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Bar complex of the Orlik–Solomon algebra

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Abstract

Let $\mathcal{A}$ be an arrangement of complex hyperplanes and $M_{\mathcal{A}}$ the complement of the union of hyperplanes in $\mathcal{A}$. The Orlik–Solomon algebra of $\mathcal{A}$ determines a subcomplex of the de Rham complex of the loop space of $M_{\mathcal{A}}$, which is called the bar complex of the Orlik–Solomon algebra. The dual of this complex is isomorphic to the tensor algebra of the homology of $M_{\mathcal{A}}$ equipped with a derivation arising from the product structure of the Orlik–Solomon algebra. Based on this construction we give an explicit description of Chen's iterated integrals of logarithmic forms depending only on the homotopy class of a loop.

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Introduction

Let $\mathcal{A}$ be an arrangement of complex hyperplanes in $\mathbb{C}^n$. In the present article we suppose that the arrangement $\mathcal{A}$ is central. We denote by $M_{\mathcal{A}}$ the complement of the union of hyperplanes in $\mathcal{A}$. We consider iterated integrals in the sense of Chen of smooth differential forms on $M_{\mathcal{A}}$ as differential forms on the loop space of $M_{\mathcal{A}}$. We have a subcomplex $B^\bullet(M_{\mathcal{A}})$ of the de Rham complex of the loop space of $M_{\mathcal{A}}$ consisting of such iterated integrals. The complex $B^\bullet(M_{\mathcal{A}})$ is called the bar construction for the de Rham complex of $M_{\mathcal{A}}$. As was shown in a more general situation by Chen, the 0th cohomology of the complex $B^\bullet(M_{\mathcal{A}})$ determines the completed group ring over $\mathbb{R}$ of the fundamental group of $M_{\mathcal{A}}$.

It was proved by Brieskorn [1] that the cohomology ring of $M_{\mathcal{A}}$ is generated by the logarithmic 1-forms with poles of order one on the hyperplanes in $\mathcal{A}$. The structure of the

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subalgebra of the de Rham complex generated by such logarithmic forms was determined by Orlik and Solomon [6] and we call it the Orlik–Solomon algebra. In this paper we focus on the iterated integrals of differential forms contained in the Orlik–Solomon algebra.

It turns out that such iterated integrals determine a subcomplex of $B^\bullet(M_{\mathcal{A}})$ defined over the ring of integers $\mathbb{Z}$ and that the cohomology of this complex tensored with $\mathbb{R}$ is isomorphic to the cohomology of the above complex $B^\bullet(M_{\mathcal{A}})$. We shall call this complex the bar complex of the Orlik–Solomon algebra. We have a convenient description of the dual of the bar complex of the Orlik–Solomon algebra as a tensor algebra on the homology of $M_{\mathcal{A}}$ with a derivation defined by the product structure of the Orlik–Solomon algebra. This construction permits us to describe the cohomology of the complex $B^\bullet(M_{\mathcal{A}})$ in a purely algebraic way. Based on this method we describe explicitly a basis of the space of iterated integrals of the logarithmic 1-forms depending only on the homotopy class of a loop. Some of the results for arrangements obtained by means of Sullivan's minimal models (see [3,4]) can be reformulated in these terminologies.

The paper is organized in the following way. In Section 1 we introduce the notion of the bar complex of the Orlik–Solomon algebra and show that its cohomology is isomorphic to the cohomology of $B^\bullet(M_{\mathcal{A}})$. In Section 2 we give an algebraic description of the dual of the bar complex of the Orlik–Solomon algebra. Section 3 is devoted to the study of the cohomology of the bar complex of the Orlik–Solomon algebra and iterated integrals. Throughout this paper $\mathbb{K}$ denotes the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$.

1. Bar complex

Let $\mathcal{A}$ be a central arrangement of hyperplanes in $\mathbb{C}^n$. By definition $\mathcal{A}$ is a set of finite number of complex hyperplanes in $\mathbb{C}^n$ through the origin. We denote by $M_{\mathcal{A}}$ the complementary space defined by

$$M_{\mathcal{A}} = \mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}} H.$$

We put $\mathcal{A} = \{H_1, \ldots, H_r\}$ and denote by f_j a linear form defining H_j. Let us consider the logarithmic differential forms

$$\omega_j = \frac{1}{2\pi\sqrt{-1}} \frac{\mathrm{d}f_j}{f_j}, \quad 1 \leqslant j \leqslant r. \tag{1.1}$$

