ELSEVIER

Topology and its Applications 118 (2002) 113–129

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

Geometric relationship between cohomology of the complement of real and complexified arrangements

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Received 11 November 1999; received in revised form 19 June 2000

Abstract

Let $\mathcal{A}_{\mathbb{R}}$ be a real hyperplane arrangement and let $\mathcal{A}_{\mathbb{C}}$ be its complexification. Let $M_{\mathbb{R}}$ and $M_{\mathbb{C}}$ be the respective complements. Then $M_{\mathbb{R}}$ is the disjoint union of convex chambers whose number is given by its only Betti number, $b_0(M_{\mathbb{R}})$. A real arrangement and its complexification satisfy the M-property: $b_0(M_{\mathbb{R}}) = \sum_q b_q(M_{\mathbb{C}})$, the number of chambers in $M_{\mathbb{R}}$ equals the sum of the Betti numbers of $M_{\mathbb{C}}$. The no-broken-circuit set, **nbc**, is a field independent combinatorial object. It has been used to label a basis for $H^*(M_{\mathbb{C}})$ but not to label the chambers of $M_{\mathbb{R}}$ in a way that makes the M-property explicit. In this paper we use the **nbc** set to label a combinatorial object in the nerve of the arrangement, which is field independent. This allows for simultaneous choices of **nbc** bases in $H^*(M_{\mathbb{R}})$ and $H^*(M_{\mathbb{C}})$. We also explore the geometrical connections between these bases. © 2002 Elsevier Science B.V. All rights reserved.

AMS classification: 52C35; 55N99; 05B35

Keywords: Arrangements of hyperplanes; Generalized Mayer–Vietoris spectral sequence; Broken circuit; Nerve

1. Introduction

Let V be an ℓ-dimensional vector space over either $\mathbb{R}$ or $\mathbb{C}$. A *hyperplane arrangement* $\mathcal{A} = \{H_k\}_{k=1}^n$ is a finite collection of affine hyperplanes in V. It is real or complex according to the underlying field. Given a real arrangement $\mathcal{A}_{\mathbb{R}} = \{H_k\}_{k=1}^n$, we define its *complexification* $\mathcal{A}_{\mathbb{C}}$ as follows: $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$, and the hyperplanes are $(H_k)_{\mathbb{C}} = H_k \otimes \mathbb{C}$. The union of an arrangement is defined by $N(\mathcal{A}) = \bigcup_{H \in \mathcal{A}} H$. The complement of an arrangement is defined by $M(\mathcal{A}) = V - N(\mathcal{A})$. Let $\widehat{X}$ denote the one-point

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PII: S0166-8641(01)00045-1

compactification of the space X with ∞ being the point at infinity. Note that $M = \widehat{V} - \widehat{N}$ as well. We use $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ to denote the union and complement of a real hyperplane arrangement $\mathcal{A}_{\mathbb{R}}$, and $N_{\mathbb{C}}$ and $M_{\mathbb{C}}$ to denote the union and complement of its complexification $\mathcal{A}_{\mathbb{C}}$.

The real complement $M_{\mathbb{R}}$ is the disjoint union of convex chambers whose number is given by its only Betti number, $b_0(M_{\mathbb{R}})$. It is known [11, 5.95] that a real hyperplane arrangement and its complexification satisfy the M-property: $b_0(M_{\mathbb{R}}) = \sum_q b_q(M_{\mathbb{C}})$, the number of chambers in $M_{\mathbb{R}}$ equals the sum of the Betti numbers of $M_{\mathbb{C}}$. Björner [1] showed that there is a set bijection from the no-broken-circuit set, **nbc**, to a basis for the cohomology of the complement of a complex hyperplane arrangement, but **nbc** has not been used to label the chambers of $M_{\mathbb{R}}$ in a way that makes the M-property explicit. In this paper we use the **nbc** set to label combinatorial objects in the nerve of the arrangement, which is field independent. This allows for simultaneous choices of **nbc** bases in $H^*(M_{\mathbb{R}})$ and $H^*(M_{\mathbb{C}})$. We also explore the geometrical connections between these bases. Field independent methods were first used in the topology of arrangements by Goresky and MacPherson [7, Part III] as an application of their work with stratified Morse Theory. Our paper may be viewed as a refinement of that application of their work.

Section 2 considers the combinatorics of arrangements. The traditional source of combinatorial information is the intersection poset. We propose to use the nerve instead. Although the intersection poset and the nerve contain the same information and each can be recovered from the other, we argue that the nerve is a more natural object to study. We define the map $\iota : \bigoplus_{X\in L} \mathbb{Z}[\mathbf{nbc}_X] \to \bigoplus_{X\in L} \bigoplus_{p\geqslant 0} H_p(K[X], K(X))$ from the **nbc** set to our main combinatorial object, the relative homology of certain subcomplexes of the nerve, and show that the map is an isomorphism by providing an essentially self-contained proof that $\mathbf{nbc}_X$ forms a basis for $H_p(K[X], K(X))$, $p = r_L(X) - 1$. These results are field independent. We give two additional means for calculating the Möbius function for the intersection poset, one of which allows local computations using the pullback of elements of the intersection poset to the nerve poset. We end the section by showing how $\mathbf{nbc}_X$ splits under deletion and restriction. Some constructions in this section have analogs in the literature [13,14]. We decided to include them here to make this paper selfcontained.

Section 3 considers the topology of arrangements. We use the generalized Mayer–Vietoris spectral sequence where the combinatorial object, $\bigoplus_{X\in L} \bigoplus_{p\geqslant 0} H_p(K[X], K(X))$ studied in Section 2 is the main character and the field dimension enters for the first time as a spectral sequence dimension. In [8], it was shown that the spectral sequence collapses at the second term, that the $E^2_{p,\varepsilon q}$ term can be described by $\bigoplus_{X\in L_q} \bigoplus_{p\geqslant 0} H_p(K[X], K(X))$ where ε is either 1 or 2 depending upon whether the arrangement is real or complexified real. Also, in [8], it was shown that the reduced homology of the one-point compactification of the union can be described by the second term of the spectral sequence, and by Alexander duality so can the cohomology of the complement. We then use **nbc** to construct an explicit basis for $\widetilde{H}_{\ell-1}(\widehat{N}_{\mathbb{R}})$ for a real arrangement by choosing caps. We then show that we can choose caps consistently for $\bigoplus_t \widetilde{H}^t(M_{\mathbb{C}})$ for its associated complexified real arrangement. We end the section with another chamber labeling argument using

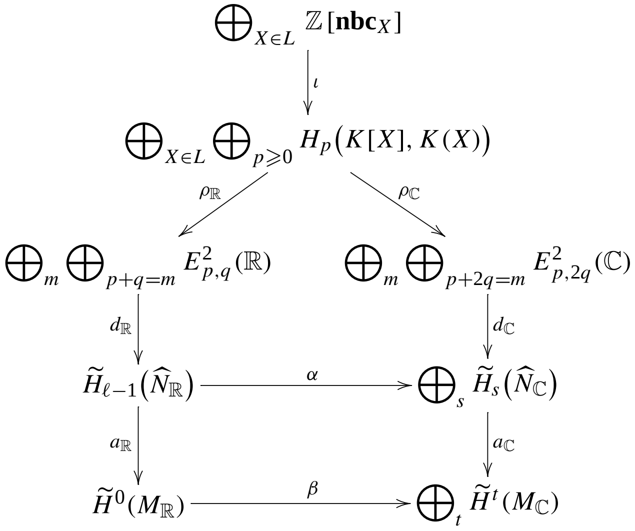


Fig. 1. Main Diagram.

deletion and restriction which labels individual chambers by **nbc** rather than collections of chambers by the spectral sequence argument.

Section 4 contains an example illustrating the capping process used in the spectral sequence argument and the individual chamber labeling process.

Our work may be summarized in the commutative diagram in Fig. 1, all of whose objects are defined in the body of the paper. All (co)homology groups have integer coefficients. We show that all arrows represent isomorphisms, some standard and some constructed in this paper, and make these isomorphisms explicit.

