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The mixed Hodge structure on the fundamental group of a complement of hyperplanes

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Dedicated to the memory of Professor Nobuo Sasakura

Abstract

The complement of an arrangement of hyperplanes is a good example of a mixed Hodge structure on the fundamental group of an algebraic variety. We compute its isomorphism class using iterated integrals and then show that the cross ratio of a arrangement is a combinatorial and projective invariant. So cross ratio equivalent arrangements are isomorphic. Moreover, an isomorphism of polarized mixed Hodge structures on the fundamental group induces cross ratio equivalent arrangements. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let V be an algebraic variety over $\mathbb{C}$. We fix a point x of V and consider the truncation

$$\mathbb{Z}\pi_1(V,x)/J^{s+1}$$

of the group algebra of the fundamental group $\pi_1(V,x)$ over $\mathbb{Z}$ by some power of its augmentation ideal J. Its dual space is isomorphic to the space

$$H^0\bigl(B_s(V),x\bigr)$$

of iterated integrals with length $\leqslant s$ that are homotopy functionals. Using the bar construction of iterated integrals, the Hodge and weight filtration on $H^0(B_s(V),x)$

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can be defined. Then $\mathbb{Z}\pi_1(V,x)/J^{s+1}$ has a mixed Hodge structure (cf. Hain [7] and Morgan [14]).

In this paper we compute invariants of mixed Hodge structures on the fundamental group of a complement of hyperplanes in complex projective space (see [10]). Furthermore we study what can be implied by isomorphisms of mixed Hodge structures. For example, if $V = \mathbb{P}^1 - \{a_1, \dots, a_n\}$ then it means that they are biholomorphic (see [7]).

A finite set $\mathcal{A}$ of hyperplanes in the N-dimensional complex projective space $\mathbb{P}^N$ is called an arrangement of hyperplanes. Let b be a base point of its complement $M(\mathcal{A}) = \mathbb{P}^N - \bigcup_{H\in\mathcal{A}} H$. We call $(\mathcal{A}, b)$ a pointed arrangement. Denote by $L_p(\mathcal{A})$ the set of nonempty intersections of elements of $\mathcal{A}$ whose codimension is p. For 3 hyperplanes H_i $(i = 1,2,3)$ with $\operatorname{codim}(H_1 \cap H_2 \cap H_3) = 2$, there is a unique projective line $l \cong \mathbb{P}^1$ in the dual projective space $(\mathbb{P}^N)^*$ which passes through 3 dual points associated to H_i. The cross ratio for the above H_i $(i = 1,2,3)$ is defined by the usual cross ratio in l of 3 dual points and the point which is the intersection of the line l and a dual hyperplane associated to b. Two pointed arrangements will be called *cross ratio equivalent* if they have same L_p $(p = 1,2)$ and their cross ratios coincide (see Definitions 9 and 10 in detail).

The main result in this paper is the following.

Main Theorem (Corollary 19 and Theorem 25). *Let $(\mathcal{A}, b)$, $(\mathcal{A}', b')$ be two pointed arrangements of hyperplanes. There is a ring isomorphism*

$$\varphi : \mathbb{Z}\pi_1\big(M(\mathcal{A}), b\big)/J^3 \to \mathbb{Z}\pi_1\big(M\big(\mathcal{A}'\big), b'\big)/J^3$$

which induces an isomorphism of mixed Hodge structures and preserve the polarization on $(J/J^2)^*$, if and only if, $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$ are cross ratio equivalent.

The outline of the proof is as follows. We consider the mixed Hodge structure on $\mathbb{Z}\pi_1(M,b)/J^{s+1}$. First there is an exact sequence

$$0 \to H^1(M) \to \operatorname{Hom}\big(J/J^3, \mathbb{Z}\big) \to K \to 0$$

of mixed Hodge structures, where K is the kernel of the cup product $H^1 \otimes H^1 \to H^2$. This exact sequence gives a extension of Hodge structures. Due to the extension theory of mixed Hodge structures (Carlson [2]), the set of congruence classes of extensions of K by H^1 forms an Abelian group $\operatorname{Ext}(K, H^1)$, and there is an Abelian group isomorphism

$$\psi : \operatorname{Ext}\big(K, H^1\big) \to \operatorname{Hom}\big(K, H^1\big)_{\mathbb{C}} / \operatorname{Hom}\big(K, H^1\big)_{\mathbb{Z}}.$$

We shall choose suitable bases of each H^1 and K, and give a concrete description of $\psi((J(M(\mathcal{A}, b)/J^3)^*)$. Its isomorphism class depends only on cross ratios of arrangements. Therefore if two pointed arrangements are cross ratio equivalent, then mixed Hodge structures on their fundamental groups are isomorphic (see Theorem 17 and Corollary 19).

Next we can define the polarization of $H^1(M(\mathcal{A}), \mathbb{Z}) \cong (J/J^2)^*$. For two pointed arrangements $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$, if there is a isomorphism of first cohomologies that preserve the polarization then it transfers one's canonical basis to the other's canonical basis, using properties of root systems of type A_n. This implies the following. If a ring isomorphism

$$\varphi : \mathbb{Z}\pi_1\big(M(\mathcal{A}), b\big)/J^3 \to \mathbb{Z}\pi_1\big(M\big(\mathcal{A}'\big), b'\big)/J^3$$

induces an isomorphism of mixed Hodge structures and preserves the polarization on $(J/J^2)^* \cong H^1$, then $\mathcal{A}$ and $\mathcal{A}'$ have the same intersection set L_2. Therefore they have the same H^1, K and $\mathrm{Hom}(K, H^1)_{\mathbb{Z}}$. Also the isomorphism of J/J^3 induces the one of $\psi((J/J^3)^*)$, and then their cross ratios coincide. This implies Main Theorem (Theorem 25).

2. The mixed Hodge structure on π_1 for arrangements

2.1. Extensions of mixed Hodge structures

First we review the extension of mixed Hodge structures (cf. [2,7]).

An *extension of Hodge structures* is an exact sequence

$$0 \to A \to E \to B \to 0$$

of mixed Hodge structures. Two extensions

$$0 \to A \to E_j \to B \to 0, \quad j = 1, 2,$$

are *congruent* if there is an isomorphism of mixed Hodge structures $\Phi : E_1 \to E_2$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Phi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & A & \longrightarrow & E_2 & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

commutes. The set of congruence classes of extensions of B by A forms an Abelian group that we shall denote by $\mathrm{Ext}(B, A)$.