It was shown by Brieskorn [1] that the cohomology ring of $M_{\mathcal{A}}$ is generated by the de Rham cohomology classes of $\omega_j,\ 1 \leqslant j \leqslant r$. We denote by $\Omega_{\mathcal{A}}$ the $\mathbb{Z}$ subalgebra of the algebra of smooth differential forms on $M_{\mathcal{A}}$ generated by the logarithmic forms $\omega_j,\ 1 \leqslant j \leqslant r$. The algebra $\Omega_{\mathcal{A}}$ is isomorphic to the cohomology ring $H^*(M_{\mathcal{A}}, \mathbb{Z})$. We shall say that a subset $\{H_{i_1}, \ldots, H_{i_p}\}$ of $\mathcal{A}$ is dependent if the condition

$$\operatorname{codim}_{\mathbb{C}}[H_{i_1} \cap \cdots \cap H_{i_p}] < p$$

is satisfied. We denote by $E_{\mathcal{A}}$ the exterior algebra over $\mathbb{Z}$ with generators $e_j,\ 1 \leqslant j \leqslant r$, modulo the ideal generated by

$$\sum_{s=1}^{p}(-1)^{s-1}e_{i_1} \wedge \cdots \wedge \widehat{e_{i_s}} \wedge \cdots \wedge e_{i_p} \tag{1.2}$$

for any dependent family $\{H_{i_1}, \ldots, H_{i_p}\} \subset \mathcal{A}$ with $i_1 < \cdots < i_p$. In [6] Orlik and Solomon proved that we have an isomorphism of algebras

$$\alpha : E_{\mathcal{A}} \cong \Omega_{\mathcal{A}}$$

given by $\alpha(e_j) = \omega_j,\ 1 \leqslant j \leqslant r$. We call $E_{\mathcal{A}}$ and also $\Omega_{\mathcal{A}}$ the Orlik–Solomon algebra of the hyperplane arrangement $\mathcal{A}$. We denote by $\Omega^j_{\mathcal{A}}$ the $\mathbb{Z}$ submodule of $\Omega_{\mathcal{A}}$ consisting of j-forms and we regard $\Omega_{\mathcal{A}} = \bigoplus_j \Omega^j_{\mathcal{A}}$ as a subcomplex of the de Rham complex of smooth differential forms on $M_{\mathcal{A}}$. Here the differential of the complex $\Omega_{\mathcal{A}}$ is identically zero.

Following [2], we recall the definition of the bar construction for the de Rham complex. Let $A = \bigoplus_{j \geqslant 0} A^j$ be a differential graded algebra. We denote by A^{*-1} the differential graded algebra whose degree j part is given by A^{j+1} if $j \geqslant 1$ and by A^1/dA^0 if $j = 0$. Let M be a smooth manifold and we denote by $\mathcal{E}(M)$ the de Rham complex of smooth differential forms with values in $\mathbb{K}$ defined on M. We consider the tensor algebra

$$T^\bullet\big(\mathcal{E}^{*-1}(M)\big) = \bigoplus_{k \geqslant 0}\Big[\bigotimes^k \mathcal{E}^{*-1}(M)\Big]$$

over the field $\mathbb{K}$. By extending the degree for an element of $\mathcal{E}^{*-1}(M)$ multiplicatively, we put on $T^\bullet(\mathcal{E}^{*-1}(M))$ a structure of a graded algebra. We set

$$B^{p,-q}(M) = \Big[\bigotimes^q \mathcal{E}^{*-1}(M)\Big]^{p-q},$$

where the right hand side stands for the degree $p - q$ part of the q-fold tensor product of $\mathcal{E}^{*-1}(M)$. We have a direct sum decomposition

$$T^\bullet\big(\mathcal{E}^{*-1}(M)\big) = \bigoplus_{p,q} B^{p,-q}(M).$$

We denote by $T^j(\mathcal{E}^{*-1}(M))$ the degree j part of the complex $T^\bullet(\mathcal{E}^{*-1}(M))$ with respect to the above grading. We have

$$T^j\big(\mathcal{E}^{*-1}(M)\big) = \bigoplus_{p-q=j} B^{p,-q}(M).$$

Let us notice that an element of $B^{p,-q}(M)$ is represented by a linear combination of $\varphi_1 \otimes \cdots \otimes \varphi_q$ where φ_j is a differential p_j form such that $\sum_{j=1}^q p_j = p$.

