

## Deformations of hypersolvable arrangements

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Received 24 November 1999; received in revised form 10 June 2000

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### Abstract

The hypersolvable class of arrangements introduced in [Topology 37 (6) (1998) 1135–1164] contains the fiber-type ones, the generic ones and many others. By extension of the deformation method proposed in [Topology 37 (6) (1998) 1135–1164], we prove that every hypersolvable arrangement can be deformed into a fiber-type one, with the same fundamental group and the same collinearity relations. © 2002 Elsevier Science B.V. All rights reserved.

*AMS classification:* Primary 32S22, Secondary 20F34

*Keywords:* Supersolvable arrangement; Hypersolvable arrangement; Deformation

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### 0. Introduction

Let  $\mathcal{A}$  be a complex arrangement in a finite dimensional complex vector space  $V$ . Set  $M_{\mathcal{A}} = V \setminus \bigcup_{H \in \mathcal{A}} H$ , the complement of  $\mathcal{A}$  (or  $M_{\mathcal{A}}^V$ , to emphasize the ambient space  $V$ ). There is a nice interplay between the topology of the complement  $M_{\mathcal{A}}$  and various combinatorial and algebraic aspects of the arrangement  $\mathcal{A}$ .

We introduced in [6] the combinatorial class of *hypersolvable arrangements* which contains fiber-type arrangements, generic arrangements and a lot of others. This class offers the best understanding of the properties of the complement  $M_{\mathcal{A}}$ . It is precisely our *deformation method* which enabled us in [6] to obtain significant results on the topology of  $M_{\mathcal{A}}$ .

The hypersolvable arrangements are inductively built by means of *solvable extensions* (Definition 1.4). Consider such a solvable arrangement pair (extension) in  $V$ ,  $\mathcal{B} \subset \mathcal{A}$

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(Definition 1.1). Then [6]: either  $\text{rank}(\mathcal{A}) = \text{rank}(\mathcal{B}) + 1$ , and there is a fibration with fiber a punctured complex plane,  $M_{\mathcal{A}} \rightarrow M_{\mathcal{B}}$  (up to homotopy), or  $\text{rank}(\mathcal{A}) = \text{rank}(\mathcal{B})$ , and after a deformation of  $\mathcal{A}$  into an arrangement  $\tilde{\mathcal{A}}$  in  $V \times \mathbb{C}$ , there is a similar fibration,  $M_{\tilde{\mathcal{A}}} \rightarrow M_{\mathcal{B}}$  (up to homotopy). In the first case, we say that the given solvable extension is *nonsingular*, and in the second case we speak about a *singular* extension.

In this paper, we extend the deformation method. Our main result is as follows. Let  $s$  be the number of singular extensions of a *composition series* of a hypersolvable arrangement  $\mathcal{A}$  (see Definition 1.4). Then there is a deformation  $\tilde{\mathcal{A}}$  of  $\mathcal{A}$  in  $V \times \mathbb{C}^s$  such that  $\tilde{\mathcal{A}}$  is fiber-type,  $\mathcal{A}$  and  $\tilde{\mathcal{A}}$  having *the same collinearity relations*. As a direct consequence we obtain that every hypersolvable arrangement is a slice of a fiber-type arrangement, with no new collinearity relations (and with the same fundamental group). In particular it is a fiber-type truncation, when it has rank three.

## 1. Hypersolvable and fibered composition series

The *hyperplane arrangements* we will consider in this paper,  $\mathcal{A} = \{H_1, \dots, H_n\}$ , will be finite collections of linear hyperplanes passing through the origin of a finite dimensional complex vector space  $V$ . We are going to review in this section some basic facts on *hypersolvable* and *fiber-type* arrangements, following [6]. It will be convenient in the sequel to view the hyperplanes of  $\mathcal{A}$  as points in the dual projective space  $\mathbb{P}(V^*)$ . We will thus rewrite  $\mathcal{A} = \{\alpha_1, \dots, \alpha_n\}$ , where each  $\alpha_H$  is the defining equation of the hyperplane  $H$ .

