Then  $\pi(\mathcal{A}, -1) = 0$ , and the coefficient  $\tilde{\mu}_q$  of  $x^q$  in  $\pi(\mathcal{A}, x)/(x+1)$  is the rank of  $H^q(U, \mathbb{Z})$ , and the  $q$ th Whitney number of the lattice  $L(\mathbf{d}\mathcal{A})$  [14].

**Proposition 3.8.**  $Z_q$  is a free  $\mathbb{Z}$ -module of rank  $\tilde{\mu}_q$ , and  $H_q(F, \mathbb{Z})/(1-t) = E_{0q}^\infty = Z_q$ .

**Proof.** Since  $Fl_q$  is a free  $\mathbb{Z}$ -module, so is the submodule  $Z_q$ . The complex  $(Fl_\bullet \otimes \mathbb{Q}, \delta)$  is dual to  $(A^\bullet \otimes \mathbb{Q}, \omega)$  from (2.2), and the latter is exact by [23, Proposition 2.1]. It follows that  $\text{rank } Z_q + \text{rank } Z_{q-1} = \text{rank } Fl_q$  for each  $q$ . The numbers  $\tilde{\mu}_q$  satisfy the same identity, so  $\tilde{\mu}_q = \text{rank } Z_q$ . To complete the proof, we note that the rank of  $E_{pq}^2$  is zero for  $p < 0$ ; it follows that  $E_{0q}^\infty = E_{0q}^2 = Z_q$  as well.  $\square$

### 4. Convergence

The proof of Theorem 1.3 follows by examining the  $E^2$  term of the spectral sequence above. Pictorially,



From (2.2), we have  $H_p(Fl_\bullet, \delta) \cong H^{p+1}(A^\bullet, \omega)^\vee$ .

**Lemma 4.1.** *If  $\gcd\{a(H) : H \in \mathcal{A}\} = 1$ , then  $H^1(A^\bullet, \omega) = 0$ .*

**Proof.** Define  $\partial^q : A^q \rightarrow A^{q+1}$  by  $\partial(x) = \omega \wedge x$ .  $H^1(A^\bullet, \omega)$  has rank zero, so  $\ker \partial^1 = A^1 \cap \mathbb{Q}(\text{im } \partial^0)$ . Since  $\text{im } \partial^0$  is generated by  $\sum_H a(H)e_H$  and the coefficients  $a(H)$  have no common prime factor,  $\text{im } \partial^0 = \ker \partial^1$ .  $\square$

From the position of the zeros in the spectral sequence, then,  $E_{-1,2}^\infty = E_{-1,2}^2$ , and the first part of the theorem is proven:

$$\text{Gr}_{-1}H_1(F, \mathbb{Z}) \cong H^2(A^\bullet(\mathcal{A}), \partial)^\vee.$$

The second part of the theorem asserts that if  $H_\bullet(F, \mathbb{Z})$  is torsion-free, then the spectral sequence collapses at the  $E^2$  term. We note that Van Straten makes a parallel observation in a different context in [19]. There, a spectral sequence of sheaves of differential forms arises from the Gauss–Manin connection, with the same shape as the spectral sequence (3.2). He finds that its degeneration at the  $E_2$  term is equivalent to a lack of torsion over the structure sheaf.

**Lemma 4.2.** *If an  $R$ -module  $M$  has no integer torsion and  $(1 - t^N)M = 0$  for some  $N > 0$ , then  $\text{Gr}_k M = \text{Gr}_1 M$  for all  $k \leq -1$ , where  $\text{Gr}_k = (1 - t)^{-k}M / (1 - t)^{-k+1}M$ .*