Lemma 1 (Carlson [2]). *We assume that A and B are pure Hodge structures of weight m and n, respectively, with $n > m$. There is an Abelian group isomorphism*

$$\psi : \mathrm{Ext}(B, A) \to \frac{\mathrm{Hom}(B, A)_{\mathbb{C}}}{F^0 \mathrm{Hom}(B, A)_{\mathbb{C}} + \mathrm{Hom}(B, A)_{\mathbb{Z}}}.$$

We can express the mixed Hodge structure on $(J/J^3)^*$ as an extension. We shall denote the integration value of an iterated integral I on a path γ by $\langle I, \gamma\rangle$.

Lemma 2 [7]. *Suppose that (X, x) is a path connected, pointed topological space. If $H_1(X, \mathbb{Z})$ is torsion-free, then there is an exact sequence*

$$0 \to H^1_{\mathbb{Z}}(X) \xrightarrow{i} \mathrm{Hom}_{\mathbb{Z}}\big(J(X, x)/J^3, \mathbb{Z}\big) \xrightarrow{p} K_{\mathbb{Z}}(X) \to 0.$$

Here $K_{\mathbb{Z}}(X)$ is the kernel of the cup product $H^1_{\mathbb{Z}}(X) \otimes H^1_{\mathbb{Z}}(X) \to H^2_{\mathbb{Z}}(X)$ and $i(z)(g-1) = \langle z, g\rangle$, where $g \in \pi_1(X, x)$ and $z \in H^1(X)$. If $\phi \in (J/J^3)^*$ and α, β are loops based at x, then

$$p(\phi)\big([\alpha] \otimes [\beta]\big) = \big\langle \phi, \big(\{\alpha\} - 1\big)\big(\{\beta\} - 1\big)\big\rangle.$$

Let $\mathcal{A}$ be an arrangement of hyperplanes and b a base point of its complement $M = M(\mathcal{A})$. A pair $(\mathcal{A}, b)$ is called a *pointed arrangement*. We consider the fundamental group $\pi_1(M(\mathcal{A}), b)$ and the augmentation ideal $J = J(M, b)$ of the group ring of $\pi_1(M, b)$ over $\mathbb{Z}$. We recall that $\mathbb{Z}\pi_1(M(\mathcal{A}), b)/J^{s+1}$ has a mixed Hodge structure. By the above there is an extension

$$0 \to H^1(M) \to \big(J/J^3\big)^* \to K \to 0$$

of $(J/J^3)^*$, where K is the kernel of the cup-product $H^1(M) \otimes H^1(M) \to H^2(M)$. Note that the mixed Hodge structure on the cohomology $H^i(M)$ is pure in general. Moreover any element of $H^i(M)$ has Hodge type (i, i) (see [16]). Since the first cohomology is pure of weight 2 and the kernel K of its cup product is pure of weight 4, there is an isomorphism

$$\psi : \mathrm{Ext}\big(K, H^1\big) \to \frac{\mathrm{Hom}(K, H^1)_{\mathbb{C}}}{\mathrm{Hom}(K, H^1)_{\mathbb{Z}}} \cong \frac{(K^* \otimes H_1^*)_{\mathbb{C}}}{(K^* \otimes H_1^*)_{\mathbb{Z}}}.$$

We want to get a description of $\psi((J/J^3)^*)$, and note that its coefficients are given by integration values of iterated integrals.

2.2. Basis for arrangements

In order to give the description of the extension isomorphism $\psi((J/J^3)^*)$ of mixed Hodge structures (see Lemmas 1 and 2), in this section we shall find basis of $H^1(M)$, $H_1(M)$, K and K^*, where K is the kernel of the cup-product $H^1 \otimes H^1 \to H^2$. We must take care of the order of an arrangement (cf. [10]).

Let $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^N$. We define the logarithmic differential form ω_i associated to H_i by

$$\omega_i = \frac{1}{2\pi\sqrt{-1}}\,\mathrm{d}\log(h_i)$$

here $H_i = \ker h_i$. Brieskorn [1] showed that the cohomology of the complement $M(\mathcal{A})$ has a basis

$$\{\omega_i,\ \text{for } 1 \leqslant i \leqslant n\}$$

(for details, see [15]). Choose loops σ_i based at b around H_i such that $\int_{\sigma_i} \omega_j = \delta_{ij}$, where δ_{ij} is Kronecker's delta. Since $[\sigma_i]$'s is the dual basis of $H_1(M(\mathcal{A}), \mathbb{Z})$, $\{(\sigma_i - 1),\ \text{for } 1 \leqslant i \leqslant n\}$ is a basis of $J/J^2 \cong H_1$ and $\{\omega_i,\ \text{for } 1 \leqslant i \leqslant n\}$ is the dual basis of $(J/J^2)^* \cong H^1$.

We want to find a basis of $K^* \cong J^2/J^3$. For $H_i, H_j \in \mathcal{A}$ with $H_i \cap H_j \neq \emptyset$, we denote by k_{ij} the maximum of indices of hyperplanes containing $H_i \cap H_j$: $k_{ij} = \mathbf{max}\{1 \leqslant k \leqslant n \mid H_i \cap H_j \subset H_k\}$. We then obtain the following proposition.

Proposition 3. *The set*

$$\begin{Bmatrix} \omega_i\omega_j + \omega_j\omega_{k_{ij}} + \omega_{k_{ij}}\omega_i, & \textit{for } (i,j) \in C(\mathcal{A}) \\ \omega_i\omega_j, & \textit{for } (i,j) \in P(\mathcal{A}) \\ \omega_i\omega_j + \omega_j\omega_i, & \textit{for } 1 \leqslant i < j \leqslant n \\ \omega_i\omega_i, & \textit{for } 1 \leqslant i \leqslant n \end{Bmatrix}$$

is a basis of K and

$$\begin{Bmatrix} [\sigma_i - 1, \sigma_j - 1], & \textit{for } (i,j) \in \Xi(\mathcal{A}) \\ (\sigma_j - 1)(\sigma_i - 1), & \textit{for } 1 \leqslant i \leqslant j \leqslant n \end{Bmatrix}$$

is the dual basis of K^*. Here

$$\begin{aligned} C(\mathcal{A}) &= \big\{(i,j) \mid i < j,\ H_i \cap H_j \neq \emptyset \text{ and } j \neq k_{ij}\big\}, \\ P(\mathcal{A}) &= \big\{(i,j) \mid i < j,\ H_i \cap H_j = \emptyset\big\}, \\ \Xi(\mathcal{A}) &= C(\mathcal{A}) \cup P(\mathcal{A}). \end{aligned}$$

We recall some properties of iterated integrals.