For a differential k form φ we set $J\varphi = (-1)^k\varphi$. With this convention we introduce the following two differentials. We define $d_1 : B^{p,-q}(M) \to B^{p+1,-q}(M)$ by

$$d_1(\varphi_1 \otimes \cdots \otimes \varphi_q) = \sum_{i=1}^{q}(-1)^i J\varphi_1 \otimes \cdots \otimes J\varphi_{i-1} \otimes d\varphi_i \otimes \varphi_{i+1} \otimes \cdots \otimes \varphi_q$$

and $d_2 : B^{p,-q}(M) \to B^{p,-q+1}(M)$ by

$$d_2(\varphi_1 \otimes \cdots \otimes \varphi_q) = \sum_{i=1}^{q-1} (-1)^{i-1} J\varphi_1 \otimes \cdots \otimes J\varphi_{i-1} \otimes \big[(J\varphi_i) \wedge \varphi_{i+1}\big] \otimes \varphi_{i+2} \otimes \cdots \otimes \varphi_q.$$

By the differentials d_1 and d_2 the algebra $T^\bullet(\mathcal{E}^{*-1}(M)) = \bigoplus_{p,q} B^{p,-q}(M)$ is equipped with a structure of a double complex. The associated total complex with the differential $d = d_1 + d_2$ is denoted by $B^\bullet(\mathcal{E}(M))$, which we shall call the bar construction for the de Rham complex of M. In the following the complex $B^\bullet(\mathcal{E}(M))$ is also denoted simply by $B^\bullet(M)$.

Let us denote by $L_x M$ the loop space of M with a fixed base point $x \in M$. The complex $B^\bullet(\mathcal{E}(M))$ is related to the de Rham complex of the loop space $L_x M$ by means of Chen's iterated integrals in the following way. We denote by

$$\pi_i : M^q = \underbrace{M \times \cdots \times M}_{q} \to M$$

the projection on the ith factor. For differential forms $\varphi_1, \ldots, \varphi_q$ on M, we put

$$\varphi_1 \times \cdots \times \varphi_q = \pi_1^*\varphi_1 \wedge \cdots \wedge \pi_q^*\varphi_q.$$

Let Δ_q denote the q-simplex defined by

$$\Delta_q = \big\{(t_1, \cdots, t_q);\ 0 \leqslant t_1 \leqslant \cdots \leqslant t_q \leqslant 1\big\}.$$

We have an evaluation map

$$e : L_x M \times \Delta_q \to M^q$$

defined by

$$e(\gamma; t_1, \ldots, t_q) = \big(\gamma(t_1), \ldots, \gamma(t_q)\big).$$

Chen's iterated integral of the differential forms $\varphi_1, \ldots, \varphi_q$ is defined using the integration along the fiber of the projection map $p : L_x M \times \Delta_q \to L_x M$ by

$$\int_{\Delta_q} e^*(\varphi_1 \times \cdots \times \varphi_q). \tag{1.3}$$

We denote the integral (1.3) by

$$\int \varphi_1 \cdots \varphi_q.$$

We consider the above integral as a differential form of degree $p - q$ on the loop space $L_x M$. In particular, if $\varphi_1, \ldots, \varphi_q$ are 1-forms, the integral (1.3) defines a function on the loop space $L_x M$. We denote by

$$\int_\gamma \varphi_1 \cdots \varphi_q$$

the value of this function at the loop γ. Let $\mathcal{E}(L_xM)$ denote the de Rham complex of the loop space L_xM. We have a map

$$\iota : B^\bullet(M) \to \mathcal{E}(L_xM)$$

defined by

$$\iota(\varphi_1 \otimes \cdots \otimes \varphi_q) = \int \varphi_1 \cdots \varphi_q. \tag{1.4}$$

It was shown by Chen [2] that ι is a cochain map. In [2] Chen proved also that ι is injective if the manifold M is connected. Therefore by means of the map ι the bar complex $B^\bullet(M)$ can be considered as a subcomplex of the de Rham complex of the loop space $\mathcal{E}(L_xM)$.

Now let us apply the above construction to the Orlik–Solomon algebra $\Omega_{\mathcal{A}}$. Namely, we put

$$B^\bullet(\Omega_{\mathcal{A}}) = \bigoplus_{k \geqslant 0} \left(\bigotimes^k \left(\Omega_{\mathcal{A}}^{*-1}\right) \right).$$

The algebra $B^\bullet(\Omega_{\mathcal{A}})$ has a structure of a graded algebra over $\mathbb{Z}$. The differential d_1 is identically zero in this case and therefore the differential d is given by

$$\begin{aligned} &d(\varphi_1 \otimes \cdots \otimes \varphi_q) \\ &\quad = \sum_{i=1}^{q-1} (-1)^{i-1} J\varphi_1 \otimes \cdots \otimes J\varphi_{i-1} \otimes \left[(J\varphi_i) \wedge \varphi_{i+1}\right] \otimes \varphi_{i+2} \otimes \cdots \otimes \varphi_q. \end{aligned}$$

We shall call the complex $B^\bullet(\Omega_{\mathcal{A}})$ the bar complex of the Orlik–Solomon algebra.