**Proof.** For convenience, let  $u = 1 - t$ , and consider the short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & u^k M & \longrightarrow & u^{k-1} M & \longrightarrow & \text{Gr}_{k-1} M \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & u^{k+1} M & \longrightarrow & u^k M & \longrightarrow & \text{Gr}_k M \longrightarrow 0
 \end{array}$$

where the vertical maps are given by multiplication by  $u$ . The first two are clearly surjections; it follows that the third one is as well. On the other hand, let  $K_k = \ker(u^k M \xrightarrow{u} u^{k+1} M)$ . By the Snake Lemma, we have another short exact sequence

$$0 \rightarrow K_k \rightarrow K_{k-1} \rightarrow G_k \rightarrow 0,$$

where  $G_k = \ker(\text{Gr}_{k-1} M \rightarrow \text{Gr}_k M)$ . To show that  $\text{Gr}_k M$  is isomorphic to  $\text{Gr}_{k-1} M$  for  $k > 1$ , it is enough to show that  $K_{k-1} = 0$  for  $k > 1$ , after which our lemma follows by induction.

This is equivalent to showing that if  $x \in M$  and  $u^2 x = 0$ , then  $ux = 0$ . Suppose not. Let  $v = (1 - t^N)/u$ . Since  $uvx = 0$ , then  $u(au + bv)x = 0$  for all  $a, b \in R$ . Since the integer  $N$  is in the ideal generated by  $u$  and  $v$ ,  $Nux = 0$ . Since  $M$  is torsion-free,  $ux = 0$ .  $\square$

For the purposes of the next lemma, set  $E_p^k(r) = E_{p,r-p}^k$ , to show the grading by total degree in the spectral sequence (3.2).

**Lemma 4.3.** *For  $k \geq 1$ , all  $p, r$  and  $p \leq s \leq 0$ , there is a natural surjection  $E_s^k(r) \rightarrow E_p^k(r)$ . The surjection is an isomorphism if  $s, p \leq 1 - k$ . Furthermore, suppose that the differential  $d^k$  is nonzero on some  $E_p^k(r)$ . Then  $E_{1-k}^{k+1}(r) \not\cong E_{-k}^{k+1}(r)$ , for some  $r$ .*

**Proof.** Use induction on  $k$  for which  $d^k$  is nonzero, with  $k = 1$  given by Corollary 3.7. For  $k > 1$ , the  $E^k$  term consists of chain complexes

$$0 \rightarrow E_p^k(r) \rightarrow E_{p-k}^k(r-1) \rightarrow E_{p-2k}^k(r-2) \rightarrow \dots \rightarrow E_{p-rk}^k(0) \rightarrow 0, \tag{4.1}$$

for each  $p, -k < p \leq 0$ , and  $r \geq 0$ .

By hypothesis, the image of some  $d_p^k(r)$  is nonzero. Since the target lies left of the line  $p = -k + 1$ , we have a commutative square

$$\begin{array}{ccc} E_0^k(r) & \xrightarrow{d_0^k(r)} & E_{-k}^k(r) \\ \downarrow & & \downarrow \cong \\ E_p^k(r) & \xrightarrow{d_p^k(r)} & E_{p-k}^k(r) \end{array}$$

where the image of  $d_0^k(r)$  is also nonzero. Now compare the chain complexes (4.1) for  $p = 0$  and  $p = 1 - k$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & E_{1-k}^k(r-1) & \longrightarrow & E_{1-2k}^k(r-2) & \longrightarrow & \dots \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_0^k(r) & \xrightarrow{d_0^k(r)} & E_{-k}^k(r-1) & \longrightarrow & E_{-2k}^k(r-2) \longrightarrow \dots \end{array}$$

The vertical maps are isomorphisms, by induction. In homology we get a short exact sequence

$$0 \rightarrow \text{im } d_0^k(r) \rightarrow E_{1-k}^{k+1}(r-1) \rightarrow E_{-k}^{k+1}(r-1) \rightarrow 0.$$

Since the left-hand term is nonzero and each term is a finite Abelian group, we see that the map  $E_{1-k}^{k+1}(r-1) \rightarrow E_{-k}^{k+1}(r-1)$  is a surjection, but not an isomorphism.