Lemma 4. *Let M be a smooth manifold, ω_1, ω_2 smooth* 1*-forms on M and σ_1, σ_2 loops based at $x \in M$. Then*

$$\begin{aligned} \big\langle \omega_1\omega_2, (\sigma_1 - 1)(\sigma_2 - 1)\big\rangle &= \int_{\sigma_1} \omega_1 \cdot \int_{\sigma_2} \omega_2, \\ \big\langle \omega_1\omega_2, [\sigma_1 - 1, \sigma_2 - 1]\big\rangle &= \int_{\sigma_1} \omega_1 \cdot \int_{\sigma_2} \omega_2 - \int_{\sigma_2} \omega_1 \cdot \int_{\sigma_1} \omega_2. \end{aligned}$$

Proof. These formulas are obtained by straight forward calculations using the shuffle product formula $\int_\gamma \omega_1 \cdot \int_\gamma \omega_2 = \int_\gamma \omega_1\omega_2 + \int_\gamma \omega_2\omega_1$. □

We shall use the following notations. For an arrangement $\mathcal{A}$ of hyperplanes, denote by $L_2(\mathcal{A})$ the set of nonempty intersections of elements of $\mathcal{A}$ whose codimension is 2. For $X \in L_2(\mathcal{A})$, define the subset $\mathcal{A}_X$ of $\mathcal{A}$ by $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$, and let $|\mathcal{A}_X|$ be its cardinality.

Proof of Proposition 3. Due to the above Lemma 4 we can check easily duality of two sets using formulas of iterated integrals, since σ_i are dual of ω_i. Let $\mathfrak{B}_K$ be the first set in Proposition 3. Then we shall verify that $\mathfrak{B}_K$ is a basis of K.

The following is well known. There are only relations

$$\omega_i \wedge \omega_j + \omega_j \wedge \omega_k + \omega_k \wedge \omega_i = 0$$

for H_i, H_j, H_k with $\operatorname{codim}(H_i \cap H_j \cap H_k) = 2$,

$$\omega_i \wedge \omega_j = 0$$

for H_i, H_j with $H_i \cap H_j = \emptyset$, and

$$\omega_i \wedge \omega_j = \omega_j \wedge \omega_i$$

for H_i, H_j with normal crossing. We can choose independent relations $\mathfrak{B}_K$, using the order of hyperplanes, by the same argument as in [10] (cf. [11–13]). Namely $\mathfrak{B}_K$ is an independent set in $H^1 \otimes H^1$.

Next we can check that the cardinality of $\Xi(\mathcal{A})$ is

$$\left|\Xi(\mathcal{A})\right| = \frac{n(n-1)}{2} - \sum_{X \in L_2(\mathcal{A})} \big(|\mathcal{A}_X| - 1\big)$$

and then the cardinality of $\mathfrak{B}_K$ is

$$\left|\Xi(\mathcal{A})\right| + \left|\big\{(i, j),\ 1 \leqslant i \leqslant j \leqslant n\big\}\right| = n^2 - \sum_{X \in L_2(\mathcal{A})} \big(|\mathcal{A}_X| - 1\big).$$

This is the dimension of K from the following lemma. □

Lemma 5.

$$\dim\big(H^2\big(M(\mathcal{A})\big)\big) = \sum_{X \in L_2(\mathcal{A})} \big(|\mathcal{A}_X| - 1\big).$$

Proof. In general, the dimensions of the cohomology groups $H^k(M(\mathcal{A}))$ are obtained from the Poincaré polynomial $\pi(\mathcal{A}, t)$ of the arrangement $\mathcal{A}$, defined as follows (see [15]). For $X \in L(\mathcal{A})$, set

$$ch[V, X] = \big\{c = (X_{i_1}, \ldots, X_{i_p}) \mid \mathbb{C}^N = V = X_{i_1} \supset \cdots \supset X_{i_p} = X,\ X_{i_k} \in L(\mathcal{A})\big\}$$

and denote p by $|c|$. We define the Möbius function by

$$\mu(X) = \sum_{c \in ch[V,X]} (-1)^{|c|-1}$$

and the Poincaré polynomial by

$$\pi(\mathcal{A}, t) = \sum_{X \in L(\mathcal{A})} \mu(X)(-t)^{\operatorname{codim} X}.$$

Since $\mu(X) = |\mathcal{A}_X| - 1$ for $X \in L_2(\mathcal{A})$, the assertion is verified. □

2.3. Calculation of iterated integrals

Since the coefficients in the description of $\psi((J/J^3)^*)$ are given by calculating certain iterated integrals, we compute concretely those iterated integrals, as follows.

Lemma 6. *Set*

$$\omega_i = \frac{1}{2\pi\sqrt{-1}} \frac{\mathrm{d}z}{z - z_i} \quad (i = 1, 2, 3).$$

Suppose that, for $i = 1, 2, 3$, γ_i is a loop based at z_0 in $\mathbb{C}$, running anti-clockwisely around z_i, and which is nullhomotopic in $\mathbb{C} - \{z_j\}$, for $j \neq i$. Then

$$\int_{\gamma_1} \omega_1\ \omega_2 = \frac{1}{2\pi\sqrt{-1}} \log\left(\frac{z_0 - z_2}{z_1 - z_2}\right),$$

$$\int_{\gamma_1} \{\omega_1\ \omega_2 + \omega_2\ \omega_3 + \omega_3\ \omega_1\} = \frac{1}{2\pi\sqrt{-1}} \log\left(\frac{z_0 - z_2}{z_1 - z_2} \cdot \frac{z_1 - z_3}{z_0 - z_3}\right).$$

Proof. We can compute them easily by line integrals (cf. [10]). □

Let H_i $(i = 1, 2, 3)$ be hyperplanes in $\mathbb{C}^N$ with $\operatorname{codim}(H_1 \cap H_2 \cap H_3) = 2$, b a base point of $\mathbb{C}^N - \bigcup_{i=1,2,3} H_i$, and

$$\omega_i = \frac{1}{2\pi\sqrt{-1}} \operatorname{d}\log h_i,$$

where $H_i = \ker h_i$. Set γ_i to be a loop based at b in $\mathbb{C}^N$ whose monodromy around H_i is equal to $2\pi\sqrt{-1}$. Since $\operatorname{codim}(H_1 \cap H_2 \cap H_3) = 2$, assume that

$$c_1 h_1 + c_2 h_2 + c_3 h_3 = 0$$

for some nonzero constants c_i. Then

Proposition 7.