2. Combinatorics

2.1. Nerve and intersection poset

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a hyperplane arrangement in an ℓ-dimensional vector space V over the field $\mathcal{F}$, where $\mathcal{F}$ is $\mathbb{R}$ or $\mathbb{C}$, and let $\Sigma = \{1, \ldots, n\}$. Assume that $\mathcal{A}$ contains ℓ linearly independent hyperplanes. We use [11] for basic definitions and results. In this paper, both inclusion and reverse inclusion will be used for partial ordering certain sets. We will call the reader's attention to the particular order used in each case.

Definition 2.1. The *nerve* K of $\mathcal{A}$ is the abstract simplicial complex with vertex set Σ whose simplexes are the nonempty subsets σ of Σ such that $H_\sigma = \bigcap_{k\in\sigma} H_k$ is nonempty. We partially order K by inclusion to make K the *nerve poset*. Let $K^{(p)}$ denote the set of all p-dimensional simplexes of K. Let K^* be the poset K together with the unique minimal element $\emptyset$. Define its rank function by $r_K(\emptyset) = 0$ and $r_K(\sigma) = |\sigma| = \dim\sigma + 1$ for $\sigma \in K$.

It will be clear from the context when we view K as a poset and when we view it as a simplicial complex. The next definition differs from [11].

Definition 2.2. The *intersection poset* L of $\mathcal{A}$ is the set of all intersections of nonempty subcollections of $\mathcal{A}$ ordered by reverse inclusion. Let L^* be the poset L together with the unique minimal element V, the intersection of the empty subcollection. Define its rank function by $r_L(X) = \operatorname{codim}_{\mathcal{F}}(X)$ for $X \in L^*$.

Definition 2.3. Define the *poset map* $\varphi : K \to L$ by $\varphi(\sigma) = H_\sigma = \bigcap_{k \in \sigma} H_k$. Let $\varphi^{-1}(X) = \{\sigma \in K \mid \varphi(\sigma) = X\}$ be the *pullback* of $X \in L$. Then $\varphi^{-1}(X)$ is a subposet of K^* and contains a unique largest simplex which we call ζ_X. Call $\varphi^{-1}(X)$ trivial if it consists of the singleton ζ_X.

The main combinatorial objects of this study are the following.

Definition 2.4. Define two subcomplexes of K by

$$K[X] = \{\sigma \in K \mid H_\sigma \leqslant X\} = \bigcup_{Y \leqslant X} \varphi^{-1}(Y),$$

$$K(X) = \{\sigma \in K \mid H_\sigma < X\} = \bigcup_{Y < X} \varphi^{-1}(Y).$$

Lemma 2.5. *Let $\sigma \in \varphi^{-1}(X)$. Then σ is a minimal element of $\varphi^{-1}(X)$ if and only if $r_K(\sigma) = r_L(X)$.*

Proof. One direction is a direct consequence of at least k hyperplanes being necessary to obtain a space of codimension k as intersection.

The other direction uses a short induction argument on $r_L(\varphi(\cdot))$. Let σ be minimal in $\varphi^{-1}(X)$, then $\varphi(\sigma \setminus \{i\}) = X' < X$ for $i \in \sigma$ and, in particular, $r_L(\varphi(\sigma \setminus \{i\})) = r_L(\varphi(\sigma)) - 1$. With $\sigma \setminus \{i\}$ being minimal in $\varphi^{-1}(X')$ we conclude by induction on $r_L(\varphi(\cdot))$ that $r_L(\varphi(\sigma)) = r_L(\varphi(\sigma \setminus \{i\})) + 1 = r_K(\sigma)$. □

Lemma 2.6. *If $\varphi^{-1}(X)$ is nontrivial, $\sigma \in K^{(t)}$ is a nonminimal element of $\varphi^{-1}(X)$, and $\varphi^{-1}(Y)$ is trivial for all $Y < X$, then each face $\partial_k\sigma$ of σ is in $\varphi^{-1}(X)$, $0 \leqslant k \leqslant t$.*

Proof. Let $\sigma \in K^{(t)}$ be a nonminimal element of $\varphi^{-1}(X)$ and assume that $\partial_k\sigma \in \varphi^{-1}(Y)$ for some $Y < X$. Since $\varphi^{-1}(Y)$ is trivial by assumption, $\partial_k\sigma$ is a minimal element. By Lemma 2.5, $r_L(Y) = t$. However, since $\sigma \in K^{(t)}$ and σ is not minimal, $t = r_L(Y) < r_L(X) \leqslant t$. Thus, each $\partial_k\sigma \in \varphi^{-1}(X)$. □

Definition 2.7. Let $\mathcal{P}$ be a poset with rank function r. The *join* $X \vee Y$ of two elements $X, Y \in \mathcal{P}$ is the minimal element of the set $\{Z \in \mathcal{P} \mid Z \geqslant X,\ Z \geqslant Y\}$ when it exists. The *meet* $X \wedge Y$ of two elements $X, Y \in \mathcal{P}$ is the maximal element of the set $\{Z \in \mathcal{P} \mid Z \leqslant X, Z \leqslant Y\}$ when it exists. The *atoms* of $\mathcal{P}$ are those elements whose rank is 1. A poset $\mathcal{P}$ with rank function r is a *geometric poset* if:

(1) $\mathcal{P}$ has a unique minimal element B,
(2) every element of $\mathcal{P} \setminus \{B\}$ is the join of atoms,
(3) for every $X \in \mathcal{P}$, all maximal linearly ordered subsets $B = X_0 < X_1 < \cdots < X_p = X$ have the same cardinality with $r(X) = p$,
(4) if $X, Y \in \mathcal{P}$ and $X \wedge Y, X \vee Y \in \mathcal{P}$, then $r(X \wedge Y) + r(X \vee Y) \leqslant r(X) + r(Y)$.

A geometric poset is a *geometric lattice* if it also has a unique maximal element T.

Definition 2.8. Let $\varphi^{-1}(X)^\circ$ be the set $\varphi^{-1}(X)$ partially ordered by reverse inclusion together with the unique maximal element $\emptyset$. We use $\prec$ for this order. Note that ζ_X is the unique minimal element of $\varphi^{-1}(X)^\circ$. Define its rank function by $r_\varphi(\sigma) = r_K(\zeta_X) - r_K(\sigma)$ for $\sigma \in \varphi^{-1}(X)$ and $r_\varphi(\emptyset) = r_K(\zeta_X) - r_L(X) + 1$.

Lemma 2.9.
(1) *The poset K^*, ordered by inclusion, is a geometric poset.*
(2) *The poset L^*, ordered by reverse inclusion, is a geometric poset.*
(3) *The poset $\varphi^{-1}(X)^\circ$, ordered by reverse inclusion, is a geometric lattice.*

Proof. (1) The atoms of K^* are the vertices of K. Since K is a simplicial complex, any simplex σ is the join of its vertices, and all maximal linearly ordered chains between $\emptyset$ and σ have the same cardinality. Since $\sigma \wedge \tau = \sigma \cap \tau$ and $\sigma \vee \tau = \sigma \cup \tau$ when defined, $r_K(\sigma \wedge \tau) + r_K(\sigma \vee \tau) = r_K(\sigma) + r_K(\tau)$. Therefore, K^* is a geometric poset.

(2) The atoms of L^* are the hyperplanes. Any element $X \in L$ is the intersection of hyperplanes and, hence, the join of those hyperplanes. Here

$$X \wedge Y = \bigcap \{Z \in L \mid X \cup Y \subseteq Z\}$$

and $X \vee Y = X \cap Y$ if not empty. Since $r_L(Y \cap H) - r_L(Y) \in \{0, 1\}$ for $Y \in L$ and $H \in \mathcal{A}$, all maximal linearly ordered sets between V and X have the same cardinality. Since $\operatorname{codim}_{\mathcal{F}}(X + Y) \geqslant \operatorname{codim}_{\mathcal{F}}(X \wedge Y)$,

$$r_L(X \wedge Y) + r_L(X \vee Y) \leqslant r_L(X) + r_L(Y).$$

Therefore, L^* is a geometric poset.