We establish the surjections  $E_s^{k+1}(r) \rightarrow E_p^{k+1}(r)$  for all  $p \leq s \leq 0$  using the same argument.  $\square$

The second part of Theorem 1.3 is now proven by combining the lemmas: since the monodromy operator has order  $N$ ,  $1 - t^N$  annihilates  $H_*(F, \mathbb{Z})$ . If the latter is torsion-free, then, the differentials  $d^k$  must be zero for  $k \geq 2$ ; that is,

$$\text{Gr}_p H_{p+q}(F, \mathbb{Z}) = E_{pq}^\infty = E_{pq}^2 = H^{q+1}(A^*, \partial)^\vee$$

for all  $p < 0$ .

### 5. Homology in positive characteristic

The same spectral sequence can be considered over a finite field  $\mathbb{F}_\ell$ , where  $\ell$  is prime, converging to  $H_*(F, \mathbb{F}_\ell)$ . The arguments producing the  $E^2$  term were independent of characteristic, so Corollary 3.7 continues to hold. However, in this case, the spectral sequence will differ from the integral version beyond the  $E^2$  term by necessarily having nonzero higher differentials, and a different approach is needed. This reflects that the monodromy operator need not be semisimple in positive characteristic.

The ring  $\mathbb{F}_\ell[t, t^{-1}]$  is a P.I.D., and  $H_*(F, \mathbb{F}_\ell)$  is finite-dimensional over  $\mathbb{F}_\ell$ , so it decomposes as a direct sum of modules of the form  $\mathbb{F}_\ell[t, t^{-1}]/p(t)^j$ , where  $p(t)$  is irreducible and  $p(t)^j$  divides  $1 - t^N$ . Here, the filtration by powers of  $u = 1 - t$  detects only the monodromy eigenvalue 1:

$$\text{Gr}_k(\mathbb{F}_\ell[t, t^{-1}]/p(t)^j) = \begin{cases} \mathbb{F}_\ell, & \text{if } p(t) = 1 - t \text{ and } -j < k \leq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{5.1}$$

That is, the dimension of  $E_{pq}^\infty = \text{Gr}_p H_{p+q}(F, \mathbb{F}_\ell)$  is the number of Jordan blocks for the eigenvalue 1 that are of size at least  $1 - p$ . Clearly there exists some  $p_0$  for which  $E_{pq}^\infty = 0$  for all  $q$  when  $p \leq p_0$ , and there exists some integer  $r_0$  so that  $E_{pq}^\infty = E_{pq}^r$  for all  $r \geq r_0$ .

We note that  $\phi_k(t) = (1 - t)^{\ell^r - \ell^{r-1}}$  over  $\mathbb{F}_\ell[t, t^{-1}]$  if  $k = \ell^r$ . If  $k$  is not a power of  $\ell$ , on the other hand,  $\phi_k(t)$  is not divisible by  $1 - t$ . If  $\ell$  does not divide  $N$ , then,  $E_{pq}^\infty = \text{Gr}_p H_q(F, \mathbb{F}_\ell) = 0$  for  $p < 0$ , so the only interesting primes  $\ell$  are the factors of  $N$ .

For  $p \leq 0$ , we define a generating function by  $P_p^r(x) = \sum_q (\dim_{\mathbb{F}_\ell} E_{pq}^r) x^{p+q}$ . Theorem 3.5 shows that  $P_p^1(t) = \pi(\mathcal{A}, x)$  for all  $p \leq 0$ , where  $\pi(\mathcal{A}, x)$  is the Poincaré polynomial of  $\mathcal{A}$ . From Corollary 3.7, we have

$$P_p^2(x) = \sum_{q \geq 0} (\dim_{\mathbb{F}_\ell} H^p(A^*, \omega)) x^q \tag{5.2}$$

for all  $p \leq -1$ . By the nature of the filtration, the rank of the map  $d^r : E_{p,k-p}^r \rightarrow E_{p-r,k-p+r-1}^r$  is the same for all  $p \leq 0$ , given a fixed choice of  $r$  and  $k$ ; similarly,  $P_p^r(x)$

is the same for all  $p \leq 1 - r$ . Using this and the fact that  $P_p^r(x) = 0$  for  $p \leq p_0$  and  $r \geq r_0$ , a generating function argument shows that