$$\int_{\gamma_1} \{\omega_1\ \omega_2 + \omega_2\ \omega_3 + \omega_3\ \omega_1\} = \frac{1}{2\pi\sqrt{-1}} \log\left(-\frac{c_2 h_2(b)}{c_3 h_3(b)}\right).$$

Proof. We consider an immersion ι of $\mathbb{C}$ in $\mathbb{C}^N$ which induces an immersion

$$\mathbb{C} - \{z_1, z_2, z_3\} \overset{\iota}{\hookrightarrow} \mathbb{C}^N - \bigcup_{i=1,2,3} H_i,$$

where $H_i \cap \mathbb{C} = z_i$ with $b = z_0 \in \mathbb{C}$. We can assume that for $i = 1, 2, 3$, $\widetilde{\gamma}_i$ is a loop based at z_0 in $\mathbb{C}$ running anti-clockwisely around z_i, and which is nullhomotopic in $\mathbb{C} - \{z_j\}$ for $j \neq i$ and $\gamma_i = \iota \circ \widetilde{\gamma}_i$, so $\iota^* \omega_i = \operatorname{d}\log(z - z_i)$. By Lemma 6 we get

$$\int_{\gamma_1} \{\omega_1\omega_2 + \omega_2\omega_3 + \omega_3\omega_1\} = \frac{1}{2\pi\sqrt{-1}} \log\left(\frac{z_0 - z_2}{z_1 - z_2} \cdot \frac{z_1 - z_3}{z_0 - z_3}\right).$$

On the other hand we can write $h_i \circ \iota = \alpha_i(z - z_i)$ for some constants α_i. Then we can see

$$\widetilde{\alpha}_1(z - z_1) + \widetilde{\alpha}_2(z - z_2) + \widetilde{\alpha}_3(z - z_3) = 0,$$

where $\widetilde{\alpha}_i = c_i \alpha_i$, namely $\widetilde{\alpha}_1 + \widetilde{\alpha}_2 + \widetilde{\alpha}_3 = 0$ and $\widetilde{\alpha}_1 z_1 + \widetilde{\alpha}_2 z_2 + \widetilde{\alpha}_3 z_3 = 0$. Also note that $z_1 - z_i = (z_0 - z_i) - (z_0 - z_1)$ and

$$z_0 - z_i = \frac{1}{\alpha_i} h_i \circ \iota(z_0) = \frac{1}{\widetilde{\alpha}_1} c_i h_i(b), \quad i = 1, 2, 3.$$

The formula in the statement now follows. □

2.4. The cross ratio equivalence

Let $\mathbb{P} = \mathbb{P}^N$ be the complex projective space of dimension N. We can define the cross ratio of four distinct points lying on a line in $\mathbb{P}$ as follows. Take homogeneous coordinates $[s : t]$ of the line $\mathbb{P}^1$. Let $p_i = [s_i : t_i]$ $(i = 1, 2, 3, 4)$ be four points on this line. Their *cross ratio* is defined by

$$\lambda = \lambda(p_1, p_2, p_3, p_4) = \frac{s_1t_3 - s_3t_1}{s_1t_4 - s_4t_1} \cdot \frac{s_2t_4 - s_4t_2}{s_2t_3 - s_3t_2}.$$

This cross ratio is invariant under projective transformations of $\mathbb{P}$, and by permutation of the four points, the cross ratio takes six values:

$$\left\{\lambda, \lambda^{-1}, 1 - \lambda, (1 - \lambda)^{-1}, \lambda^{-1}(\lambda - 1), \lambda(\lambda - 1)^{-1}\right\}.$$

We consider the following cross ratio of an arrangement. Let H_1, H_2, H_3 be distinct three hyperplanes in $\mathbb{P}$ with $\operatorname{codim}(H_1 \cap H_2 \cap H_3) = 2$ and b a point of $\mathbb{P}$ not lying in H_i $(i = 1, 2, 3)$. We denote by $\mathbb{P}^*$ the projective dual space of $\mathbb{P}$. Let $p_i = H_i^*$ be the point of $\mathbb{P}^*$ associated to the hyperplane H_i of $\mathbb{P}$ for $i = 1, 2, 3$ and $H_b = b^*$ the hyperplane of $\mathbb{P}^*$ associated to the point b of $\mathbb{P}$. Then p_1, p_2 and p_3 lie on some line and do not lie in H_b. The intersection of this line and the hyperplane H_b is a point of $\mathbb{P}^*$ and so this point is denoted by p_b. Then the *cross ratio* is defined by

$$\lambda_{123} = \lambda(H_1, H_2, H_3 : b) = \lambda(p_3, p_2, p_1, p_b).$$

This cross ratio has the following expression. We take homogeneous coordinates z of $\mathbb{P}$ and denote by $h_i(z)$ the homogeneous linear polynomial defining H_i for $i = 1, 2, 3$. Since $\operatorname{codim}(H_1 \cap H_2 \cap H_3) = 2$, there are nonzero constants c_1, c_2, c_3 satisfying

$$c_1h_1(z) + c_2h_2(z) + c_3h_3(z) = 0$$

(or we can take the normalization h_1, h_2, h_3 as $h_1(z) + h_2(z) + h_3(z) = 0$).

Lemma 8.

$$\lambda = \lambda(H_1, H_2, H_3 : b) = -\frac{c_2h_2(b)}{c_3h_3(b)}.$$

For a permutation of the three hyperplanes, this cross ratios takes six values:

$$\lambda_{123} = \lambda, \qquad \lambda_{231} = (1 - \lambda)^{-1}, \qquad \lambda_{312} = \lambda^{-1}(\lambda - 1),$$
$$\lambda_{132} = \lambda^{-1}, \qquad \lambda_{213} = (1 - \lambda), \qquad \lambda_{321} = \lambda(\lambda - 1)^{-1}.$$

Proof. Taking homogeneous coordinates, the result is obtained by straight calculations using the above condition. □

We recall the following notations (see [15]). For an arrangement $\mathcal{A}$ of hyperplanes, denote by $L(\mathcal{A})$ the set of nonempty intersections of elements of $\mathcal{A}$ and by $L_p(\mathcal{A})$ the set of elements of $L(\mathcal{A})$ whose codimension is p. We define the subset $L_{\leqslant p}(\mathcal{A}) = \bigcup_{i \leqslant p} L_i(\mathcal{A})$

of $L(\mathcal{A}) = \bigcup_i L_i(\mathcal{A})$. For $X \in L(\mathcal{A})$, define the following subset $\mathcal{A}_X$ of $\mathcal{A}$ by $\mathcal{A}_X = \{H \in \mathcal{A} \mid X \subseteq H\}$.