By an *extension*,  $\mathcal{B} \subset \mathcal{A}$ , we mean an arrangement  $\mathcal{A}$ , together with a proper subarrangement  $\mathcal{B}$ . Set  $\overline{\mathcal{B}} = \mathcal{A} \setminus \mathcal{B}$ . We say that the extension is *closed* if no point  $a \in \overline{\mathcal{B}}$  lies on a projective line passing through  $\alpha$  and  $\beta$ , where  $\alpha, \beta \in \mathcal{B}, \alpha \neq \beta$ . We say that the extension is *complete* if for any  $a, b \in \overline{\mathcal{B}}, a \neq b$ , there is  $\gamma \in \mathcal{B}$  which is collinear with  $a$  and  $b$ . If the extension is complete and closed then obviously the point  $\gamma$  above is unique. We will thus write in the sequel  $\gamma = f(a, b)$ . The key combinatorial definition is as follows.

**Definition 1.1** [6, Chapter 1]. The extension  $\mathcal{B} \subset \mathcal{A}$  is *solvable* if it is closed and complete and for any distinct points  $a, b, c \in \overline{\mathcal{B}}$  either  $f(a, b) = f(b, c) = f(a, c)$  or the points  $f(a, b), f(b, c)$  and  $f(a, c)$  are collinear.

One may associate to any arrangement  $\mathcal{A}$  its *complement*,  $M_{\mathcal{A}}^V := V \setminus \bigcup_{i=1}^n H_i$ , and its *cardinality*,  $|\mathcal{A}| := n$ . Let  $\mathcal{B}$  be another arrangement, in  $W$ , where  $\dim W = \dim V - 1$ . The following basic topological definition was introduced by Falk and Randell.

**Definition 1.2** [3, Chapter 1]. The complement  $M_{\mathcal{A}}^V$  is *linearly fibered* over  $M_{\mathcal{B}}^W$  if there is a linear surjection,  $p: V \rightarrow W$ , inducing a topological fibration,  $\pi: M_{\mathcal{A}}^V \rightarrow M_{\mathcal{B}}^W$ , with fiber  $\mathbb{C} \setminus \{m \text{ points}\}$ , where  $m = |\mathcal{A}| - |\mathcal{B}|$ .

There is a close relation between solvable extensions and linear fibrations.

Consider a nonsingular solvable extension in  $V$ ,  $\mathcal{B} \subset \mathcal{A}$ . Set  $X_{\mathcal{A}} = \bigcap_{H \in \mathcal{A}} H$ , and  $V_{\mathcal{A}} = V/X_{\mathcal{A}}$ . Then obviously  $\dim V_{\mathcal{A}} = \text{rank}(\mathcal{A})$ , and  $\mathcal{A}$  may be canonically viewed as an arrangement in  $V_{\mathcal{A}}$  (as its defining equations factor through the canonical projection,  $V \rightarrow V_{\mathcal{A}}$ ). Since  $\mathcal{B} \subset \mathcal{A}$ , plainly  $X_{\mathcal{A}} \subset X_{\mathcal{B}}$ , and the canonical projection,  $p: V_{\mathcal{A}} \rightarrow V_{\mathcal{B}}$ , induces by restriction a map  $\pi: M_{\mathcal{A}}^{V_{\mathcal{A}}} \rightarrow M_{\mathcal{B}}^{V_{\mathcal{B}}}$ .

Conversely, start with a linear fibration, as in Definition 1.2. The linear surjection  $p$  induces an embedding,  $p^*: \mathbb{P}(W^*) \hookrightarrow \mathbb{P}(V^*)$ , and thus  $\mathcal{B}$  may be canonically viewed as an arrangement in  $V$ . Then the property of  $p$  to preserve arrangement complements easily translates to the fact that  $\mathcal{B}$  is a subarrangement of  $\mathcal{A}$ .

With these preliminary remarks, one has the following result.

**Proposition 1.3.**

- (i) If  $\mathcal{B} \subset \mathcal{A}$  is solvable and nonsingular, then the above map  $\pi$  is a topological fibration, with fiber  $\mathbb{C} \setminus \{m \text{ points}\}$ , where  $m = |\overline{\mathcal{B}}|$ .
- (ii) If  $\pi: M_{\mathcal{A}}^V \rightarrow M_{\mathcal{B}}^W$  is a linear fibration, with fiber  $\mathbb{C} \setminus \{m \text{ points}\}$ , then either  $\mathcal{B} = \mathcal{A}$  (if  $m = 0$ ), or  $\mathcal{B} \subset \mathcal{A}$  is a nonsingular solvable extension (if  $m > 0$ ).