$$P_{1-r}^\infty(x) = \frac{P_s^r(x)}{x+1} \quad \text{for all } s \leq 1 - r.$$

Setting  $r = 1$  gives  $P_0^\infty(x) = \pi(\mathcal{A}, x)/(x+1)$  and recovers the known result that

$$\dim_{\mathbb{F}_\ell} \frac{H_q(F, \mathbb{F}_\ell)}{1-t} = \tilde{\mu}_q.$$

Since  $N \equiv 0 \pmod{\ell}$ , the 1-form  $\omega \in A^1(\mathcal{A}) \otimes \mathbb{F}_\ell$  has a preimage  $\lambda$  in  $A^1(\mathbf{d}\mathcal{A}) \otimes \mathbb{F}_\ell$  under the natural map. By [23] there is a short exact sequence for each  $q$ :

$$0 \rightarrow H^q(A^\bullet(\mathbf{d}\mathcal{A}), \lambda) \rightarrow H^{q+1}(A^\bullet(\mathcal{A}), \omega) \rightarrow H^{q+1}(A^\bullet(\mathbf{d}\mathcal{A}), \lambda) \rightarrow 0.$$

Counting dimensions with (5.2),

$$P_{-1}^\infty(x) = \frac{P_{-1}^2(x)}{1+x} = \sum_{q=0}^{n-1} (\dim_{\mathbb{F}_\ell} H^q(\mathbf{d}\mathcal{A}, \lambda)) x^q,$$

which yields the following.

**Theorem 5.1.** *The dimension of  $H^q(\mathbf{d}\mathcal{A}, \lambda)$  over  $\mathbb{F}_\ell$  is the number of Jordan blocks of the monodromy operator, of size at least two, for the eigenvalue 1 on  $H_q(F, \mathbb{F}_\ell)$ .*

**Proof.** We apply the isomorphism (5.1) to  $H_q(F, \mathbb{F}_\ell)$  to interpret the dimension of  $E_{-1, q+1}^2$ , then apply the generating function above.  $\square$

This result should be compared with Theorem 1.4, where the left-hand side of the inequality equals the dimension of the eigenspace of  $H_q(F, \mathbb{Q})$  for a primitive  $\ell$ th root of unity. In particular, an example of a complexified arrangement where the inequality is strict may be interesting from the point of view of understanding the relationship between the modular and rational representations of the monodromy, and the possible existence of  $\ell$ -torsion in  $H_q(F, \mathbb{Z})$ .

## 6. Conclusions

Theorem 1.3 gives some information about the integral representation of the cyclic group  $\mathbb{Z}/N\mathbb{Z}$  on  $H_*(F, \mathbb{Z})$ . Over  $R_{\mathbb{Q}} = \mathbb{Q}[t, t^{-1}]$ , there is a decomposition

$$H_q(F, \mathbb{Q}) \cong \bigoplus_{r|N} \left( \frac{R_{\mathbb{Q}}}{\phi_r} \right)^{a_r}, \quad (6.1)$$

where each  $a_r \geq 0$  and  $\phi_r$  is the cyclotomic polynomial of order  $r$ . It is known, for example, that the Milnor fibres of two plane curves with an isolated singularity could have the same rational structure, above, but differ integrally [13].

It is not clear, however, what structural constraints, if any, are imposed by restriction to hyperplane arrangements. Accordingly, the results here are insufficient to characterize  $H_*(F, \mathbb{Z})$  or even  $H_*(F, \mathbb{F}_\ell)$  for primes  $\ell$  in terms of the Orlik–Solomon algebra. However, we see that any example of two distinct matroids with isomorphic Milnor fibre homology would have to be rather subtle. The examples below use Theorem 1.3 to motivate a conjecture.