There exists an injective homomorphism of complexes

$$i : B^\bullet(\Omega_{\mathcal{A}}) \to B^\bullet\left(\mathcal{E}(M_{\mathcal{A}})\right).$$

We have the following theorem.

Theorem 1.5. *The inclusion map $i : B^\bullet(\Omega_{\mathcal{A}}) \to B^\bullet(\mathcal{E}(M_{\mathcal{A}}))$ induces an isomorphism of the cohomology*

$$i_* : H^j\left(B^\bullet(\Omega_{\mathcal{A}})\right) \otimes \mathbb{K} \cong H^j\left(B^\bullet(M_{\mathcal{A}})\right)$$

for any j.

Proof. We put

$$\mathcal{F}^{-q} B^\bullet\left(\mathcal{E}(M_{\mathcal{A}})\right) = \bigoplus_{q' \leqslant q} B^{p,-q'}\left(\mathcal{E}(M_{\mathcal{A}})\right),$$

which defines a decreasing filtration

$$\mathbb{K} = \mathcal{F}^0 \subset \mathcal{F}^{-1} \subset \mathcal{F}^{-2} \subset \cdots \tag{1.6}$$

on $B^\bullet(\mathcal{E}(M_{\mathcal{A}}))$. We use the argument of the spectral sequence associated with this filtration as in [2]. The E_1-term of the spectral sequence is described as

$$E_1^{p,-q}\left(\mathcal{E}(M_{\mathcal{A}})\right) \cong H^{p-q}\left(\bigotimes^q \mathcal{E}^{*-1}(M_{\mathcal{A}}) \right). \tag{1.7}$$

The right hand side of (1.7) is isomorphic to the degree $p-q$ part of the q-fold tensor product of the de Rham cohomology ring

$$\left[\bigotimes^q H_{DR}^{*-1}(M_{\mathcal{A}})\right]^{p-q},$$

where the degree for the de Rham cohomology is shifted by 1 as before. Similarly we define a decreasing filtration on the Orlik–Solomon algebra by putting

$$\mathcal{F}^{-q} B^\bullet(\Omega_{\mathcal{A}}) = \bigoplus_{q' \leqslant q} B^{p,-q'}(\Omega_{\mathcal{A}}).$$

The inclusion map $i : B^\bullet(\Omega_{\mathcal{A}}) \to B^\bullet(\mathcal{E}(M_{\mathcal{A}}))$ preserves the above filtrations. Since the differential acts trivially on

$$E_0^{p,-q}(\Omega_{\mathcal{A}}) \cong B^{p,-q}(\Omega_{\mathcal{A}})$$

we have

$$E_1^{p,-q}(\Omega_{\mathcal{A}}) \cong B^{p,-q}(\Omega_{\mathcal{A}}),$$

which is isomorphic to the degree $p-q$ part of $\bigotimes^q \Omega_{\mathcal{A}}^{*-1}$. Applying a result due to Brieskorn [1] and the de Rham theorem, we have an isomorphism

$$\Omega_{\mathcal{A}} \otimes \mathbb{K} \cong H_{DR}^*(M_{\mathcal{A}}).$$

Hence we have an isomorphism

$$E_1^{p,-q}(\Omega_{\mathcal{A}}) \otimes \mathbb{K} \cong E_1^{p,-q}\big(\mathcal{E}(M_{\mathcal{A}})\big),$$

which induces an isomorphism

$$E_\infty^{p,-q}(\Omega_{\mathcal{A}}) \otimes \mathbb{K} \cong E_\infty^{p,-q}\big(\mathcal{E}(M_{\mathcal{A}})\big).$$