Definition 9. Two arrangements $\mathcal{A}$ and $\mathcal{A}'$ are called *L-equivalent* if there is a one-to-one correspondence between $L(\mathcal{A})$ and $L(\mathcal{A}')$ that preserves inclusions. Furthermore two arrangements $\mathcal{A}$ and $\mathcal{A}'$ are called *L_p-equivalent* if there is a one-to-one correspondence between $L_{\leqslant p}(\mathcal{A})$ and $L_{\leqslant p}(\mathcal{A}')$ that preserves inclusions.

Definition 10. Let $\overline{\mathcal{A}}$ be a projective arrangement of hyperplanes in $\mathbb{P} = \mathbb{P}^N$, and b a base point of its complement $M(\overline{\mathcal{A}}) = \mathbb{P} - \bigcup_{H \in \overline{\mathcal{A}}} H$. Two pointed arrangements $(\overline{\mathcal{A}}, b)$ and $(\overline{\mathcal{A}'}, b')$ are *cross ratio equivalent* if and only if they are L_2-equivalent and their cross ratios of arrangements coincide. Namely for $H_1, H_2, H_3 \in \overline{\mathcal{A}}$ with $\operatorname{codim}(H_1 \cap H_2 \cap H_3) = 2$, let $H_1', H_2', H_3' \in \overline{\mathcal{A}'}$ be the corresponding hyperplanes. Then

$$\lambda(H_1, H_2, H_3 : b) = \lambda\bigl(H_1', H_2', H_3' : b'\bigr).$$

Remark 11. We can rephrase the cross ratio equivalence relation, as follows. Let $(\mathcal{A}, b)$ be a pointed arrangement in $\mathbb{P}$. For $X \in L_2(\mathcal{A})$, assume that $\mathcal{A}_X = \{H_1, \ldots, H_k\}$ and define

$$\mathcal{P}(X) = X^* - \bigl\{H^* \mid H \in \mathcal{A}_X\bigr\} \cup \bigl\{b^* \cap X^*\bigr\} \cong \mathbb{P}^1 - \{p_1, \ldots, p_k, p_b\}.$$

Pointed arrangements are cross ratio equivalent if and only if they are L_2-equivalent and for any $X \in L_2$, the corresponding $\mathcal{P}(X)$'s are biholomorphic.

Next we shall consider the affine case. Let $\mathbb{P}^N$ be the complex projective space which is a compactification of $\mathbb{C}^N$. For an affine arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ in $\mathbb{C}^N$, consider the projective arrangement $\overline{\mathcal{A}}$ of hyperplanes in $\mathbb{P}^N$ defined by $\overline{\mathcal{A}} = \{\overline{H_0}, \overline{H_1}, \ldots, \overline{H_n}\}$, where $\overline{H_i}$ is the projective closure of H_i, $1 \leqslant i \leqslant n$, and $\overline{H_0} = \mathbb{P}^N - \mathbb{C}^N$. Note that $M = M(\mathcal{A}) = M(\overline{\mathcal{A}})$.

Definition 12. The cross ratio of an affine arrangement is defined by one of its projective arrangements. In the affine case the cross ratio has also the expression in Lemma 8. Two pointed affine arrangements are called *cross ratio equivalent* if their pointed projective arrangements are cross ratio equivalent.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement in $\mathbb{C}^N$ and $\overline{\mathcal{A}} = \{\overline{H_0}, \overline{H_1}, \ldots, \overline{H_n}\}$ be its projective arrangement in $\mathbb{P}^N$. Recall the combinatorial data $\Xi(\mathcal{A})$ in Proposition 3. We define a cross ratio for $(i, j) \in \Xi(\mathcal{A})$ by

$$\lambda_{ij}(\mathcal{A}) = \begin{cases} \lambda\bigl(\overline{H_i}, \overline{H_j}, \overline{H_{k_{ij}}} : b\bigr) & \text{if } H_i \cap H_j \neq \emptyset, \\ \lambda\bigl(\overline{H_i}, \overline{H_j}, \overline{H_0} : b\bigr) & \text{if } H_i \cap H_j = \emptyset. \end{cases}$$

Proposition 13. *Two pointed affine arrangements $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$ are cross ratio equivalent if and only if they are L_2-equivalent and $\lambda_{ij}(\mathcal{A}) = \lambda_{ij}(\mathcal{A}')$ for any $(i, j) \in \Xi(\mathcal{A}) = \Xi(\mathcal{A}')$.*

Proof. The necessity is clear. If $\emptyset \neq H_i \cap H_j = K_{ij}$ and $\mathcal{A}_{K_{ij}} = \{H_{k_1}, \ldots, H_{k_{ij}}\}$ ($k_1 < \cdots < k_{ij}$) then their dual points $p_{k_1}, \ldots, p_{k_{ij}}$ are on one line. Also, if $H_i \cap H_j = \emptyset$ then $\operatorname{codim}(\overline{H_i} \cap \overline{H_j} \cap \overline{H_0}) = 2$ in $\mathbb{P}^N$. Therefore, by the property of the usual cross ratio, the proposition follows. □

Example 14. In the case of arrangements in general position, cross ratio equivalence means that they have the same number of hyperplanes.

Example 15. The braid arrangement in $\mathbb{C}^n$ is defined by the hyperplanes $z_i - z_j = 0$ ($1 \leqslant i < j \leqslant n$). For a base point z, its cross ratio can be written as

$$\lambda_{ijk} = \frac{z_i - z_j}{z_k - z_j}.$$

Example 16. Given 4 generic points in $\mathbb{P}^2$, we consider the arrangement of 6 lines determined by any two points. This is defined by the equation $xyz(x-y)(y-z)(z-x) = 0$. For a base point $[x : y : z]$, we obtain cross ratios

$$\lambda_{[001]} = \frac{x}{y}, \qquad \lambda_{[010]} = \frac{z}{x}, \qquad \lambda_{[100]} = \frac{y}{z}, \qquad \lambda_{[111]} = \frac{x-y}{z-y},$$

and in this case, cross ratio equivalence means biholomorphism.