(3) If the rank of $\varphi^{-1}(X)^\circ$ is one, then $\varphi^{-1}(X)^\circ = \{\zeta_X, \emptyset\}$ is clearly a geometric lattice. So without loss of generality assume that the rank of $\varphi^{-1}(X)^\circ$ is greater than one. The set of atoms of $\varphi^{-1}(X)^\circ$ is the set of faces of the minimal element ζ_X, $\{\partial_k(\zeta_X) \in \varphi^{-1}(X)\}$. The element $\sigma \in \varphi^{-1}(X)$ is the join of the elements of $\{\partial_k(\zeta_X) \in \varphi^{-1}(X)^\circ \mid \partial_k(\zeta_X) \prec \sigma\}$. Since pullbacks do not have any gaps, that is, if $\sigma < \eta < \tau$ with $\sigma, \tau \in \varphi^{-1}(X)$, then $\eta \in \varphi^{-1}(X)$, all maximal linearly ordered chains between $\emptyset$ and σ have the same cardinality. Note that for $\sigma, \tau \in \varphi^{-1}(X)$ $\sigma \wedge_K \tau$ may not be in $\varphi^{-1}(X)$ when viewed as an element of K^*. When this occurs, the corresponding element $\sigma \vee_\varphi \tau$ in $\varphi^{-1}(X)^\circ$ is $\emptyset$. Thus, $r_\varphi(\sigma \wedge \tau) + r_\varphi(\sigma \vee \tau) \leqslant r_\varphi(\sigma) + r_\varphi(\sigma)$. Since $\varphi^{-1}(X)^\circ$ has ζ_X as its unique minimal element and $\emptyset$ as its unique maximal element, $\varphi^{-1}(X)^\circ$ is a geometric lattice. $\square$

2.2. The Möbius function

Definition 2.10. Let $\mathcal{P}$ be a poset with unique minimal element B. Let $\mu_{\mathcal{P}}:\mathcal{P}\to\mathbb{Z}$ be the *Möbius function* defined recursively by $\mu_{\mathcal{P}}(B)=1$ and $\sum_{Y\leqslant X}\mu_{\mathcal{P}}(Y)=0$ for $X\in\mathcal{P}\setminus\{B\}$.

We have three Möbius functions $\mu_{L^*}:L^*\to\mathbb{Z}$, $\mu_{K^*}:K^*\to\mathbb{Z}$, and $\mu_{\varphi^{-1}(X)^\circ}:\varphi^{-1}(X)^\circ\to\mathbb{Z}$. Next we establish relationships among them.

Lemma 2.11.
(1) *For* $\sigma\in K$, $\mu_{K^*}(\sigma)=(-1)^{r_K(\sigma)}$.
(2) *For* $\sigma\in\varphi^{-1}(X)$, $\mu_{\varphi^{-1}(X)^\circ}(\sigma)=(-1)^{r_\varphi(\sigma)}$.

Proof. (1) We prove this by induction on the dimension of σ. For a vertex $\{k\}$, $\mu_{K^*}(\{k\})=-\mu_{K^*}(\emptyset)=-1$. Assume that, for $\tau\in K^{(p)}$ with $p<r$, $\mu_{K^*}(\tau)=(-1)^{r_K(\tau)}$. Let $\sigma\in K^{(r)}$. Since the set $\{\tau\in K\mid\tau\leqslant\sigma\}$ forms a simplex,

$$\mu_{K^*}(\sigma)=-\sum_{\emptyset\leqslant\tau<\sigma}(-1)^{r_K(\tau)}=-\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}=(-1)^n,$$

where $n=r_K(\sigma)$.

(2) If $\sigma\in\varphi^{-1}(X)^\circ$ with $\zeta_X\prec\sigma\prec\emptyset$, then $\{\tau\in\varphi^{-1}(X)^\circ\mid\zeta_X\prec\tau\preceq\sigma\}$ can be viewed as a simplex since $\eta\in\varphi^{-1}(X)$ when $\sigma<\eta<\tau$ with $\sigma,\tau\in\varphi^{-1}(X)$. If we view its atoms $\partial_k\zeta_X$, with $\partial_k\zeta_X\preceq\sigma$, as vertices, using the above argument in (1), we get the corresponding result $\mu_{\varphi^{-1}(X)^\circ}(\sigma)=(-1)^{r_\varphi(\sigma)}$. $\square$

Definition 2.12. For $X\in L$, let
(1) $\mu_L(X)=\mu_{L^*}(X)$,
(2) $\mu_K(X)=\sum_{\sigma\in\varphi^{-1}(X)}\mu_{K^*}(\sigma)$,
(3) $\mu_\varphi(X)=\mu_{\varphi^{-1}(X)^\circ}(\emptyset)$.

Lemma 2.13.
(1) $\mu_K(X)=\sum_{\sigma\in\varphi^{-1}(X)}(-1)^{r_K(\sigma)}$.
(2) $\mu_\varphi(X)=(-1)^{\dim K[X]}\sum_{\sigma\in\varphi^{-1}(X)}(-1)^{r_K(\sigma)}$.

Proof. (1) From Lemma 2.11 we have

$$\mu_K(X)=\sum_{\sigma\in\varphi^{-1}(X)}\mu_{K^*}(\sigma)=\sum_{\sigma\in\varphi^{-1}(X)}(-1)^{r_K(\sigma)}.$$

(2) Note that the set $\{\sigma\in\varphi^{-1}(X)^\circ\mid\zeta_X\preceq\sigma\prec\emptyset\}=\varphi^{-1}(X)$. From Lemma 2.11 we have $\mu_\varphi(X)=\mu_{\varphi^{-1}(X)^\circ}(\emptyset)=-\sum_{\zeta_X\preceq\sigma\prec\emptyset}\mu_{\varphi^{-1}(X)^\circ}(\sigma)=-\sum_{\sigma\in\varphi^{-1}(X)}(-1)^{r_K(\zeta_X)-r_K(\sigma)}=(-1)^{r_K(\zeta_X)-1}\sum_{\sigma\in\varphi^{-1}(X)}(-1)^{r_K(\sigma)}$. $\square$

Theorem 2.14. *Let* $\mu_K,\mu_L,\mu_\varphi:L\to\mathbb{Z}$ *be defined as above. Then* $\mu_K=\mu_L=\pm\mu_\varphi$.

Proof. Clearly, $|\mu_K| = |\mu_\varphi|$ by Lemma 2.13. We prove $\mu_K = \mu_L$ by induction on r_L. Let H be a hyperplane. Since $\varphi^{-1}(H)$ is trivial, $\mu_K(H) = \mu_L(H) = -1$. Assume that $\mu_K(Y) = \mu_L(Y)$ for all Y with $r_L(Y) < p$. Let $r_L(X) = p$. Since $K[X]$ is a simplex, $\sum_{\sigma \in K[X]} \mu_{K^*}(\sigma) = -\mu_{K^*}(\emptyset) = -1$. Thus

$$\begin{aligned}\mu_K(X) &= \sum_{\sigma \in \varphi^{-1}(X)} \mu_{K^*}(\sigma) = \sum_{\sigma \in K[X]} \mu_{K^*}(\sigma) - \sum_{\sigma \in K(X)} \mu_{K^*}(\sigma)\\ &= -1 - \sum_{\sigma \in K(X)} \mu_{K^*}(\sigma) = -1 - \sum_{Y<X} \mu_K(Y)\\ &= -1 - \sum_{Y<X} \mu_L(Y) = -\sum_{V \leqslant Y < X} \mu_{L^*}(Y) = \mu_L(X). \qquad \square\end{aligned}$$

In the rest of this paper we use $\mu(X)$ to denote $\mu_L(X) = \mu_K(X)$. In practice we use formula (1) of Lemma 2.13 which generally involves a smaller poset than the recursive definition.

2.3. Homology

Recall that $K[X]$ is a simplex and $K(X)$ is a subcomplex of $K[X]$ when viewed as subcomplexes of K. Moreover, $K[X] = K(X) \sqcup \varphi^{-1}(X)$ is a disjoint union when viewed as subposets of K. Since $\varphi^{-1}(X) = K[X] - K(X)$ as underlying set of a poset, we can capture the homological data for $\varphi^{-1}(X)$ by finding the relative homology of the pair $(K[X], K(X))$.

Definition 2.15. Let $\mathcal{P}$ be a poset. The *order complex* of $\mathcal{P}$, denoted by $\Delta(\mathcal{P})$, is the simplicial complex with vertex set $\mathcal{P}$ and whose simplexes span the vertices of linearly ordered subsets (chains) of $\mathcal{P}$.

The following theorem is proven in [6,12].