**Proof.** (i) Choosing coordinates, we may suppose that  $p: V_{\mathcal{A}} \rightarrow V_{\mathcal{B}}$  is the projection from  $\mathbb{C}^r$  to the first  $r - 1$  coordinates. Therefore, all the defining equations of  $\mathcal{B}$  are independent of  $z_r$ . Furthermore, we may also suppose that

$$\alpha_H(z_1, \dots, z_r) = z_r - \alpha'_H(z_1, \dots, z_{r-1}), \quad \forall H \in \overline{\mathcal{B}}. \tag{1.1}$$

Indeed, if there is  $H \in \overline{\mathcal{B}}$  with  $\alpha_H$  independent of  $z_r$ , then the completeness property of the extension may be used to infer that  $\alpha_H$  should be independent of  $z_r$ , for all  $H \in \overline{\mathcal{B}}$ . This would imply that  $\text{rank}(\mathcal{A}) \leq r - 1$ , contradicting the fact that the extension is nonsingular. In the presence of property (1.1), it is straightforward to check that the completeness property of the extension  $\mathcal{B} \subset \mathcal{A}$  is equivalent to

$$\alpha'_{H_1} - \alpha'_{H_2} \in \mathcal{B}, \quad \forall H_1, H_2 \in \overline{\mathcal{B}}, H_1 \neq H_2. \tag{1.2}$$

Denoting by  $M_m$  the complement of the braid arrangement in  $\mathbb{C}^m$ , one may use (1.2) to construct a map,  $\alpha': M_{\mathcal{B}}^{V_{\mathcal{B}}} \rightarrow M_m$ . It is then easy to see that the pullback by  $\alpha'$  of the well-known fibration,  $M_{m+1} \rightarrow M_m$ , with fiber  $\mathbb{C} \setminus \{m \text{ points}\}$ , coincides with  $\pi$ .

(ii) The case  $m = 0$  is clear, so we may assume that  $\mathcal{B}$  is a proper subarrangement of  $\mathcal{A}$ ; see Definition 1.2. Choose coordinates, as before. If there is  $H \in \overline{\mathcal{B}}$  with  $\alpha_H$  independent of  $z_r$ , then we may find  $y \in M_{\mathcal{B}}^W$  such that  $\alpha_H(y, t) = 0$ , for all  $t \in \mathbb{C}$ . Consequently the fiber of  $\pi$  over  $y$ ,  $F_y$ , is void. This contradiction shows that property (1.1) must be satisfied.

This in turn implies that

$$F_y = \{t \in \mathbb{C} \mid t \neq \alpha'_H(y), \forall H \in \overline{\mathcal{B}}\}$$

for every  $y \in M_{\mathcal{B}}^W$ . We infer from the fibration property (see Definition 1.2) that

$$\alpha'_{H_1}(y) \neq \alpha'_{H_2}(y), \quad \forall H_1, H_2 \in \overline{\mathcal{B}}, H_1 \neq H_2, \forall y \in M_{\mathcal{B}}^W. \tag{1.3}$$

The completeness of the extension  $\mathcal{B} \subset \mathcal{A}$  follows from (1.3), via (1.1)–(1.2). The extension is also closed, since all defining equations of  $\mathcal{B}$  are independent of  $z_r$  (use

again (1.1)). By [6, Lemmas 1.3(i) and 1.7(i)], we know that either  $\text{rank}(\mathcal{A}) = \text{rank}(\mathcal{B})$ , or the extension is solvable and nonsingular, as asserted. We may see that the first case is impossible, by resorting to (1.1) once more, and we are done.  $\square$

Solvable extensions are the building blocks for the *hypersolvable* arrangements introduced in [6].

**Definition 1.4** [6, Chapter 1].  $\mathcal{A}$  is *hypersolvable* if it has a *hypersolvable composition series*, i.e., an ascending chain of arrangements,  $\mathcal{A}_1 \subset \cdots \subset \mathcal{A}_i \subset \mathcal{A}_{i+1} \subset \cdots \subset \mathcal{A}_l = \mathcal{A}$ , where  $\text{rank}(\mathcal{A}_1) = 1$  and all extensions  $\mathcal{A}_i \subset \mathcal{A}_{i+1}$  are solvable.

Likewise, it follows from Proposition 1.3 that the *fiber-type* arrangements of [3] may be described using nonsingular solvable extensions. See [10] for another (equivalent) combinatorial description.

**Definition 1.5** [3, Chapter 1].  $\mathcal{A}$  is *fiber-type* if it has a hypersolvable composition series which is *fiber-type*, that is all extensions  $\mathcal{A}_i \subset \mathcal{A}_{i+1}$  are *fibred* (i.e., solvable and nonsingular).