2.5. Description of $\psi((J/J^3)^*)$

Let $\mathcal{A}$ be an arrangement of hyperplanes. Recall that $\mathbb{Z}\pi_1(M(\mathcal{A}), b)/J^{s+1}$ has a mixed Hodge structure, that there is an extension $0 \to H^1 \to (J/J^3)^* \to K \to 0$ of K by H^1 and that we have the isomorphism $\psi : \operatorname{Ext}(K, H^1) \cong (K^* \otimes H_1^*)_{\mathbb{C}}/(K^* \otimes H_1^*)_{\mathbb{Z}}$. The isomorphism ψ associated with the mixed Hodge structure on $(J/J^3)^*$ with respect to the basis in Proposition 3 can be written as follows.

Theorem 17.

$$\begin{aligned}
&\psi\big((J/J^3)^*\big) \\
&= \frac{1}{2\pi\sqrt{-1}} \sum_{(i,j)\in C(\mathcal{A})} \left\{ \begin{array}{r} \log(\lambda_{ij})\ [\sigma_i - 1, \sigma_j - 1] \otimes (\sigma_i - 1)^* \\ + \log\big((1-\lambda_{ij})^{-1}\big)\ [\sigma_i - 1, \sigma_j - 1] \otimes (\sigma_j - 1)^* \\ + \log\big(\lambda_{ij}^{-1}(\lambda_{ij} - 1)\big)\ [\sigma_i - 1, \sigma_j - 1] \otimes (\sigma_{k_{ij}} - 1)^* \end{array} \right\} \\
&\quad + \frac{1}{2\pi\sqrt{-1}} \sum_{(i,j)\in P(\mathcal{A})} \left\{ \begin{array}{r} \log(\lambda_{ij})\ [\sigma_i - 1, \sigma_j - 1] \otimes (\sigma_i - 1)^* \\ + \log\big((1-\lambda_{ij})^{-1}\big)\ [\sigma_i - 1, \sigma_j - 1] \otimes (\sigma_j - 1)^* \end{array} \right\} \\
&\hspace{20em} \bmod (K^* \otimes H_1^*)_{\mathbb{Z}},
\end{aligned}$$

where $\lambda_{ij} = \lambda_{ij}(\mathcal{A})$ are the cross ratios for $(i, j) \in \Xi(\mathcal{A})$.

Remark 18. Due to Proposition 3, we get the basis

$$[\sigma_i - 1, \sigma_j - 1] \otimes (\sigma_k - 1)^*, \quad \text{for } (i, j) \in \Xi(\mathcal{A}) \text{ and } 1 \leqslant k \leqslant n,$$
$$(\sigma_i - 1)(\sigma_j - 1) \otimes (\sigma_k - 1)^*, \quad \text{for } 1 \leqslant i \leqslant j \leqslant n \text{ and } 1 \leqslant k \leqslant n$$

of $K^* \otimes H_1^*$. But the coefficients of $(\sigma_i - 1)(\sigma_j - 1) \otimes (\sigma_k - 1)^*$ of $\psi((J/J^3)^*)$ vanish in $K^* \otimes H_1^*$ modulo $\mathbb{Z}$.

Proof. Due to Proposition 7 and Lemma 8, we can compute the following iterated integrals

(1) For each $(i, j) \in C(\mathcal{A})$ and $\sigma \in BH_1$,

$$2\pi\sqrt{-1}\int_\sigma \omega_i\omega_j + \omega_j\omega_{k_{ij}} + \omega_{k_{ij}}\omega_i = \begin{cases} \log(\lambda_{ij}), & \text{if } \sigma = \sigma_i, \\ \log\big((1-\lambda_{ij})^{-1}\big), & \text{if } \sigma = \sigma_j, \\ \log\big(\lambda_{ij}^{-1}(\lambda_{ij} - 1)\big), & \text{if } \sigma = \sigma_{k_{ij}}, \\ 0, & \text{otherwise.} \end{cases}$$

(2) For each $(i, j) \in P(\mathcal{A})$ and $\sigma \in BH_1$,

$$2\pi\sqrt{-1}\int_\sigma \omega_i\omega_j = \begin{cases} \log(\lambda_{ij}), & \text{if } \sigma = \sigma_i, \\ \log\big((1-\lambda_{ij})^{-1}\big), & \text{if } \sigma = \sigma_j, \\ 0, & \text{otherwise.} \end{cases}$$

(3) For each $1 \leqslant i < j \leqslant n$ and $\sigma \in BH_1$,

$$2\pi\sqrt{-1}\int_\sigma \omega_i\omega_j + \omega_j\omega_i = 0.$$

(4) For each $1 \leqslant i \leqslant n$ and $\sigma \in BH_1$,

$$2\pi\sqrt{-1}\int_\sigma \omega_i\omega_i = \begin{cases} 1, & \text{if } \sigma = \sigma_i, \\ 0, & \text{otherwise.} \end{cases}$$

This yields the theorem. □

This gives the first half of the Main Theorem, in the following way.

Corollary 19. *If $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$ are cross ratio equivalent pointed arrangements of hyperplanes, then there is a ring isomorphism*

$$\varphi : \mathbb{Z}\pi_1\big(M(\mathcal{A}), b\big)/J^3 \to \mathbb{Z}\pi_1\big(M\big(\mathcal{A}'\big), b'\big)/J^3$$

which induces an isomorphism of mixed Hodge structures.

Proof. Let $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$ be cross ratio equivalent pointed arrangements of hyperplanes. First since they are L_2-equivalent, there is a bijection from $\mathcal{A}$ to $\mathcal{A}'$ that induces a bijection from $L_2(\mathcal{A})$ to $L_2(\mathcal{A}')$. Fixing a hyperplane at infinity, we may assume that they are affine arrangements.

The morphism $\varphi : \mathbb{Z}\pi_1(M(\mathcal{A}), b)/J^3 \to \mathbb{Z}\pi_1(M(\mathcal{A}'), b')/J^3$ is defined by carrying a loop around a hyperplane $H \in \mathcal{A}$ to a loop around the hyperplane $H' \in \mathcal{A}'$ corresponding to H. This induces isomorphisms of $J/J^2 \cong H_1$ and $J^2/J^3 \cong K^*$, by Proposition 3, and so φ is the desired ring isomorphism.