Theorem 2.16. *For any geometric lattice $\mathcal{P}$, with minimal element B, maximal element T, rank $r \geqslant 2$ and Möbius function $\mu_\mathcal{P}$, the reduced homology of the order complex $\Delta(\mathcal{P} \setminus \{T, B\})$ is given by*

$$\widetilde{H}^p\big(\Delta\big(\mathcal{P} \setminus \{T, B\}\big)\big) = \begin{cases} \mathbb{Z}^{|\mu_\mathcal{P}(T)|}, & \text{if } p = r-2,\\ 0, & \text{if } p \neq r-2.\end{cases}$$

Using Theorem 2.16 we can describe the homology groups of the pair $(K[X], K(X))$.

Theorem 2.17. *The relative homology of the pair $(K[X], K(X))$ is given by*

$$H_p\big(K[X], K(X)\big) = \begin{cases} \mathbb{Z}^{|\mu(X)|}, & \text{if } p = r_L(X) - 1,\\ 0, & \text{if } p \neq r_L(X) - 1.\end{cases}$$

Proof. If $\varphi^{-1}(X) = \{\zeta_X\}$ is trivial, then $K(X)$ is the boundary of the simplex $K[X]$ so we get

$$H_p\big(K[X], K(X)\big) = \begin{cases} \mathbb{Z}, & \text{if } p = \dim K[X], \\ 0, & \text{if } p \neq \dim K[X], \end{cases}$$

and $\dim K[X] = r_L(X) - 1$ by Lemma 2.5. If $\varphi^{-1}(X)$ is nontrivial, then the rank of $\varphi^{-1}(X)^\circ$ is greater than one. Since $K[X]$ is a simplex, the long exact sequence of the pair $(K[X], K(X))$ gives

$$H_p\big(K[X], K(X)\big) \cong \widetilde{H}_{p-1}\big(K(X)\big) \cong \widetilde{H}_{p-1}\big(\Delta\big(K(X)\big)\big),$$

where the second isomorphism comes from the fact that $\Delta(K(X))$ is the barycentric subdivision of $K(X)$. By Alexander duality,

$$H_p\big(K[X], K(X)\big) \cong \widetilde{H}^{\dim K[X]-p-1}\big(\Delta\big(\varphi^{-1}(X)^\circ \setminus \{\emptyset, \zeta_X\}\big)\big).$$

Using Theorem 2.16 with $|\mu(X)| = |\mu_{\varphi^{-1}(X)^\circ}(\emptyset)|$, we get

$$\begin{aligned} H_p\big(K[X], K(X)\big) &= \widetilde{H}^{r_K(\zeta_X)-p-2}\big(\Delta\big(\varphi^{-1}(X)^\circ \setminus \{\emptyset, \zeta_X\}\big)\big) \\ &= \begin{cases} \mathbb{Z}^{|\mu(X)|}, & \text{if } p = r_L(X) - 1, \\ 0, & \text{if } p \neq r_L(X) - 1. \end{cases} \end{aligned}$$

The relevant dimension is $r_K(\zeta_X) - p - 2 = r_{\varphi^{-1}(X)^\circ}(\emptyset) - 2$ so $p = r_K(\zeta_X) - r_{\varphi^{-1}(X)^\circ}(\emptyset) = \dim K[X] = r_L(X) - 1$. □

2.4. The no-broken-circuit set

The only contribution to the relative homology of the pair $(K[X], K(X))$ occurs in the row of minimal elements of $\varphi^{-1}(X)$ when ordered by inclusion. The number of generators in that dimension is given by the absolute value of the Möbius function which we can obtain in terms of local information from $\varphi^{-1}(X)$. We give an explicit basis for $H_{r_L(X)-1}(K[X], K(X)) = \mathbb{Z}^{|\mu(X)|}$ using the no-broken-circuit set, see [1,11]. For the first time we use the natural *linear order* in $\mathcal{A}$: $H_i < H_j$ if $i < j$. We replace the usual notion of a standard $(p+1)$-tuple $(H_{i_0}, \ldots, H_{i_p})$, $i_0 < \cdots < i_p$, with the p-simplex $\sigma = \{i_0, \ldots, i_p\}$.

Definition 2.18. Call $\sigma \in K^{(p)}$ *independent* if $r_L(H_\sigma) = p + 1$, and *dependent* if $r_L(H_\sigma) < p + 1$. Thus $\sigma \in \varphi^{-1}(X)$ is independent if and only if it is minimal in $\varphi^{-1}(X)$ when ordered by inclusion. We say that $\sigma \in \varphi^{-1}(X)$ is *semiminimal* if one of its faces $\partial_k\sigma$ is minimal in $\varphi^{-1}(X)$. We call $\sigma \in \varphi^{-1}(X)$ a *circuit* if it is semiminimal and all of its faces $\partial_k\sigma$, $0 \leqslant k \leqslant p+1$, are in $\varphi^{-1}(X)$. This agrees with the usual notion of a standard $(p+1)$-tuple being minimally dependent.

Note that by Lemma 2.6, in a nontrivial pullback, every dependent simplex contains circuits since they must contain semiminimal elements of a possibly lower nontrivial pullback where all of the pullbacks below that one are trivial.

Definition 2.19. We call $\sigma \in K^{(p)}$ a *broken circuit* if there exists some circuit $\tau \in K^{(p+1)}$ such that $\sigma = \partial_{p+1}\tau$. We call σ *χ-independent* if it does not contain any broken circuits. Note that χ-independence implies independence. For $X \in L$, we call

$$\mathbf{nbc}_X = \{\sigma \in \varphi^{-1}(X) \mid \sigma \text{ is } \chi\text{-independent}\}$$

the *no-broken-circuit set* of $\varphi^{-1}(X)$.

Theorem 2.20. *If $X \in L$, then $\mathbf{nbc}_X$ forms a basis for $H_{r_L(X)-1}(K[X], K(X))$.*

Proof. Let $p = r_L(X) - 1$. The set $K^{(p)} \cap \varphi^{-1}(X)$ forms a basis for the cycle group $Z_p(K[X], K(X))$ since $K^{(p-1)} \cap K[X] \subseteq K(X)$ by Lemma 2.5. Since $\sigma \in \mathbf{nbc}_X$ is χ-independent and hence independent, $\sigma \in K^{(p)}$ and is therefore a cycle in $Z_p(K[X], K(X))$.

Since χ-independence implies independence, every simplex of $K^{(p+1)} \cap \varphi^{-1}(X)$ contains a broken circuit and therefore one of its faces in $K^{(p)} \cap K[X]$ must also contain a broken circuit as well. Thus, any boundary in $B_p(K[X], K(X))$ has elements which contain broken circuits. The argument is completed by noting that $|\mathbf{nbc}_X| = |\mu(X)|$ by [2, Proposition 7.4.5]. □

Definition 2.21. The *no-broken-circuit set* is $\mathbf{nbc} = \bigcup_{X\in L} \mathbf{nbc}_X$. In our convention, $V \notin L$ and $\emptyset \notin \mathbf{nbc}$.

This provides the isomorphism

$$\iota : \bigoplus_{X\in L} \mathbb{Z}[\mathbf{nbc}_X] \to \bigoplus_{X\in L} \bigoplus_{p\geqslant 0} H_p\big(K[X], K(X)\big)$$

of the main diagram in the introduction.

We emphasize that $K(\mathcal{A}_\mathbb{R}) = K(\mathcal{A}_\mathbb{C})$ and $L(\mathcal{A}_\mathbb{R}) = L(\mathcal{A}_\mathbb{C})$ as posets with rank functions. Thus the pair $(K[X], K(X))$ and the sets $\mathbf{nbc}_X$, $\mathbf{nbc}$ are field independent.

2.5. Deletion and restriction

Next we need the properties of **nbc** under deletion and restriction [2, Proposition 7.4.5]. We agree to delete the minimal element.

Definition 2.22. Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a nonempty arrangement. Let $\mathcal{A}' = \mathcal{A} \setminus \{H_1\}$ be the deletion of the hyperplane H_1 from $\mathcal{A}$. Let $\mathcal{A}'' = \{H \cap H_1 \mid H \cap H_1 \neq \emptyset, H \in \mathcal{A}'\}$ be the restriction of $\mathcal{A}$ to the affine space H_1. The linear order in $\mathcal{A}'$ is inherited from $\mathcal{A}$. Define $\mathcal{A}'' \to \Sigma$ by $K \mapsto \max\{j \mid K \subset H_j\}$. This gives $\mathcal{A}''$ a linear order.