**Example 1.6.** Besides fiber-type arrangements, the hypersolvable class contains many others, such as Hattori's [5] *generic* class; see [6, Remark 1.11].

As a very simple example, consider the following rank 3 arrangement  $\mathcal{A} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$  in  $\mathbb{C}^3$ , where  $\alpha_1 = x, \alpha_2 = x + y, \alpha_3 = x - y, \alpha_4 = x + y + z, \alpha_5 = x - y + z$ . It is straightforward to check that

$$\mathcal{A}_1 := \{\alpha_1\} \subset \mathcal{A}_2 := \mathcal{A}_1 \cup \{\alpha_2, \alpha_3\} \subset \mathcal{A}_3 := \mathcal{A}_2 \cup \{\alpha_4\} \subset \mathcal{A}_4 := \mathcal{A}$$

is a hypersolvable composition series of length  $l = 4$  (where  $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \mathcal{A}_3$  is a fiber-type series, but the last extension,  $\mathcal{A}_3 \subset \mathcal{A}_4$ , is singular). Obviously  $\mathcal{A}$  is not generic (since  $\{\alpha_1, \alpha_2, \alpha_3\}$  are collinear). One can see that  $\mathcal{A}$  is not fiber-type, just by noting that its length is greater than its rank (use [6, Theorem D]).

See [6] for more examples of hypersolvable and nonhypersolvable arrangements.

## 2. Deformation results

We begin by describing the framework of our key deformation result. Let  $\mathcal{B}$  be an arrangement in  $V$ , given by equations  $\{\alpha_H \in V^* \setminus \{0\} \mid H \in \mathcal{B}\}$ . Denote by  $q$  the canonical projection,  $q: V^* \times \mathbb{C} \rightarrow V^*$ . We will say that an arrangement  $\tilde{\mathcal{B}} = \{\tilde{\alpha}_H \in (V^* \times \mathbb{C}) \setminus \{0\} \mid H \in \mathcal{B}\}$  *lifts*  $\mathcal{B}$  to  $V \times \mathbb{C}$  if  $q(\tilde{\alpha}_H) = \alpha_H$ , for every  $H \in \mathcal{B}$ . We will say that  $\mathcal{B}$  and  $\tilde{\mathcal{B}}$  *have the same collinearity relations* (and write  $\mathcal{L}_2(\mathcal{B}) = \mathcal{L}_2(\tilde{\mathcal{B}})$ ) if  $\text{rank}\{\alpha_H, \alpha_{H'}, \alpha_{H''}\} = \text{rank}\{\tilde{\alpha}_H, \tilde{\alpha}_{H'}, \tilde{\alpha}_{H''}\}$  for all distinct hyperplanes  $H, H', H''$  of  $\mathcal{B}$ . We may spell out our basic deformation result.

**Proposition 2.1.** *Let  $\mathcal{B} \subset \mathcal{A}$  be a solvable extension in  $V$ . Suppose that  $\tilde{\mathcal{B}}$  lifts  $\mathcal{B}$  to  $V \times \mathbb{C}$ , and  $\mathcal{L}_2(\mathcal{B}) = \mathcal{L}_2(\tilde{\mathcal{B}})$ . Then one may find  $\tilde{\alpha}_H \in (V^* \times \mathbb{C}) \setminus \{0\}$  such that  $q(\tilde{\alpha}_H) = \alpha_H$ , for every  $H \in \tilde{\mathcal{B}}$ , and with the property that  $\mathcal{L}_2(\mathcal{A}) = \mathcal{L}_2(\tilde{\mathcal{A}})$ , where  $\tilde{\mathcal{A}} = \tilde{\mathcal{B}} \cup \{\tilde{\alpha}_H \mid H \in \tilde{\mathcal{B}}\}$ .*

**Proof.** The constructions and the proofs from [6, Proposition 4.6] work perfectly in this more general context. One has just to replace the trivial lift considered there,  $\tilde{\mathcal{B}}_0 = \{(\alpha_H, 0) \mid H \in \mathcal{B}\}$ , by the given lift  $\tilde{\mathcal{B}}$ . The reason why no other changes are needed is the fact that the very notion of solvability (see Definition 1.1) depends only on collinearity relations. The other additional hypotheses made in our previous work (besides the triviality of the lift of  $\mathcal{B}$ ), namely  $\text{rank}(\mathcal{B}) = \text{rank}(\mathcal{A}) = \dim V$ , are not necessary to obtain the conclusions of Proposition 2.1 above.  $\square$

**Remark 2.2.** The fact that  $\mathcal{L}_2(\mathcal{A}) = \mathcal{L}_2(\tilde{\mathcal{A}})$  clearly implies that the solvability property is inherited by the lifted extension,  $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$ . It readily follows that  $\tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$  is nonsingular, if  $\mathcal{B} \subset \mathcal{A}$  is nonsingular. In the particular case treated before ( $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}_0$ ), Proposition 4.6(i) from [6] shows that the lifted extension may always be taken to be nonsingular, even if  $\mathcal{B} \subset \mathcal{A}$  is singular.