Thus the inclusion map i induces an isomorphism

$$i_* : H^j\big(B^\bullet(\Omega_{\mathcal{A}})\big) \otimes \mathbb{K} \cong H^j(B^\bullet\big(M_{\mathcal{A}})\big)$$

for any j. This completes the proof. □

2. Derivation on the homology

Let $1, \omega_1, \ldots, \omega_m$ be a basis of the Orlik–Solomon algebra $\Omega_{\mathcal{A}}$ as a free module over $\mathbb{Z}$. Here we order the basis in such a way that ω_j, $1 \leqslant j \leqslant r$, is given by the logarithmic 1-form defined in (1.1). We denote by $[\omega_j]$ the de Rham cohomology class of the differential form ω_j. We suppose that ω_j is a p_j-form. We choose a basis $X_1, \ldots, X_m$ of the reduced homology $\overline{H}_*(M_{\mathcal{A}}, \mathbb{Z})$ so that the condition $\langle [\omega_i], X_j \rangle = \delta_{ij}$ is satisfied for $1 \leqslant i, j \leqslant m$,

We denote by $H_{*-1}(M_{\mathcal{A}}, \mathbb{Z})$ the graded algebra whose degree j part is defined to be the homology group $H_{j+1}(M_{\mathcal{A}}, \mathbb{Z})$ if $j \geqslant 0$ and is defined to be 0 if $j < 0$. Let us consider the tensor algebra

$$T_\bullet\big(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z})\big) = \bigoplus_{k \geqslant 0} \left(\bigotimes^k H_{*-1}(M_{\mathcal{A}}, \mathbb{Z})\right),$$

which has a structure of a graded algebra by extending the grading of $H_{*-1}(M_{\mathcal{A}}, \mathbb{Z})$ multiplicatively on the tensor product. The graded algebra $T_\bullet(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z}))$ is identified with the free non-commutative associative graded algebra generated by $X_1, \ldots, X_m$ over $\mathbb{Z}$ with

$$\deg X_j = p_j - 1, \quad 1 \leqslant j \leqslant m.$$

We denote by $T_j(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z}))$ the degree j part of the above graded algebra $T_\bullet(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z}))$.

We define the bracket on $T_\bullet(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z}))$ by

$$[X, Y] = XY - (-1)^{pq} YX$$

for homogeneous elements X and Y of degree p and q, respectively. Let us note that the above bracket satisfies the graded anti-commutative relation

$$[X, Y] + (-1)^{pq}[Y, X] = 0$$

and the graded Jacobi identity

$$(-1)^{rp}\big[[X, Y], Z\big] + (-1)^{pq}\big[[Y, Z], X\big] + (-1)^{qr}\big[[Z, X], Y\big] = 0$$

for homogeneous elements X, Y, Z of degree p, q and r, respectively.

By a derivation on $T_\bullet(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z}))$ we mean a $\mathbb{Z}$ linear endomorphism of degree -1 with $\partial \circ \partial = 0$ such that, for homogeneous elements u and v the condition

$$\partial(uv) = (\partial u)v + (-1)^{\deg u} u(\partial v) \tag{2.1}$$

is satisfied. It follows that ∂ is uniquely determined by ∂X_k, $1 \leqslant k \leqslant m$. We introduce a derivation on $T_\bullet(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z}))$ in the following way. When the wedge product $\omega_i \wedge \omega_j$ is written as

$$\omega_i \wedge \omega_j = \sum_k c_{ij}^k \omega_k, \quad c_{ij}^k \in \mathbb{Z},\ 1 \leqslant i < j \leqslant m,$$

we define ∂X_k by

$$\partial X_k = \sum_{1 \leqslant i < j \leqslant m} (-1)^{p_i} c_{ij}^k\, [X_i, X_j] \tag{2.2}$$

if $\deg X_k > 0$ and by $\partial X_k = 0$ if $\deg X_k = 0$. Let us note that ∂X_k is also expressed as

$$\partial X_k = \frac{1}{2} \sum_{1 \leqslant i, j \leqslant m} (-1)^{p_i} c_{ij}^k\, [X_i, X_j]$$

for $\deg X_k > 0$. The property $\partial \circ \partial = 0$ might be verified directly by means of the graded Jacobi identity and the graded associativity of the Orlik–Solomon algebra, but it is a consequence of the fact that the differential d of the bar complex satisfies $d \circ d = 0$ and Proposition 2.3.

The above derivation ∂ is also characterized by Chen's formal connection in the following way. Let ω be an element of $\Omega_{\mathcal{A}} \otimes T_\bullet(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z}))$ defined by

$$\omega = \sum_{j=1}^m \omega_j \otimes X_j.$$

We define $\kappa \in \Omega_{\mathcal{A}} \otimes T_\bullet(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z}))$ as

$$\kappa = -\sum_{1 \leqslant i < j \leqslant m} \left[(-1)^{p_i} \omega_i \wedge \omega_j\right] \otimes X_i X_j.$$

We see that the derivation ∂ satisfies

$$\partial \omega + \kappa = 0,$$

which shows that ω is a formal connection in the sense of Chen and that κ is its curvature. Let us observe that the derivation ∂ is quadratic in our case.