Note that 1 is not the label of any hyperplane in $\mathcal{A}'$ or $\mathcal{A}''$. Let $K(\mathcal{A}')$ be the nerve of $\mathcal{A}'$, $\mathbf{nbc}(\mathcal{A}')_X$ and $\mathbf{nbc}(\mathcal{A}')$ be the corresponding no-broken circuit sets of $\mathcal{A}'$. Similarly, let $K(\mathcal{A}'')$ be the nerve of $\mathcal{A}''$, $\mathbf{nbc}(\mathcal{A}'')_X$ and $\mathbf{nbc}(\mathcal{A}'')$ be the corresponding no-broken circuit sets of $\mathcal{A}''$.

Lemma 2.23. *Define* $\nu : \mathbf{nbc}(\mathcal{A}'') \cup \{\emptyset_{\mathcal{A}''}\} \to \mathbf{nbc}(\mathcal{A})$ *by* $\nu(\sigma) = \sigma \cup \{1\}$ *and* $\nu(\emptyset_{\mathcal{A}''}) = 1$. *There are disjoint unions*:

(1) $\mathbf{nbc}(\mathcal{A}) \cup \{\emptyset_{\mathcal{A}}\} = (\mathbf{nbc}(\mathcal{A}') \cup \{\emptyset_{\mathcal{A}'}\}) \sqcup \nu(\mathbf{nbc}(\mathcal{A}'') \cup \{\emptyset_{\mathcal{A}''}\})$.

(2) *For* $X \in L \setminus \{H_1\}$, $\mathbf{nbc}(\mathcal{A})_X = \mathbf{nbc}(\mathcal{A}')_X \sqcup \nu\mathbf{nbc}(\mathcal{A}'')_X$.

Proof. (1) Since $K(\mathcal{A}') = \{\sigma \in K(\mathcal{A}) \mid 1 \notin \sigma\}$, the set of circuits of $\mathcal{A}'$ is the same as the set of circuits of $\mathcal{A}$ that lie in $K(\mathcal{A}')$. Similarly, the set of broken circuits of $\mathcal{A}'$ is the same as the set of broken circuits of $\mathcal{A}$ that lie in $K(\mathcal{A}')$. Hence, $\mathbf{nbc}(\mathcal{A}') = \mathbf{nbc}(\mathcal{A}) \cap K(\mathcal{A}')$.

Let $\{1\} \cup \sigma$ contain a broken circuit in $\mathcal{A}$. Since broken circuits are generated by deleting the largest vertex from a circuit, a broken circuit in $\{1\} \cup \sigma$ is of the form $\{1\} \cup \tau$ where τ is a face of σ. If $\sigma \in K(\mathcal{A}'')$, then σ contains the broken circuit τ in $\mathcal{A}''$. Thus, $\nu\mathbf{nbc}(\mathcal{A}'') \subseteq \mathbf{nbc}(\mathcal{A}) \cap \nu K(\mathcal{A}'')$. By the same reasoning, if σ contains a broken circuit in $\mathcal{A}''$, then $\{1\} \cup \sigma$ contains a broken circuit in $\mathcal{A}$. Hence, $\nu\mathbf{nbc}(\mathcal{A}'') = \mathbf{nbc}(\mathcal{A}) \cap \nu K(\mathcal{A}'')$.

Since $\{1\}$ is a trivial equivalence class in $K(\mathcal{A})$, $\{1\} \in \mathbf{nbc}(\mathcal{A})$ and corresponds with $\nu(\emptyset_{\mathcal{A}''})$. Let $\{1\} \cup \sigma \in \mathbf{nbc}(\mathcal{A}) \setminus K(\mathcal{A}')$ with $\sigma \neq \emptyset$. Assume that $\sigma \notin K(\mathcal{A}'')$. Then there is an index $j \in \sigma$ with $\{j\} \notin K(\mathcal{A}'')$. Since $\{j\} \notin K(\mathcal{A}'')$, there is another index k, $k > j$, with $\{k\} \in K(\mathcal{A}'')$ and $X = H_k \cap H_1 = H_j \cap H_1$. However, since $X = H_{\{j,k\}} = H_{\{1,j,k\}}$ as well, $\{1, j, k\}$ is a circuit in $\mathcal{A}$, and $\{1, j\}$ is a broken circuit contradicting the fact that $\{1\} \cup \sigma \in \mathbf{nbc}(\mathcal{A})$. Thus, $\mathbf{nbc}(\mathcal{A}) \setminus K(\mathcal{A}') \subseteq \nu K(\mathcal{A}'') \cup \nu(\emptyset_{\mathcal{A}''})$. Since $\emptyset_{\mathcal{A}}$ corresponds with $\emptyset_{\mathcal{A}'}$, the result for (1) follows.

The result for (2) follows from (1) intersecting each no-broken-circuit set with $\varphi^{-1}(X)$ for $X \in L \setminus \{H_1\}$. □

3. Topology

3.1. The spectral sequence

The generalized Mayer–Vietoris spectral sequence is used to find the homology of the union of a finite collection of spaces using the intersections of those spaces and the nerve of the collection. For a general description of this spectral sequence, see [5]. Since affine subspaces intersect in affine subspaces, the generalized Mayer–Vietoris spectral sequence is an ideal tool to find the homology groups of the union of a subspace arrangement, which is a finite collection of affine subspaces in a vector space V. Since hyperplane arrangements are special cases of subspace arrangements, we can say a bit more about the generalized Mayer–Vietoris spectral sequence for hyperplane arrangements. Recall that $\widehat{X}$ is the one-point compactification of X.

Definition 3.1. Let $\{C_{p,q}, \partial, \delta\}$ be the double complex defined by

$$C_{p,q} = \bigoplus_{\sigma \in K^{(p)}} \widetilde{C}_q(\widehat{H}_\sigma)$$

with boundary maps: $\delta : C_{p,q} \to C_{p,q-1}$ defined by the boundary map on $\widehat{H}_\sigma$ given by

$\delta : \widetilde{C}_q(\widehat{H}_\sigma) \to \widetilde{C}_{q-1}(\widehat{H}_\sigma)$, and $\partial : C_{p,q} \to C_{p-1,q}$ induced by the boundary map on σ given by $\partial(\sigma) = \sum_{k=1}^{p}(-1)^k \partial_k \sigma$.

The first two terms of the spectral sequence are general results for the generalized Mayer–Vietoris spectral sequence. In [8], the E^2 term was shown to be precisely the homology of combinatorial data described in Theorem 2.17. The following lemma summarizes these results for the generalized Mayer–Vietoris spectral sequence for subspace arrangements.

Lemma 3.2. *Let $\mathcal{A} = \{H_k\}_{k=1}^n$ be a subspace arrangement with nerve K and intersection poset L. The first three terms of the generalized Mayer–Vietoris spectral sequence are given below*:

$$E^0_{p,q} = \bigoplus_{\sigma \in K^{(p)}} \widetilde{C}_q\big(\widehat{H}_\sigma\big), \quad \text{with } d^0 = \delta,$$

$$E^1_{p,q} = \bigoplus_{\sigma \in K^{(p)}} \widetilde{H}_q\big(\widehat{H}_\sigma\big), \quad \text{with } d^1 = \partial_*, \text{ and}$$

$$E^2_{p,q} \cong \bigoplus_{\substack{X \in L \\ \dim_{\mathbb{R}} X = q}} H_p\big(K[X], K(X)\big).$$

The last isomorphism in this lemma gives rise to the isomorphisms $\rho_{\mathbb{R}}$ and $\rho_{\mathbb{C}}$ in the main diagram in the introduction. The differential map

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

is defined by the next lemma, proven in [5].

Lemma 3.3. *For an element $c_0 \in C_{p,q}$ and $r > 0$, $d^r[c_0] = 0$ if and only if $\delta(c_0) = 0$ and there exist elements $c_k \in C_{p-k,q+k}$, $1 \leqslant k \leqslant r$, such that $\partial(c_{k-1}) = (-1)^{p-k+1}\delta(c_k)$. In particular, $d^{r+1}[c_0] = d^{r+1}[c_0 + \cdots + c_r] = [\partial(c_r)]$.*

Definition 3.4. We define *capping* as a particular choice of elements $c_k \in C_{p-k,q+k}$, $1 \leqslant k \leqslant r$, satisfying Lemma 3.3.