Let  $\mathcal{A}$  be an arrangement in  $V$ , given by equations  $\{\alpha_H \in V^* \setminus \{0\} \mid H \in \mathcal{A}\}$ . The preceding result suggests the following definition.

**Definition 2.3.** A vertical deformation of  $\mathcal{A}$  in  $V \times \mathbb{C}^s$  is an algebraic family of arrangements in  $V \times \mathbb{C}^s$  of the form  $\tilde{\mathcal{A}}(t) = \{\tilde{\alpha}_H(t) = (\alpha_H, tv_H) \mid H \in \mathcal{A}\}$ , where  $v_H \in \mathbb{C}^s$  and  $t \in \mathbb{C}$ .

Let  $\mathcal{A}$  be a hypersolvable arrangement in  $V$ , with a composition series of length  $l$  as in Definition 1.4. Let  $s$  be the number of singular extensions of the series. We may now state our main result in this paper.

**Theorem 2.4.** *There is a vertical deformation,  $\{\tilde{\mathcal{A}}(t)\}_{t \in \mathbb{C}}$ , of  $\mathcal{A}$  in  $V \times \mathbb{C}^s$ , such that:*

- (i)  $\tilde{\mathcal{A}}(t)$  is fiber-type of rank  $l$ , for all  $t \neq 0$ , and
- (ii)  $\mathcal{L}_2(\mathcal{A}) = \mathcal{L}_2(\tilde{\mathcal{A}}(t))$ , for all  $t \in \mathbb{C}$ .

**Proof.** Plainly all arrangements  $\tilde{\mathcal{A}}(t)$  with  $t \neq 0$  are projectively equivalent (see Definition 2.3). It is therefore enough to construct  $\tilde{\mathcal{A}}(1)$  and to prove that  $\tilde{\mathcal{A}}(1)$  is fiber-type of rank  $l$  and  $\mathcal{L}_2(\mathcal{A}) = \mathcal{L}_2(\tilde{\mathcal{A}}(1))$ .

Let  $\mathcal{A}_{i_k} \subset \mathcal{A}_{i_{k+1}}$ ,  $i_1 < \dots < i_s$ , be the singular extensions in a fixed composition series of  $\mathcal{A}$ . Set also  $i_{s+1} = l$ . We are going to construct by induction vertical deformations  $\tilde{\mathcal{A}}^k(t)$  of  $\mathcal{A}$  in  $V \times \mathbb{C}^{k-1}$  such that, setting  $\tilde{\mathcal{A}}^k_{i_k}(t) := \{\tilde{\alpha}^k_H(t) \mid H \in \mathcal{A}_{i_k}\}$ ,

$$\tilde{\mathcal{A}}^k_{i_k}(t) \text{ is fiber-type of rank } i_k, \text{ for all } t \neq 0, \text{ and} \tag{2.1}$$

$$\mathcal{L}_2(\tilde{\mathcal{A}}^k(t)) = \mathcal{L}_2(\mathcal{A}), \text{ for all } t \in \mathbb{C}. \tag{2.2}$$

If  $k = 1$  there is nothing to prove (see Definition 1.5). Having constructed  $\tilde{\mathcal{A}}^k(t)$ , we will proceed to construct  $\tilde{\mathcal{A}}^{k+1}(t)$  by iterated extensions, using induction on  $j$ ,  $j \geq i_k$ . We are

thus going to assume that we have a vertical deformation  $\tilde{\mathcal{A}}_j^{k+1}(t)$  of  $\mathcal{A}_j$  in  $V \times \mathbb{C}^k$  such that:

$$\tilde{\mathcal{A}}_{i_k}^{k+1}(t) = \tilde{\mathcal{A}}_{i_k}^k(t), \quad \text{for all } t \in \mathbb{C}, \tag{2.3}$$

$$\mathcal{L}_2(\tilde{\mathcal{A}}_j^{k+1}(t)) = \mathcal{L}_2(\mathcal{A}_j), \quad \text{for all } t \in \mathbb{C}, \text{ and} \tag{2.4}$$

$$\tilde{\mathcal{A}}_{i_{k+1}}^{k+1}(t) \text{ is fiber-type of rank } i_k + 1, \text{ for all } t \neq 0 \text{ (if } j > i_k). \tag{2.5}$$