Proposition 2.3. *The bar complex $B^\bullet(\Omega_{\mathcal{A}})$ and the complex $T_\bullet(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z}))$ with the derivation ∂ are dual to each other.*

Proof. The canonical pairing

$$\Omega^j_{\mathcal{A}} \times H_j(M_{\mathcal{A}}, \mathbb{Z}) \to \mathbb{Z}$$

induces the duality pairing

$$\langle\ ,\ \rangle : B^\bullet(\Omega_{\mathcal{A}}) \times T_\bullet\big(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z})\big) \to \mathbb{Z}.$$

We are going to prove that

$$\langle d\varphi, X\rangle = \langle \varphi, \partial X\rangle \tag{2.4}$$

holds for $\varphi \in B^\bullet(\Omega_{\mathcal{A}})$ and $X \in T_\bullet(H_{*-1}(M_{\mathcal{A}}, \mathbb{Z}))$. First we show the equality

$$\big\langle d(\omega_i \otimes \omega_j), X_k\big\rangle = \langle \omega_i \otimes \omega_j, \partial X_k\rangle \tag{2.5}$$

for any $1 \leqslant i, j, k \leqslant m$. We have

$$\begin{aligned}\langle \omega_i \otimes \omega_j, \partial X_k\rangle &= \frac{1}{2}\sum_{s,t}\big\langle \omega_i \otimes \omega_j, (-1)^{p_s} c^k_{st}[X_s, X_t]\big\rangle \\ &= \frac{1}{2}\big((-1)^{p_i} c^k_{ij} - (-1)^{p_j}(-1)^{(p_i-1)(p_j-1)} c^k_{ji}\big) = (-1)^{p_i} c^k_{ij}.\end{aligned}$$

This shows the desired assertion. To prove Proposition 2.3 it will be enough to show the equality (2.4) when φ and X are of the form $\varphi = \omega_{i_1} \otimes \cdots \otimes \omega_{i_p}$ and $X = X_{j_1} \cdots X_{j_{p-1}}$. In this case we have

$$\begin{aligned}\langle \mathrm{d}\varphi, X\rangle &= \sum_{k=1}^{p} (-1)^{k-1} \langle J\omega_{i_1}, X_{j_1}\rangle \cdots \langle J\omega_{i_{k-1}}, X_{j_{k-1}}\rangle \\ &\quad \times \big\langle (J\omega_{i_k}) \wedge \omega_{i_{k+1}}, X_{j_k}\big\rangle \langle \omega_{i_{k+2}}, X_{j_{k+1}}\rangle \cdots \langle \omega_{i_p}, X_{j_{p-1}}\rangle.\end{aligned}$$

On the other hand, using the the formula (2.1) and the fact that ∂X_{j_k} is quadratic, we have

$$\begin{aligned}\langle \varphi, \partial X\rangle &= \sum_{k=1}^{p} (-1)^{\deg X_{j_1} + \cdots + \deg X_{j_{k-1}}} \langle \omega_{i_1}, X_{j_1}\rangle \cdots \\ &\quad \times \langle \omega_{i_{k-1}}, X_{j_{k-1}}\rangle \langle \omega_{i_k} \otimes \omega_{i_{k+1}}, \partial X_{j_k}\rangle \langle \omega_{i_{k+2}}, X_{j_{k+1}}\rangle \cdots \langle \omega_{i_p}, X_{j_{p-1}}\rangle.\end{aligned}$$

Combining with the assertion (2.5), we have our proposition. □

Corollary 2.6. *The bar cohomology $H^j(B^\bullet(M_{\mathcal{A}}))$ and the jth homology of the complex $T_\bullet(H_{*-1}(M_{\mathcal{A}},\mathbb{K}))$ with the derivation ∂ are dual to each other.*

3. Description of the 0th homology

In the following we write $T_\bullet(M_{\mathcal{A}})$ for the complex $T_\bullet(H_{*-1}(M_{\mathcal{A}},\mathbb{Z}))$ with the derivation ∂ defined in the previous section. In this section we focus on the 0th homology $H_0(T_\bullet(M_{\mathcal{A}}))$.