Next we turn to the maps $d_{\mathbb{R}}$ and $d_{\mathbb{C}}$ of the main diagram induced by the differential maps d^r of the spectral sequence. In [8], it was shown that the generalized Mayer–Vietoris spectral sequence for subspace arrangements collapses at the second term, $E^2_{p,q} = E^\infty_{p,q}$, and that there is a direct isomorphism relating the E^2 term and the reduced homology of the one point compactification of the union of the arrangement, $\widetilde{H}_*(\widehat{N})$, described in the following lemma. The geometric insight into why the spectral sequence collapses at the second term involves representing each c_k as cycles of spheres, choosing a b_{k+1} consisting of bounding hemispheres, hence the term capping, for the spheres of c_k, so that

$\delta(b_{k+1}) = c_k$, and such that $\partial(b_{k+1})$ is also a cycle of spheres of one higher dimension, extending the argument one step further.

Lemma 3.5. *Let $\mathcal{A}$ be a subspace arrangement. The map $d : \bigoplus_{p+q=n} E^2_{p,q} \to \widetilde{H}_n(\widehat{N})$ defined by $d[c_0] = d^{p+1}[c_0] = [\partial(c_p)]$ for $[c_0] \in E^2_{p,q}$ is an isomorphism.*

We specialize from the case of subspace arrangements to the case when $\mathcal{A}_{\mathbb{R}}$ is a real hyperplane arrangement and $\mathcal{A}_{\mathbb{C}}$ its complexification. Recall that $L(\mathcal{A}_{\mathbb{R}}) = L(\mathcal{A}_{\mathbb{C}})$ as ranked geometric posets. Dependence on the field enters here. If we let L_q denote the elements of L of rank $\ell - q$, then $X \in L_q$ has real dimension q viewed as a subspace of the real arrangement and real dimension $2q$ viewed as a subspace of the complexified arrangement.

Corollary 3.6. *Let $\mathcal{A}_{\mathbb{R}}$ be a real hyperplane arrangement in $\mathbb{R}^\ell$, and let $\mathcal{A}_{\mathbb{C}}$ be its complexification. Then the isomorphisms $d_{\mathbb{R}}$ and $d_{\mathbb{C}}$ of the main diagram are given as*

$$d_{\mathbb{R}} : \bigoplus_{p+q=n} E^2_{p,q}(\mathbb{R}) \to \widetilde{H}_n\big(\widehat{N}_{\mathbb{R}}\big), \qquad d_{\mathbb{C}} : \bigoplus_{p+2q=n} E^2_{p,2q}(\mathbb{C}) \to \widetilde{H}_n\big(\widehat{N}_{\mathbb{C}}\big).$$

In the complex case, nonzero groups can occur in different dimensions. In the real case, the only nonzero group occurs when $m = \ell - 1$. The maps $a_{\mathbb{R}}$ and $a_{\mathbb{C}}$ of the main diagram are defined by Alexander duality. It remains to describe the isomorphisms α and β. These isomorphisms depend upon the particular choices made in Lemma 3.3. Additional information about the capping is needed for a real arrangement and its complexification. This allows the definition of α and β described in the next two sections.

3.2. Real arrangements

Our first aim is to use **nbc** to construct an explicit basis for $\widetilde{H}_{\ell-1}(\widehat{N}_{\mathbb{R}})$. The complement $M_{\mathbb{R}}$ is the union of disjoint open subsets of $\mathbb{R}^\ell$, called chambers. Let $\mathrm{Ch}(\mathcal{A}_{\mathbb{R}})$ denote the set of chambers of $M_{\mathbb{R}}$.

Definition 3.7. Choose a chamber $c_\emptyset$ in $M_{\mathbb{R}}$ and label it with $\emptyset$, we call this chamber the *empty chamber*. Let $\widetilde{\mathrm{Ch}}(\mathcal{A}_{\mathbb{R}}) = \mathrm{Ch}(\mathcal{A}_{\mathbb{R}}) \setminus \{c_\emptyset\}$ be the *reduced set of chambers*. For each hyperplane $H_k \in \mathcal{A}_{\mathbb{R}}$, $\mathbb{R}^\ell - H_k$ consists of two open half-spaces. Label the open half-space H_k^- if it contains $c_\emptyset$, and label the other open half-space H_k^+. This associates a unique n-tuple with each chamber $c \in \mathrm{Ch}(\mathcal{A}_{\mathbb{R}})$ with entries in the set $J = \{+, -\}$. We agree to identify c with this unique n-tuple. Let $H_\sigma^+ = \bigcap_{k\in\sigma} H_k^+$, and let $\overline{H}_\sigma^+$ be its closure.

The object we have defined, $\mathrm{Ch}(\mathcal{A}_{\mathbb{R}})$, under this association, is a subset of covectors of the oriented matroid of the arrangement [3]. The next lemma shows that it is possible in

Lemma 3.3 to choose the caps so that the underlying set of the homology class for σ is the boundary of H_σ^+ in $\widehat{\mathbb{R}}^\ell$. This makes the map $d_\mathbb{R}$ geometrically explicit.

Lemma 3.8. *For each simplex $\sigma \in \mathbf{nbc}$, the caps can be chosen iteratively such that the homology class $[d_\mathbb{R} \circ \rho_\mathbb{R} \circ \iota(\sigma)] \in \widetilde{H}_{\ell-1}(\widehat{N}_\mathbb{R})$ has as its underlying set $(\overline{H}_\sigma^+ \setminus H_\sigma^+) \cup \infty$.*

Proof. Let $|\sigma| = m$. The homology class $[\rho_\mathbb{R} \circ \iota(\sigma)] \in E^2_{m-1,\ell-m}(\mathbb{R})$ has as its underlying set $\widehat{H}_\sigma$. The underlying sets of the caps c_k are chosen iteratively to be

$$\Bigg(\bigcup_{\substack{\tau\subseteq\sigma\\ m-k\leqslant|\tau|\leqslant m}} \big(H_\tau \cap \overline{H}_\sigma^+\big)\Bigg) \cup \infty = \Bigg(\Bigg(\bigcup_{\substack{\tau\subseteq\sigma\\ |\tau|=m-k}} H_\tau\Bigg) \cap \overline{H}_\sigma^+\Bigg) \cup \infty.$$

Hence, the underlying set of the homology class $[d_\mathbb{R} \circ \rho_\mathbb{R} \circ \iota(\sigma)] = [\partial(c_{m-1})]$ is $((\cup_{k\in\sigma} H_k) \cap H_\sigma^+) \cup \infty = (\overline{H}_\sigma^+ \setminus H_\sigma^+) \cup \infty$. □

Using Lemma 3.8, making the choice of the empty chamber, $c_\emptyset$, provides a well-defined capping sequence, which in turn gives an explicit basis for $\widetilde{H}_{\ell-1}(\widehat{N}_\mathbb{R})$ bijective with **nbc**.

Definition 3.9. For a chamber $c \in \widetilde{\mathrm{Ch}}(\mathcal{A}_\mathbb{R})$, let σ_c be the simplex spanning the vertices $k \in \Sigma = \{1,\ldots,n\}$ where the kth coordinate of the n-tuple associated with c is a "$+$". We call σ_c the coefficient simplex of the chamber c. Thus $\mathrm{Ch}(\mathcal{A}_\mathbb{R})$ is also the collection of all the σ_c's associated with chambers $c \in \widetilde{\mathrm{Ch}}(\mathcal{A}_\mathbb{R})$.

Finally, we use Alexander duality to provide a bijection of **nbc** with an explicit basis for $\widetilde{H}^0(M_\mathbb{R})$, where each **nbc** element is assigned to a linear combination of chambers in $\widetilde{\mathrm{Ch}}(\mathcal{A}_\mathbb{R})$.