We may start by defining  $\tilde{\mathcal{A}}_{i_k}^{k+1}(t)$  by (2.3), and then (2.4) follows from (2.2). Next consider the solvable extension  $\tilde{\mathcal{A}}_{i_k}^k(1) \subset \tilde{\mathcal{A}}_{i_{k+1}}^k(1)$  in  $V \times \mathbb{C}^{k-1}$ , whose solvability is guaranteed by (2.2). We may apply Proposition 2.1 to the trivial lift to get a vertical deformation  $\tilde{\mathcal{A}}_{i_{k+1}}^{k+1}(t)$  of  $\mathcal{A}_{i_{k+1}}$  in  $V \times \mathbb{C}^k$ . Moreover,  $\text{rank}(\tilde{\mathcal{A}}_{i_{k+1}}^{k+1}(1)) = \text{rank}(\tilde{\mathcal{A}}_{i_k}^k(1)) + 1 = i_k + 1$ . (See Remark 2.2.) By construction it extends  $\tilde{\mathcal{A}}_{i_k}^k(t)$ . The property (2.4) follows from (2.2), while (2.1) forces (2.5), via the remarks made in 2.2. Having constructed  $\tilde{\mathcal{A}}_j^{k+1}(t)$  for  $j > i_k$ , we may extend it to  $\tilde{\mathcal{A}}_{j+1}^{k+1}(t)$  as follows. Assume inductively that  $\tilde{\mathcal{A}}_j^{k+1}(1)$  lifts  $\tilde{\mathcal{A}}_j^k(1)$ . Apply Proposition 2.1 to  $\tilde{\mathcal{A}}_j^k(1) \subset \tilde{\mathcal{A}}_{j+1}^k(1)$  in  $V \times \mathbb{C}^{k-1}$  to obtain a vertical deformation  $\tilde{\mathcal{A}}_{j+1}^{k+1}(t)$  of  $\mathcal{A}_{j+1}$  in  $V \times \mathbb{C}^k$  extending  $\tilde{\mathcal{A}}_j^{k+1}(t)$ , which satisfies (2.4). Moreover  $\tilde{\mathcal{A}}_{j+1}^{k+1}(1)$  lifts  $\tilde{\mathcal{A}}_{j+1}^k(1)$ , by construction. For  $j = l$  we thus arrive at a vertical deformation  $\tilde{\mathcal{A}}^{k+1}(t) := \tilde{\mathcal{A}}_l^{k+1}(t)$  of  $\mathcal{A}$  in  $V \times \mathbb{C}^k$ , satisfying (2.2). Therefore  $\tilde{\mathcal{A}}_1^{k+1}(t) \subset \dots \subset \tilde{\mathcal{A}}_j^{k+1}(t) \subset \tilde{\mathcal{A}}_{j+1}^{k+1}(t) \subset \dots \subset \tilde{\mathcal{A}}_{i_{k+1}}^{k+1}(t)$  will be a hypersolvable composition series, for all  $t \in \mathbb{C}$ . If  $j \leq i_k$  then  $\tilde{\mathcal{A}}_j^{k+1}(t) \subset \tilde{\mathcal{A}}_{j+1}^{k+1}(t)$  will be nonsingular, for  $t \neq 0$ ; see (2.5). The same property also holds for  $i_k + 1 \leq j < i_{k+1}$ , since the extensions  $\mathcal{A}_j \subset \mathcal{A}_{j+1}$  are all nonsingular in this range and  $q(\tilde{\mathcal{A}}_j^{k+1}(t)) = \mathcal{A}$  (where  $q$  is the canonical projection,  $q: V^* \times \mathbb{C}^k \rightarrow V^*$ ). Consequently  $\tilde{\mathcal{A}}_{i_{k+1}}^{k+1}(t)$  is fiber-type of rank  $i_{k+1}$ , for  $t \neq 0$ , as needed in (2.1).