Let us denote by $\mathbb{Z}\langle X_1,\dots,X_r\rangle$ the free non-commutative associative algebra generated by $X_1,\dots,X_r$ over $\mathbb{Z}$. The degree 0 part $T_0(M_{\mathcal{A}})$ of the complex $T_\bullet(M_{\mathcal{A}})$ is identified with the algebra $\mathbb{Z}\langle X_1,\dots,X_r\rangle$. We define $\mathcal{J}$ to be the two sided ideal of $\mathbb{Z}\langle X_1,\dots,X_k\rangle$ generated by

$$[X_{i_s}, X_{i_1}+\cdots+X_{i_p}],\quad 1<s\leqslant p,$$

for any maximal family of hyperplanes $H_{i_1},\dots,H_{i_p}$ with $i_1<\cdots<i_p$ such that the condition

$$\operatorname{codim}_{\mathbb{C}}[H_{i_1}\cap\cdots\cap H_{i_p}]=2$$

is satisfied. Let us notice that the homology $H_0(T_\bullet(M_{\mathcal{A}}))$ has a natural structure of a Hopf algebra where the coproduct is defined by means of the natural product of the tensor algebra in $B^\bullet(M_{\mathcal{A}})$.

Proposition 3.1. *We have an isomorphism of Hopf algebras*

$$H_0\big(T_\bullet(M_{\mathcal{A}})\big)\cong\mathbb{Z}\langle X_1,\dots,X_r\rangle/\mathcal{J}.$$

Proof. We show that the image of the derivation $\partial: T_1(M_{\mathcal{A}})\to T_0(M_{\mathcal{A}})$ coincides with the ideal $\mathcal{J}$. For each maximal family of hyperplanes $H_{i_1},\dots,H_{i_p}$ with $\operatorname{codim}_{\mathbb{C}}[H_{i_1}\cap\cdots\cap H_{i_p}]=2$, we associate the 2-forms $\omega_{i_1}\wedge\omega_{i_s}$, $1<s\leqslant p$. It follows from Brieskorn's Lemma ([1], see also [6]) that the degree 2 part of the Orlik–Solomon algebra $\Omega^2_{\mathcal{A}}$ is spanned by such differential forms. We denote by $X_{i_1 i_s}$ the dual element of $\omega_{i_1}\wedge\omega_{i_s}$ in $H_2(M_{\mathcal{A}},\mathbb{Z})$. By means of the relation

$$\omega_i\wedge\omega_j+\omega_j\wedge\omega_k+\omega_k\wedge\omega_i=0$$

for any hyperplanes H_i, H_j, H_k with $\operatorname{codim}_{\mathbb{C}}[H_i\cap H_j\cap H_k]=2$, we see that $\partial X_{i_1 i_s}$ is equal to $[X_{i_s}, X_{i_1}+\cdots+X_{i_p}]$ up to sign. Since $T_1(M_{\mathcal{A}})$ is generated by $X_{i_1 i_s}$, $1<s\leqslant p$, over $T_0(M_{\mathcal{A}})$ we conclude that image $\partial=\mathcal{J}$. This completes the proof. □

We now describe a relation to the holonomy Lie algebra. Let us denote by $F(H_1(M_{\mathcal{A}}))$ the free Lie algebra over $\mathbb{Z}$ generated by $H_1(M_{\mathcal{A}},\mathbb{Z})$. We define

$$\mu: H_1(M_{\mathcal{A}},\mathbb{Z})\wedge H_1(M_{\mathcal{A}},\mathbb{Z})\to H_2(M_{\mathcal{A}},\mathbb{Z})$$

as the dual of the cup product homomorphism. We define $\mathcal{I}$ to be the ideal of the free Lie algebra $F(H_1(M_{\mathcal{A}}))$ generated by the image of μ. Here we identify the wedge product and the Lie bracket. We define the holonomy Lie algebra associated with the arrangement $\mathcal{A}$ as $F(H_1(M_{\mathcal{A}}))/\mathcal{I}$ and denote it by $\mathfrak{g}_{\mathcal{A}}$. Let us denote by $PH_0(T_\bullet(M_{\mathcal{A}}))$ the primitive part of $H_0(T_\bullet(M_{\mathcal{A}}))$ with respect to the above coproduct. The following is a consequence of Proposition 3.1.