Lemma 3.10. *For each simplex $\sigma \in \mathbf{nbc}$, the cohomology class $[a_\mathbb{R} \circ d_\mathbb{R} \circ \rho_\mathbb{R} \circ \iota(\sigma))] \in \widetilde{H}^0(M_\mathbb{R})$ is defined over the union of chambers of $\mathcal{A}$ which lie in H_σ^+, namely, $\bigcup\{c \in \widetilde{\mathrm{Ch}}(\mathcal{A}_\mathbb{R}) \mid \sigma \subseteq \sigma_c\}$.*

Proof. The result follows from Lemma 3.8 using Alexander duality and the fact that $M_\mathbb{R} \cap \overline{H}_\mathbb{R}^+ = \bigcup\{c \in \widetilde{\mathrm{Ch}}(\mathcal{A}_\mathbb{R}) \mid \sigma \subseteq \sigma_c\}$. □

Recall that we labeled the distinguished chamber $c_\emptyset$ by the empty set, which is not in **nbc** by our convention. We now have a natural labeling of $\widetilde{\mathrm{Ch}}(\mathcal{A}_\mathbb{R})$ using **nbc**, where a simplex $\sigma \in \mathbf{nbc}$ is associated with a union of chambers in $M_\mathbb{R}$. This is a bijection between **nbc** and $\widetilde{\mathrm{Ch}}(\mathcal{A}_\mathbb{R})$ because the maps $a_\mathbb{R}, d_\mathbb{R}, \rho_\mathbb{R}, \iota$ are isomorphisms. The following incidence matrix indicates the labeling of **nbc** with unions of chambers in $\widetilde{\mathrm{Ch}}(\mathcal{A}_\mathbb{R})$.

Definition 3.11. We order **nbc** and $\mathrm{Ch}(\mathcal{A}_\mathbb{R})$ lexicographically. Let $m = |\mathbf{nbc}|$. Let σ_j be the jth **nbc** element in the reverse lexicographic order for $j < m$. Let $(\sigma_c)_k$ be the kth chamber label in the lexicographic order. Let A be the $m \times m$ matrix defined by

$$a_{j,k} = \begin{cases} 1, & \text{if } \sigma_j \subseteq (\sigma_c)_k, \\ 0, & \text{if } \sigma_j \not\subseteq (\sigma_c)_k. \end{cases}$$

Note that for the **nbc** element σ_j and the jth row of A, $\{(\sigma_c)_k \mid a_{j,k} \neq 0\} = \{\sigma_c \mid \sigma_j \subseteq \sigma_c\}$. Since $H_\sigma^+ \cap M_{\mathbb{R}} = \bigcup\{c \in \widetilde{\mathrm{Ch}}(\mathcal{A}_{\mathbb{R}}) \mid \sigma \subseteq \sigma_c\}$ and $\{[H_\sigma^+ \cap M_{\mathbb{R}}] \mid \sigma \in \mathbf{nbc}\}$ forms a basis for $\widetilde{H}^0(M_{\mathbb{R}})$, the matrix A is nonsingular.

3.3. Complexified real arrangements

Next we relate the homology of the one-point compactification of the union of a real hyperplane arrangement to that of its complexification. There is a natural embedding of $\mathbb{R}$ in $\mathbb{C}$ by mapping $x \mapsto x + 0\mathrm{i}$. Under this embedding, $H_{\mathbb{R}}$ is embedded in $H_{\mathbb{C}}$, and the union of the real hyperplane arrangement $N_{\mathbb{R}}$ is embedded in the union of the complexified real arrangement $N_{\mathbb{C}}$.

Let $\sigma \in K^{(p)}$ be an element of the $\mathbf{nbc}_X$ basis for $H_p(K[X], K(X))$. Here $X = H_\sigma$ is independent of the field as an element of L, but $[\widehat{H}_\sigma^{\mathbb{R}}]$ is a generator of the $E^2_{p,q}(\mathbb{R})$ term with $\dim_{\mathbb{R}}(H_\sigma^{\mathbb{R}}) = q$ and $[\widehat{H}_\sigma^{\mathbb{C}}]$ is a generator of the $E^2_{p,2q}(\mathbb{C})$ term with $\dim_{\mathbb{R}}(H_\sigma^{\mathbb{C}}) = 2q$. Define the sets $(dH)_\sigma^{\mathbb{R}}$ and $(dH)_\sigma^{\mathbb{C}}$ by the conditions that $d_{\mathbb{R}}[\widehat{H}_\sigma^{\mathbb{R}}]$ has $(dH)_\sigma^{\mathbb{R}} \cup \infty$ as its underlying set and $d_{\mathbb{C}}[\widehat{H}_\sigma^{\mathbb{C}}]$ has $(dH)_\sigma^{\mathbb{C}} \cup \infty$ as its underlying set.

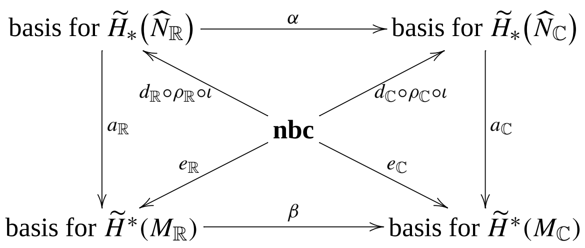
The next lemma shows that it is possible in Lemma 3.3 to choose the caps for $d_{\mathbb{C}}[\widehat{H}_\sigma^{\mathbb{C}}]$ consistently with the caps chosen for $d_{\mathbb{R}}[\widehat{H}_\sigma^{\mathbb{R}}]$.

Lemma 3.12. *For the element $\sigma \in K^{(p)}$ of the $\mathbf{nbc}_X$ basis for $H_p(K[X], K(X))$, the caps for $d_{\mathbb{C}}[\widehat{H}_\sigma^{\mathbb{C}}]$ can be chosen consistently with the caps chosen for $d_{\mathbb{R}}[\widehat{H}_\sigma^{\mathbb{R}}]$ such that $(dH)_\sigma^{\mathbb{C}} = (dH)_\sigma^{\mathbb{R}} \oplus \mathrm{i}H_\sigma^{\mathbb{R}}$.*

Proof. Choose the caps $c_k^{\mathbb{R}} \in C_{p-k,q+k}$, $1 \leqslant k \leqslant p$, for $d_{\mathbb{R}}[\widehat{H}_\sigma^{\mathbb{R}}]$, as in Lemma 3.3 such that $\partial(c_{k-1}^{\mathbb{R}}) = (-1)^{p-k+1}\delta(c_k^{\mathbb{R}})$. Let $c_k^{\mathbb{C}} \in C_{p-k,2q+k}$, $1 \leqslant k \leqslant p$, be chosen for $d[\widehat{H}_\sigma^{\mathbb{C}}]$, such that the real portion of the underlying set for each cell of $c_k^{\mathbb{C}}$ is the underlying set for the corresponding cell of $c_k^{\mathbb{R}}$ and the imaginary part of each cell is $\mathrm{i}H_\sigma^{\mathbb{R}}$. Clearly, $\partial(c_{k-1}^{\mathbb{C}}) = (-1)^{p-k+1}\delta(c_k^{\mathbb{C}})$, and the result follows. □

This proves the main assertion of the Introduction.

Theorem 3.13. *Define $\alpha(d_{\mathbb{R}}[\widehat{H}_\sigma^{\mathbb{R}}]) = d_{\mathbb{C}}[\widehat{H}_\sigma^{\mathbb{C}}]$ for $\sigma \in$ **nbc**. If $\dim(\sigma) = p$, then α takes an $(\ell - 1)$-dimensional generator to a $(2(\ell - 1) - p)$-dimensional generator. It follows from Lemma* 3.12 *that the maps in the top triangle are commuting bijections. The vertical maps are bijections induced by Alexander duality. Define the remaining maps to be bijections to make the respective parts of the diagram commute.*



3.4. Another chamber labeling

Recall that the composition $a_{\mathbb{R}} \circ d_{\mathbb{R}} \circ \rho_{\mathbb{R}} \circ \iota$ assigns to each element of **nbc** a linear combination of chambers in $\widetilde{H}^0(M_{\mathbb{R}})$. Thus it is not an obvious bijection. Next we offer a natural labeling of the set of chambers using **nbc**, where a simplex $\sigma \in \mathbf{nbc}$ is associated with an individual chamber in $M_{\mathbb{R}}$ inside the open halfspace H_σ^+ exhibiting the bijection.

Lemma 3.14. *There is a bijection* $\pi : \mathrm{Ch}(\mathcal{A}_{\mathbb{R}}) \to \mathbf{nbc} \cup \{\emptyset\}$ *such that if* $\pi(c) = \sigma$ *then* $c \subseteq H_\sigma^+$.