Recall that mixed Hodge structures on the complements are induced by extensions $0 \to H^1 \to (J/J^3)^* \to K \to 0$. Therefore Theorem 17 and Proposition 13 yield the corollary. □

3. Torelli problem

3.1. Polarization of $H^1(M(\mathcal{A}))$

First we recall the root system of type A_n. Let n be a positive integer. We shall work in the real vector space $V = \mathbb{R}^{n+1}$, with the usual inner product. Let $\varepsilon_1, \ldots, \varepsilon_{n+1}$ be a basis of $\mathbb{R}^{n+1}$ consisting of orthonormal unit vectors. The lattice $V_{\mathbb{Z}} = \mathbb{Z}^{n+1}$ is the $\mathbb{Z}$-span of this basis. We define the n-dimensional subspace E of V to be the orthogonal complement to the vector $\varepsilon_1 + \cdots + \varepsilon_{n+1}$, and set $E_{\mathbb{Z}} = V_{\mathbb{Z}} \cap E$. We take $\Phi = \{\alpha \in E_{\mathbb{Z}} \mid (\alpha, \alpha) = 2\}$. Then Φ is the root system in E of type A_n, such that $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, and that $\Delta = \{\alpha_i := \varepsilon_i - \varepsilon_{i+1} \mid 1 \leqslant i \leqslant n\}$ is a base of Φ, with Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & & 0 \\ -1 & 2 & -1 & \ddots & \\ 0 & -1 & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots & -1 \\ 0 & & 0 & -1 & 2 \end{pmatrix}.$$

The Weyl group $\mathcal{W}$ of Φ is isomorphic to the symmetric group Sym^{n+1}; the reflection with respect to α_i corresponds to the transposition $(i, i+1)$.

We now define the polarization of the cohomology of the complement of hyperplanes. Let $\mathcal{A} = \{H_1, H_2, \ldots, H_{n+1}\}$ be an arrangement of hyperplanes in $\mathbb{P}^N$, and M be its complement. The Gysin homomorphism gives a exact sequence

$$0 \to H^1(M) \to H^0\left(\bigcup_i H_i\right).$$

$H^0(\bigcup_i H_i)$ is isomorphic to $H_{2(N-1)}(\bigcup_i H_i) = \bigoplus_i H_{2(N-1)}(H_i)$ by Poincaré duality. Since the dimension of $H_{2(N-1)}(H_i)$ is one, let a_i be a generator of $H_{2(N-1)}(H_i)$, for $1 \leqslant i \leqslant n+1$. Then $\{a_i^*,\ 1 \leqslant i \leqslant n+1\}$ is a basis of $H^0(\bigcup_i H_i)$. The polarization of $H^0(\bigcup_i H_i)$ given by $\langle a_i^*, a_j^* \rangle = \delta_{ij}$ induces the pairing of $H^1(M)$. We shall call this pairing *the polarization on $H^1(M)$ for an arrangement $\mathcal{A}$*. Define logarithmic differential forms

$$\omega_{i,j} = \frac{1}{2\pi\sqrt{-1}} \,\mathrm{d}\log \frac{h_i}{h_j},$$

where $H_i = \ker h_i$. Note that $\{\omega_{i,n+1},\ 1 \leqslant i \leqslant n\}$ is a basis of $H^1(M)$ and $\{\omega_{i,i+1},\ 1 \leqslant i \leqslant n\}$ is also a basis. $H^1(M)$ is regarded as the subspace in $H^0(\bigcup_i H_i)$ with $\omega_{i,j} = a_i^* - a_j^*$. Therefore $\Phi = \{\omega_{i,j},\ i \neq j\}$ is the root system in $H^1(M)$ of type A_n.

The following results from the fact that the Cartan matrix determines a root system up to isomorphism (see [9, p. 55]).

Lemma 20. *For two arrangements $\mathcal{A}$ and $\mathcal{A}'$, if $\varphi : H^1(M(\mathcal{A})) \to H^1(M(\mathcal{A}'))$ is an isomorphism satisfying $\langle \varphi(\alpha), \varphi(\beta)\rangle = \langle \alpha, \beta\rangle$ for all $\alpha, \beta \in H^1(M(\mathcal{A}))$, then we can take indexes satisfying $\varphi(\omega_{i,j}) = \omega'_{i,j}$ for all $i \neq j$.*

As consequence, the cohomology algebra with polarization determines the combinatorics for arrangements as follows. (Recall Definition 9 of combinatorics for arrangements.)

Proposition 21. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two arrangements. There is a isomorphism $H^*(M(\mathcal{A})) \xrightarrow{\sim} H^*(M(\mathcal{A}'))$ of cohomology algebras of their complements $M(\mathcal{A})$ and $M(\mathcal{A}')$ preserving the polarization on first cohomologies, if and only if, $\mathcal{A}$ and $\mathcal{A}'$ are L-equivalent.*

Let $\mathcal{A}$ and $\mathcal{A}'$ be arrangements in $\mathbb{P}^N$ and $\mathcal{A} = \{H_1, \dots, H_{n+1}\}$. We need the following lemmas (cf. [15]). Their proofs are straightforward.

Lemma 22. *$\mathcal{A}$ and $\mathcal{A}'$ are L-equivalent, if and only if, there is bijection $\varphi : \mathcal{A} \to \mathcal{A}'$ such that for all subsets $\{H_{i_1}, \dots, H_{i_{N+1}}\}$ of $\mathcal{A}$ with $H_{i_1} \cap \cdots \cap H_{i_{N+1}} \neq \emptyset$, $\varphi(H_{i_1}) \cap \cdots \cap \varphi(H_{i_{N+1}}) \neq \emptyset$.*

Denote by $\omega_i = \frac{1}{2\pi\sqrt{-1}} \mathrm{d}\log(h_i)$ the logarithmic differential form associated to $H_i = \ker h_i$. We define a linear differential ∂ by

$$\partial \omega_1 \cdots \omega_p = \sum_{i=1}^{p} (-1)^{i-1} \omega_1 \cdots \widehat{\omega_i} \cdots \omega_p.$$

Lemma 23.

$$H_{i_1} \cap \cdots \cap H_{i_{N+1}} \neq \emptyset \quad \Longleftrightarrow \quad \partial \omega_{i_1} \cdots \omega_{i_{N+1}} = 0.$$

These lemmas lead to the above proposition.