**Example 6.1.** In [18], Suciu considers the complexified real arrangement  $\mathcal{D}$  defined by  $Q = xyz(x - y)(x - z)(y - z)(x - y - z)(x - y + z)$ . He observes that its characteristic variety contains a one-dimensional component that does not contain the trivial representation. This component contains the rational local system  $\rho = (i, i, -1, -1, i, i, -1, -1)$  (expressed by listing the action of each generator in order), so that  $\dim H^1(U, \mathbb{C}_\rho) = 1$ . However, he observes,  $H^1(A^*(\mathbf{d}\mathcal{A}), \lambda) = 0$ , where  $\lambda$  is any one-form given by weights congruent to  $(1/4, 1/4, -1/2, -1/2, 1/4, 1/4, -1/2, 1/2)$  modulo  $\mathbb{Z}^8$ . The same is true for  $A^*(\mathcal{A})$ , answering a question posed in [11].

As in [2, Example 4.6], though, nonzero integer torsion reflects the existence of this local system. Paraphrasing the example from [18], we have  $H_1(F, \mathbb{Q}) \cong R_{\mathbb{Q}}/(1 - t^4) \oplus (R_{\mathbb{Q}}/(1 - t))^6$ , where  $F$  is the Milnor fibre of  $Q = xyz^2(x - y)^2(x - z)(y - z)(x - y - z)^2(x - y + z)^2$ . Calculation with the aid of a computer shows that  $H^2(A^*, \omega) = \mathbb{Z}/4\mathbb{Z}$ .

**Example 6.2.** Consider the polynomial  $Q = x^2y^2z^2(x + y + z)(x + y - z)(x - y + z)(x - y - z)$ , which defines (a weighted version of) the non-Fano arrangement. One can calculate that  $H^2(A^*, \omega) = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ . From Theorem 1.3, one can deduce that  $H_1(F, \mathbb{Q}) \cong (R_{\mathbb{Q}}/(1 - t^2))^2 \oplus (R_{\mathbb{Q}}/(1 - t))^4$ . We recover the observation of [5] that  $H^1(U, \mathbb{C}_\rho)$  has dimension 2 for the local systems of the form  $(t^2, t^2, t^2, t, t, t, t)$ , where  $t^{10} = 1, t \neq 1$ . (The example was of interest because, for all  $\lambda$ ,  $\text{rank} H^1(A^*(\mathbf{d}\mathcal{A}) \otimes k, \lambda) \leq 1$  when the field  $k$  has characteristic zero, but equals 2 for some  $\lambda$  when  $k$  has characteristic 2.) Over  $\mathbb{F}_2$ , we have  $H^1(\mathbf{d}\mathcal{A}, \lambda) = \mathbb{F}_2^2$ . Using Theorem 5.1, then, it must be the case that  $H_1(F, \mathbb{F}_2) \cong (R_2[t, t^{-1}]/u^2)^2 \oplus (R_2/u)^4$ , where  $R_2 = \mathbb{F}_2[t, t^{-1}]$  and  $u = 1 - t$ .

**Example 6.3.** Consider the ordinary Milnor fibre of the braid arrangements  $A_n$ , studied in [4]. The characteristic polynomials of the monodromy operator (also computed in [4]) in dimension  $p$  are

$p$	0	1	2	3
	$\phi_1$	$\phi_1^9$	$\phi_1^{24}\phi_2^2$	$\phi_1^{16}\phi_2^8\phi_5^6\phi_{10}^6$

It is straightforward to calculate  $H^p(A^*, \omega)$ :

$p$	1	2	3	4
$H^p(A^*, \omega)$	0	0	$(\mathbb{Z}/2\mathbb{Z})^2$	$(\mathbb{Z}/2\mathbb{Z})^8(\mathbb{Z}/5\mathbb{Z})^6$

All other examples computed have been consistent with the conjecture that, for each  $q$ ,  $H_q(F, \mathbb{Z}) = \bigoplus_{i=1}^{\mu_q} R/(1 - t^{d_i})$  for some integers  $d_i$  dividing  $N$ ; that is, that the integral

homology is a permutation representation. If so, the main theorem would imply that  $H^{q+1}(A^\bullet, \omega) = \bigoplus_{i=1}^{\tilde{\mu}_q} \mathbb{Z}/d_i\mathbb{Z}$ .

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