We may set now  $\tilde{\mathcal{A}}(t) = \tilde{\mathcal{A}}^{s+1}(t)$ . This will be the desired deformation from the statement of our theorem.  $\square$

**Remark 2.5.** If  $\mathcal{A}_1 \subset \dots \subset \mathcal{A}_l = \mathcal{A}$  is a hypersolvable composition series, then obviously  $l = \text{rank}(\mathcal{A}) + s$ , where  $s$  is the number of singular extensions. We recall from [6, Theorem B] that the composition series length  $l$  of a hypersolvable arrangement  $\mathcal{A}$  is a well-defined invariant of  $\mathcal{A}$ . It follows that  $s = s(\mathcal{A}) = l(\mathcal{A}) - \text{rank}(\mathcal{A})$ .

### 3. Properties of hypersolvable arrangements

The first property is a direct consequence of Theorem 2.4.

**Corollary 3.1.** *Every hypersolvable arrangement  $\mathcal{A}$  in  $V$  of length  $l$  is the restriction to  $V$  of a fiber-type arrangement in  $V \times \mathbb{C}^s$  of rank  $l$ . Moreover, all dependence relations up to length 3 are preserved by restriction.*

**Proof.** Take  $\tilde{\mathcal{A}}(1)$  in  $V \times \mathbb{C}^s$  as in Theorem 2.4. Clearly the trace of  $\tilde{\mathcal{A}}(1)$  on  $V \times \{0\} \subset V \times \mathbb{C}^s$  is  $\mathcal{A}$  (see Definition 2.3).  $\square$

Let  $S = \bigoplus_{d \geq 0} S_d$  be the graded symmetric algebra of  $V^*$  and  $\text{Der}(S) = \text{Der}(S, S)$  the graded  $S$ -module of derivations of  $S$ . Let  $D(\mathcal{A}) = \{\theta \in \text{Der}(S) \mid \theta(\alpha_H) \in S\alpha_H, H \in \mathcal{A}\}$ . If the  $S$ -module  $D(\mathcal{A})$  is free then the arrangement  $\mathcal{A}$  is called *free*.

**Corollary 3.2** [2]. *Let  $\mathcal{A}$  be a hypersolvable arrangement of rank 3. Then  $\mathcal{A}$  is free  $\Leftrightarrow \mathcal{A}$  is supersolvable.*

**Proof.** We may suppose that  $\mathcal{A}$  is a free hypersolvable arrangement of rank 3 in  $\mathbb{C}^3$ . If  $\mathcal{A}$  is not supersolvable (fiber-type) then  $s(\mathcal{A}) > 0$  (see Remark 2.5). We know from Corollary 3.1 that  $\mathcal{A}$  is a (proper) 3-truncation of  $\tilde{\mathcal{A}}(1)$ , which is essential in  $\mathbb{C}^{3+s}$ . This leads to a contradiction, via Yuzvinsky’s solution of a conjecture of Ziegler, which says that free arrangements are not proper truncations ([11, Corollary 3.5]).  $\square$

**Remark 3.3.** If  $\text{rank}(\mathcal{A}) \geq 4$  then  $\mathcal{A}$  may not be a truncation of  $\tilde{\mathcal{A}}(1)$ , in the sense of [11, Definition 3.1]. Take for example  $\mathcal{A} = \bigcup_{1 \leq i < j \leq 5} \{z_i - z_j = 0\} \cup \{a := z_1 + z_2 + z_3 - 3z_5 = 0\}$ . It is a rank 4 hypersolvable arrangement in  $V = \{z \mid z_1 + z_2 + z_3 + z_4 + z_5 = 0\}$ . Here  $\tilde{\mathcal{A}}(1) = \bigcup_{1 \leq i < j \leq 5} \{z_i - z_j = 0\} \cup \{\tilde{a} := z_1 + z_2 + z_3 - 3z_5 + u = 0\}$  sits in  $V \times \mathbb{C}$ . When restricting to  $V$ , the rank of  $\{z_1 - z_5, z_2 - z_5, z_3 - z_5, \tilde{a}\}$  drops from 4 to 3, therefore the trace of  $\tilde{\mathcal{A}}(1)$  on  $V \times \{0\}$  is not a 4-truncation.

The next two important properties were first derived in [6] by combining various techniques. They become now straightforward consequences of our deformation theorem and well-known properties of fiber-type (supersolvable) arrangements.