Proof of Proposition 21. The necessity is given by the well-known fact that cohomologies of complements of L-equivalent arrangements are isomorphic (Orlik and Solomon, cf. [15]).

Let $\mathcal{A}$ and $\mathcal{A}'$ be arrangements in $\mathbb{P}^N$. Suppose that there is a isomorphism $\varphi : H^*(M(\mathcal{A})) \xrightarrow{\sim} H^*(M(\mathcal{A}'))$ preserving the polarization on first cohomologies. Since it preserves the polarization on first cohomologies, due to Lemma 20 we can assume that $\mathcal{A} = \{H_1, \dots, H_{n+1}\}$, $\mathcal{A}' = \{H'_1, \dots, H'_{n+1}\}$ and $\varphi(\omega_{i,j}) = \omega'_{i,j}$ here $\omega_{i,j}$ (respectively $\omega_{i,j}$) are logarithmic differential forms on $M(\mathcal{A})$ (respectively $M(\mathcal{A}')$). So we get the bijection $\varphi : \mathcal{A} \to \mathcal{A}'$ defined by $\varphi(H_i) = H'_i$. We choose H_{n+1} and H'_{n+1} as a hyperplane at infinity respectively and then we can consider them to be affine arrangements. Recall that the logarithmic differential forms $\omega_i = \omega_{i,n+1}$ (respectively $\omega'_i = \omega'_{i,n+1}$) associated to H_i (respectively H'_i), $1 \leqslant i \leqslant n$, generate the cohomology $H^*(M(\mathcal{A}))$ (respectively $H^*(M(\mathcal{A}')))$ (Brieskorn [1]).

For subsets $\{H_{i_1}, \ldots, H_{i_{N+1}}\}$ of $\mathcal{A}$ with $H_{i_1} \cap \cdots \cap H_{i_{N+1}} \neq \emptyset$ and not containing H_{N+1}, we have $\partial \omega_{i_1} \cdots \omega_{i_{N+1}} = 0$. By the isomorphism of cohomologies, we get $\partial \omega'_{i_1} \cdots \omega'_{i_{N+1}} = 0$ and then $H'_{i_1} \cap \cdots \cap H'_{i_{N+1}} \neq \emptyset$. In the case of subsets containing H_{N+1}, we change the hyperplane at infinity. Therefore $\mathcal{A}$ and $\mathcal{A}'$ are L-equivalent by the two lemmas above. □

In addition, the same argument gives the following.

Corollary 24. *Let $\mathcal{A}$ and $\mathcal{A}'$ be two arrangements. There is an isomorphism $H^{\leqslant p}(M(\mathcal{A})) \xrightarrow{\sim} H^{\leqslant p}(M(\mathcal{A}'))$ preserving the polarization on first cohomologies, if and only if, $\mathcal{A}$ and $\mathcal{A}'$ are L_p-equivalent, where $H^{\leqslant p} = \bigoplus_{i \leqslant p} H^i$.*

Proof. We can modify Lemma 22; $\mathcal{A}$ and $\mathcal{A}'$ are L_p-equivalent, if and only if, there is bijection $\varphi : \mathcal{A} \to \mathcal{A}'$ such that for all subsets $\{H_{i_1}, \ldots, H_{i_{p+1}}\}$ of $\mathcal{A}$ with $\operatorname{codim}(H_{i_1} \cap \cdots \cap H_{i_{p+1}}) < p+1$ (namely $\partial \omega_{i_1} \cdots \omega_{i_{p+1}} = 0$), $\operatorname{codim}(\varphi(H_{i_1}) \cap \cdots \cap \varphi(H_{i_{p+1}})) < p+1$ (namely $\partial \omega'_{i_1} \cdots \omega'_{i_{p+1}} = 0$); here ω_j (respectively ω'_j) is the logarithmic differential form associated to H_j (respectively $\varphi(H_j)$). This leads to the corollary. □

3.2. Torelli type theorem

From the above discussion, we can get a Torelli type theorem for polarized mixed Hodge structures on π_1 of pointed arrangements of hyperplanes, as follows.

Theorem 25. *Let $(\mathcal{A}, b)$, $(\mathcal{A}', b')$ be two pointed arrangements of hyperplanes. If there is a ring isomorphism*

$$\varphi : \mathbb{Z}\pi_1\big(M(\mathcal{A}), b\big)/J^3 \to \mathbb{Z}\pi_1\big(M\big(\mathcal{A}'\big), b'\big)/J^3$$

which induces an isomorphism of mixed Hodge structures and preserve the polarization on $(J/J^2)^*$, then $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$ are cross ratio equivalent.

Proof. First φ induces isomorphisms φ_1^* of $H^1 \cong (J/J^2)^*$ and φ_K^* of $K \cong (J^2/J^3)^*$, where K is the kernel of the cup-product. So the exact sequence $0 \to K \to H^1 \otimes H^1 \to H^2 \to 0$ leads to an isomorphism of H^2. Therefore there is an isomorphism of $H^{\leqslant 2}$ preserving the polarization on H^1, and so $\mathcal{A}$ and $\mathcal{A}'$ are L_2-equivalent by Corollary 24. Note that we can define cross ratios λ_{ij} and λ'_{ij} for $(i, j) \in \Xi(\mathcal{A}) = \Xi(\mathcal{A}')$ (see Proposition 3 and Definition 12).

Since φ induce the isomorphism φ^* of mixed Hodge structures on $(J/J^3)^*$, their extensions $0 \to H^1 \to (J/J^3)^* \to K \to 0$ of K by H^1 are congruent. Via the isomorphism $\psi : \operatorname{Ext}(K, H^1) \cong (K^* \otimes H_1^*)_{\mathbb{C}}/(K^* \otimes H_1^*)_{\mathbb{Z}}$, their coefficients in the description of $\psi((J/J^3)^*)$ (see Theorem 17) coincide, namely $\log(\lambda_{ij})$ is equal to $\log(\lambda'_{ij})$ in $\mathbb{C}/2\pi\sqrt{-1}\mathbb{Z}$. Therefore taking exponents of both sides, we get $\lambda_{ij} = \lambda'_{ij}$. Due to Proposition 13, we have that $(\mathcal{A}, b)$ and $(\mathcal{A}', b')$ are cross ratio equivalent. □

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