Proof. We argue by induction on $|\mathcal{A}|$. If $|\mathcal{A}| = 0$, then $\mathrm{Ch}(\mathcal{A}_{\mathbb{R}}) = \{\mathbb{R}^\ell\} = \{c_\emptyset\}$ and $\pi(c_\emptyset) = \emptyset$. Let $|\mathcal{A}| \geqslant 1$. We use deletion and restriction of Definition 2.22. The oriented half spaces in $\mathcal{A}$ determine the orientations of the half spaces in $\mathcal{A}'$ and $\mathcal{A}''$. The chamber labeled by $\emptyset$ determines unique chambers in $\mathcal{A}'$ and $\mathcal{A}''$ to be labeled by $\emptyset$. By the induction hypothesis, we have bijections $\pi' : \mathrm{Ch}(\mathcal{A}') \to \mathbf{nbc}(\mathcal{A}') \cup \{\emptyset'\}$ and $\pi'' : \mathrm{Ch}(\mathcal{A}'') \to \mathbf{nbc}(\mathcal{A}'') \cup \{\emptyset''\}$ satisfying the respective conditions. Let $c \in \mathrm{Ch}(\mathcal{A}')$.

(1) If $c \cap H_1 = \emptyset$, then $c \in \mathrm{Ch}(\mathcal{A})$. Let $\pi(c) = \pi'(c)$. This is well defined by Lemma 2.23.

(2) Otherwise, let $c'' = c \cap H_1 \neq \emptyset$. Then $c \setminus H_1$ is the union of two chambers, $c^+ = c \cap H_1^+$ and $c^- = c \cap H_1^-$, with $c^+, c^- \in \mathrm{Ch}(\mathcal{A}_{\mathbb{R}})$. Define $\pi(c^-) = \pi'(c)$ and $\pi(c^+) = \pi''(c'') \cup \{1\}$. This is well defined by Lemma 2.23. Thus c^- inherits its label from $c \in \mathrm{Ch}(\mathcal{A}')$ and c^+ inherits its label from $c'' \in \mathrm{Ch}(\mathcal{A}'')$ with the extra vertex 1 added on.

It remains to show that if $\pi(c) = \sigma$, then $c \subseteq H_\sigma^+$. In case (1), let $c \in \mathrm{Ch}(\mathcal{A}')$ and $\pi'(c) = \sigma \in \mathbf{nbc}(\mathcal{A}')$. By the induction hypothesis, $c \subseteq H_\sigma^+$. In case (2), let $c \in \mathrm{Ch}(\mathcal{A}')$ and $\pi'(c) = \sigma \in \mathbf{nbc}(\mathcal{A}')$. By the induction hypothesis, $c \subseteq H_\sigma^+$. For $c^- \in \mathrm{Ch}(\mathcal{A}_{\mathbb{R}})$, $c^- \subset c \subseteq H_\sigma^+ \cap H_1^- \subseteq H_\sigma^+$. Since $\pi(c^-) = \pi'(c) = \sigma$, the assertion holds for c^-. Finally, consider c^+. Let $c'' \in \mathrm{Ch}(\mathcal{A}'')$ and $\pi''(c'') = \sigma'' \in \mathbf{nbc}(\mathcal{A}'')$. By the induction hypothesis, $c'' \subseteq H_{\sigma''}^+$. Since $\pi(c^+) = \sigma'' \cup \{1\}$ and $c^+ \subseteq H_{\sigma''}^+ \cap H_1^+ = H_{\sigma'' \cup \{1\}}^+$, the condition also holds for c^+. $\square$

4. Example

Let $\mathcal{A}$ denote the 3-arrangement with five hyperplanes

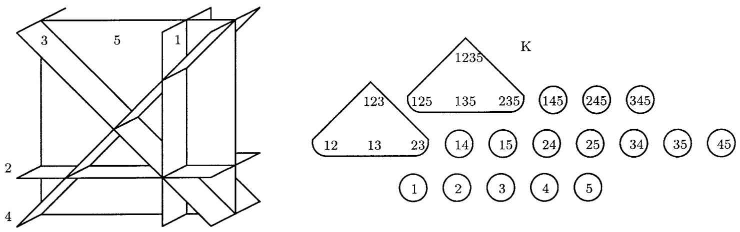


Fig. 2.

$$H_1 = \{y = 0\}, \quad H_2 = \{z = 0\}, \quad H_3 = \{y + z = 0\},$$
$$H_4 = \{y - z + 1 = 0\} \quad \text{and} \quad H_5 = \{x = 0\}.$$

Choose the empty chamber $c_\emptyset$ to be the chamber containing the point $(-1, -\frac{1}{2}, \frac{1}{4})$. The nerve K is given in Fig. 2. Each bubble represents an equivalence class.

The simplex 123 is a circuit and 12 is a broken circuit. Thus

$$\mathbf{nbc} = \{1, 2, 3, 4, 5, 13, 14, 15, 23, 24, 25, 34, 35, 45, 135, 145, 235, 245, 345\}.$$

is the no-broken-circuit set in lexicographic order. The generalized Mayer–Vietoris spectral sequences for the real and complexified real hyperplane arrangement are given below along with the reduced homology of the one-point compactification of the union.

$E^2_{p,q}(\mathbb{R})$

4			
3			
2	$\mathbb{Z}^5$		
1		$\mathbb{Z}^9$	
0			$\mathbb{Z}^5$
	0	1	2

$E^2_{p,q}(\mathbb{C})$

4	$\mathbb{Z}^5$		
3			
2		$\mathbb{Z}^9$	
1			
0			$\mathbb{Z}^5$
	0	1	2

$$\widetilde{H}_m(\widehat{N}_{\mathbb{R}}) = \begin{cases} \mathbb{Z}^{19}, & \text{if } m = 2, \\ 0, & \text{if } m \neq 2, \end{cases} \qquad \widetilde{H}_m(\widehat{N}_{\mathbb{C}}) = \begin{cases} \mathbb{Z}^5, & \text{if } m = 2, \\ \mathbb{Z}^9, & \text{if } m = 3, \\ \mathbb{Z}^5, & \text{if } m = 4, \\ 0, & \text{if } m \neq 2, 3, 4. \end{cases}$$

Table 1
Incidence matrix

	2	3	4	5	12	13	24	25	34	35	45	123	125	134	135	245	345	1235	1345
345	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
245	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
235	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
145	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
135	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1
45	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	1	0	1
35	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	1	1	1
34	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1
25	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	0
24	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0
15	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	0	1	1
14	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1
13	0	0	0	0	0	1	0	0	0	0	0	1	0	1	1	0	0	1	1
5	0	0	0	1	0	0	0	1	0	1	1	0	1	0	1	1	1	1	1
4	0	0	1	0	0	0	1	0	1	0	1	0	0	1	0	1	1	0	1
3	0	1	0	0	0	1	0	0	1	1	0	1	0	1	1	0	1	1	1
2	1	0	0	0	1	0	1	1	0	0	0	1	1	0	0	1	0	1	0
1	0	0	0	0	1	1	0	0	0	0	0	1	1	1	1	0	0	1	1

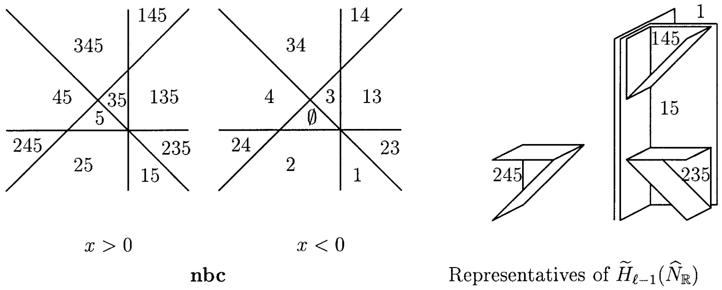


Fig. 3.

The incidence matrix of Definition 3.11 is given in Table 1. The rows are labeled by **nbc** and the columns by $\mathrm{Ch}(\mathcal{A}_{\mathbb{R}})$. The coefficient simplexes σ_c in $\mathrm{Ch}(\mathcal{A}_{\mathbb{R}})$, the labeling of the individual chambers by **nbc** in Lemma 3.14, and selected representatives of $\widetilde{H}_{\ell-1}(\widehat{N}_{\mathbb{R}})$ are given in Fig. 3.

Acknowledgements

We thank the referee for many helpful suggestions.

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