Recall the Orlik–Solomon algebra [7] of an arrangement,  $A(\mathcal{A})$ , with Poincaré polynomial  $P_{\mathcal{A}}(T)$ , and its quadratic analogue from [6],  $\bar{A}(\mathcal{A})$ , which depends only on  $\mathcal{L}_2(\mathcal{A})$ , with Poincaré polynomial  $\bar{P}_{\mathcal{A}}(T)$ . Let  $\mathcal{A}$  be hypersolvable, with composition series  $\mathcal{A}_1 \subset \dots \subset \mathcal{A}_l = \mathcal{A}$ . For convenience, we also set  $\mathcal{A}_0 = \emptyset$ .

**Proposition 3.4** [6, Chapter 3]. *If  $\mathcal{A}$  is as above, then:*

- (i)  $\bar{P}_{\mathcal{A}}(T) = \prod_{i=1}^l (1 + d_i T)$ , where  $d_i = |\mathcal{A}_i \setminus \mathcal{A}_{i-1}|$ , for  $i = 1, \dots, l$ .
- (ii)  $\mathcal{A}$  is fiber-type  $\Leftrightarrow l = \text{rank}(\mathcal{A}) \Leftrightarrow \bar{P}_{\mathcal{A}}(T) = P_{\mathcal{A}}(T)$ .

**Proof.** (i) We know from the proof of Theorem 2.4 that  $\tilde{\mathcal{A}} := \tilde{\mathcal{A}}(1)$  has a fiber-type composition series  $\tilde{\mathcal{A}}_1 \subset \dots \subset \tilde{\mathcal{A}}_l = \tilde{\mathcal{A}}$ , with  $|\tilde{\mathcal{A}}_i \setminus \tilde{\mathcal{A}}_{i-1}| = d_i$ , for all  $i$ . We may replace  $\bar{P}_{\mathcal{A}}$  by  $\bar{P}_{\tilde{\mathcal{A}}}$  (using 2.4(ii)), then  $\bar{P}_{\tilde{\mathcal{A}}}$  by  $P_{\tilde{\mathcal{A}}}$  (see [1,9]) and finally resort to the factorization result from [3, Chapter 2].

(ii) If  $\bar{P}_{\mathcal{A}}(T) = P_{\mathcal{A}}(T)$  then  $l = \text{rank}(\mathcal{A})$ , by degree inspection. This implies at once that  $\mathcal{A}$  is fiber-type, and in this case it is well-known that  $\bar{P}_{\mathcal{A}} = P_{\mathcal{A}}$ .  $\square$

**Remark 3.5.** All the results on Poincaré polynomials are available with arbitrary field coefficients.

For the last result, denote by  $\text{gr}_G^*(G) = \bigoplus_{i \geq 1} \text{gr}_G^i(G)$  the associated graded abelian group of a group  $G$  with respect to the lower central series, and by  $\mathbf{F}_d$  the free group on  $d$  generators. Let  $\mathcal{A}$  be hypersolvable, with composition series as described before.

**Proposition 3.6** [6, Chapters 4 and 5].

- (i)  $\pi_1(M_{\mathcal{A}})$  is an iterated almost-direct product of free groups,  $\mathbf{F}_{d_1} \rtimes \cdots \rtimes \mathbf{F}_{d_1}$ .
- (ii)  $\text{gr}_G^i(\pi_1(M_{\mathcal{A}}))$  is a finitely generated free Abelian group, for all  $i$ .
- (iii) Setting  $\varphi_i(\mathcal{A}) := \text{rank } \text{gr}_G^i(\pi_1(M_{\mathcal{A}}))$ , one has the following generalized Witt formula:

$$\prod_{i=1}^{\infty} (1 - T^i)^{\varphi_i(\mathcal{A})} = \overline{P}_{\mathcal{A}}(-T).$$

**Proof.** The deformation provided by Theorem 2.4 may be exploited exactly as in [6, Chapter 4], together with Randell's isotopy theorem [8] and Zariski's theorem on the  $\pi_1$  of the complement (see [4]), to replace  $\pi_1(M_{\mathcal{A}})$  by  $\pi_1(M_{\tilde{\mathcal{A}}(1)})$  in the above statement. All the assertions then follow from the corresponding results on fiber-type arrangements from [3, Chapters 2–4].  $\square$

**Question 3.7.** Find examples of non-hypersolvable arrangements satisfying the generalized Witt formula.

### Acknowledgements

The second named author benefited from a Research in Pairs stay at Oberwolfach, supported by the Volkswagen-Stiftung, during the completion of this work. The excellent facilities and the stimulating atmosphere are gratefully acknowledged.

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