Corollary 3.2. *The primitive part $PH_0(T_\bullet(M_{\mathcal{A}}))$ is isomorphic to the holonomy Lie algebra $\mathfrak{g}_{\mathcal{A}}$. The homology $H_0(T_\bullet(M_{\mathcal{A}}))$ is isomorphic to the universal enveloping algebra of the holonomy Lie algebra $\mathfrak{g}_{\mathcal{A}}$.*

Remark 3.3. We give a brief description on a relation to the fundamental group $\pi_1(M_{\mathcal{A}})$. We denote by $\mathbb{K}[\pi_1(M_{\mathcal{A}})]$ the group ring of $\pi_1(M_{\mathcal{A}})$ over $\mathbb{K}$ and by J its augmentation ideal. It is a consequence of a general result due to Chen [2] and Proposition 3.1 that we have an isomorphism of Hopf algebras

$$\mathbb{K}\big[\pi_1(M_{\mathcal{A}})\big]/J^{q+1} \cong \mathbb{K}\langle X_1, \ldots, X_r\rangle/\big(\mathcal{J} + I^{q+1}\big) \tag{3.4}$$

where I is the augmentation ideal of $\mathbb{K}\langle X_1, \ldots, X_r\rangle$. These algebras are related to the bar complex in the following way. The filtration (1.6) is compatible with the differential and induces a decreasing filtration on the cohomology $H^0(B^\bullet(M_{\mathcal{A}}))$. By means of the duality in Proposition 2.3 we observe that the algebras (3.4) are dual to $\mathcal{F}^{-q}H^0(B^\bullet(M_{\mathcal{A}}))$. The primitive part of the completed group algebra

$$\varprojlim \mathbb{K}\big[\pi_1(M_{\mathcal{A}})\big]/J^{q+1}$$

is isomorphic to the Malcev completion of $\pi_1(M_{\mathcal{A}})$ over $\mathbb{K}$.

Theorem 3.5. *The iterated integral of logarithmic* 1*-forms on a loop γ*

$$\sum_{1\leqslant i_1,\ldots,i_q\leqslant r} \int_\gamma a_{i_1\cdots i_q}\omega_{i_1}\cdots\omega_{i_q}, \quad a_{i_1\cdots i_q} \in \mathbb{Z},$$

depends only on the homotopy class of the loop γ if and only if the associated element in the bar complex of the Orlik–Solomon algebra satisfies

$$\left\langle \sum_{1\leqslant i_1,\ldots,i_q\leqslant r} a_{i_1\cdots i_q}\omega_{i_1}\otimes\cdots\otimes\omega_{i_q}, X\right\rangle = 0$$

for any $X \in \mathcal{J}$. Moreover, the above iterated integrals span $H^0(B^\bullet(M_{\mathcal{A}}))$.

Proof. We show that

$$d\left(\sum_{1\leqslant i_1,\ldots,i_q\leqslant r} \int a_{i_1\cdots i_q}\omega_{i_1}\cdots\omega_{i_q}\right) = 0 \tag{3.6}$$

holds if and only if the condition

$$\left\langle \sum_{1\leqslant i_1,\ldots,i_q\leqslant r} a_{i_1\cdots i_q}\omega_{i_1}\otimes\cdots\otimes\omega_{i_q}, X\right\rangle = 0 \tag{3.7}$$

is satisfied for any $X \in \mathcal{J}$. Since ι defined in (1.4) is an injective cochain map, the condition (3.6) is equivalent to the condition

$$d\left(\sum_{1\leqslant i_1,\ldots,i_q\leqslant r} a_{i_1\cdots i_q}\omega_{i_1}\otimes\cdots\otimes\omega_{i_q}\right)=0$$

in $B^{q,-q}(M_{\mathcal{A}})$. By means of the duality in Proposition 2.3 we see that this is equivalent to the condition

$$\left\langle\sum_{1\leqslant i_1,\ldots,i_q\leqslant r} a_{i_1\cdots i_q}\omega_{i_1}\otimes\cdots\otimes\omega_{i_q}, \partial Z\right\rangle=0$$

for any $Z \in T_1(M_{\mathcal{A}})$. Combining with Proposition 3.1, we obtain the first assertion of Theorem 3.5. We have shown that the 0th cohomology of the bar complex of the Orlik–Solomon algebra is spanned by the iterated integrals of logarithmic 1-forms with the condition (3.7). Therefore we obtain the last assertion of Theorem 3.5 by means of Theorem 1.5. This completes the proof. □

Remark 3.8. Let us consider the case when the arrangement $\mathcal{A}$ consists of diagonal hyperplanes H_{ij}, $1 \leqslant i < j \leqslant n$, defined by $z_i = z_j$, where $z_1, \ldots, z_n$ are coordinate functions for $\mathbb{C}^n$. The condition (3.7) coincides with the 4 term relation for the weight system of the Vassiliev invariants for pure braids. We refer the readers to [5] for more details on this aspect.

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