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The universal finite-type invariant for braids, with integer coefficients

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Abstract

We give a simple, explicit construction of a universal finite-type invariant for braids, which is multiplicative, at the associated graded level. Our approach is valid for arbitrary ring coefficients. A key ingredient is provided by the properties of fundamental groups of fiber-type arrangements. We examine the relation with the Vassiliev theory for links. © 2002 Elsevier Science B.V. All rights reserved.

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Introduction

The fundamental theorem from the Vassiliev (finite-type) theory for knots and links gives the construction of the *universal finite-type invariant with* $\mathbb{Q}$ *coefficients*. It is due to Kontsevich [12] (see also [1,14,18]).

The theorem relates two filtered objects. On the geometric side, one has $\mathbb{Z}[\mathcal{L}]$, the free abelian group generated by isotopy classes of oriented links. It is endowed with the Vassiliev filtration, $\{VF_k(\mathcal{L})\}_k$, obtained from resolutions of singular links. On the combinatorial side, there is $\widetilde{\mathcal{A}}^*$, the graded abelian group generated by closed chord diagrams, modulo four-term and framing independence relations. Its completion, $\widehat{\widetilde{\mathcal{A}}}$, carries the canonical filtration.

The universal finite-type invariant for links is a filtered map, $Z:\mathbb{Q}[\mathcal{L}]\to\widehat{\widetilde{\mathcal{A}}}\otimes\mathbb{Q}$, which induces an isomorphism at the associated graded level, $\mathrm{gr}^*(Z):\mathrm{gr}_V^*(\mathcal{L})\otimes\mathbb{Q}\xrightarrow{\sim}\widetilde{\mathcal{A}}^*\otimes\mathbb{Q}$.

For the time being, it is not known whether this result holds, with integer coefficients.

For each fixed n, there are similar objects, related to braids. On one hand, one has $\mathbb{Z}[B_n]$, the group ring of Artin's braid group on n strands, B_n. It carries the (multiplicative)

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Vassiliev-type filtration, $\{VF_k(B_n)\}_k$, constructed from resolutions of singular braids. On the other hand, there is the graded $\mathbb{Z}$-algebra of horizontal chord diagrams on n vertical segments, U_n^*, and the semidirect product graded algebra $U_n^* \rtimes \mathbb{Z}[\Sigma_n]$, obtained from the natural action of the symmetric group Σ_n on U_n^*. The completion, $\widehat{U_n} \rtimes \mathbb{Z}[\Sigma_n]$, is equipped with the canonical filtration, as before.

Our *universal finite-type invariant for braids, with $\mathbb{Z}$-coefficients*, is a filtered map, $M : \mathbb{Z}[B_n] \to \widehat{U_n} \rtimes \mathbb{Z}[\Sigma_n]$, inducing a graded algebra isomorphism,

$$\mathrm{gr}^*(M) : \mathrm{gr}_V^*(B_n) \xrightarrow{\sim} U_n^* \rtimes \mathbb{Z}[\Sigma_n].$$

See Theorem 1.1.

We use two key tools for the proof. Firstly, we derive in Corollary 4.2 the $\mathbb{Z}$-form of a general result of Quillen [19], for the case of the pure braid group, P_n. The theorem of Quillen establishes a strong connection between the I-adic filtration of an arbitrary group ring and the lower central series filtration of the group, with $\mathbb{Q}$-coefficients (see Theorem 4.1). Secondly, we show that the I-adic filtration of $\mathbb{Z}[P_n]$ behaves nicely with respect to the well-known normal form of pure braids; see Theorem 3.1 for the precise result, in a more general framework.

The additive isomorphism between $\mathrm{gr}_V^*(B_n)$ and $U_n^* \rtimes \mathbb{Z}[\Sigma_n]$ was obtained by Lin [13], with different methods. He also proved that the above groups are free. This implies that any additive isomorphism, $\mathrm{gr}_V^*(B_n) \xrightarrow{\sim} U_n^* \rtimes \mathbb{Z}[\Sigma_n]$, may be lifted (nonexplicitly) to a universal $\mathbb{Z}$-invariant, $\mathbb{Z}[B_n] \to \widehat{U_n} \rtimes \mathbb{Z}[\Sigma_n]$. The novelty here is twofold. Firstly, we show that $\mathrm{gr}^*(M)$ is multiplicative; see Proposition 1.5. Secondly, our construction of M is simple and explicit; see (1.6)–(1.9). This construction depends on a manageable set of parameters, with a conveniently large degree of freedom. At the same time, $\mathrm{gr}^*(M)$ is independent of the parameters; see Remark 4.5.

We also prove, in Corollary 1.4, that the filtration $\{VF_k(B_n)\}$ is separate, hence in particular our universal $\mathbb{Z}$-invariant M classifies braids. This result was previously known for $\mathbb{Q}$-coefficients; see [2,11].

We are not yet able to construct a universal link invariant, in this way. Nevertheless, we have two results in this direction at the associated graded level, with $\mathbb{Z}$-coefficients. At this level, one has the *basic geometric construction* in the finite-type theory for links. It provides a surjective map, $f : \widetilde{\mathcal{A}}^* \to \mathrm{gr}_V^*(\mathcal{L})$. One knows that $f \otimes \mathbb{Q}$ is an isomorphism, by the Kontsevich theorem, but the injectivity problem for f is open, with $\mathbb{Z}$-coefficients.

It is known that by *Artin closure*, for singular braids and for horizontal chord diagrams, one has surjective maps, $\bigoplus_n \mathrm{gr}_V^*(B_n) \to \mathrm{gr}_V^*(\mathcal{L})$, and $\bigoplus_n (U_n^* \rtimes \mathbb{Z}[\Sigma_n]) \to \widetilde{\mathcal{A}}^*$, respectively, as a consequence of the singular Alexander theorem [5].

We prove in Theorem 5.1 that $\mathrm{gr}^*(M)$ is compatible with f, via the above Artin closure operations; see also (5.11). Theorem 5.2 gives a partial factorization result, for $\mathrm{gr}^*(M)$ followed by Artin closure of horizontal chord diagrams, through braid closure. By the preceding discussion, it factors over $\mathbb{Q}$. The full factorization result over $\mathbb{Z}$ is actually equivalent to the fact that $f : \widetilde{\mathcal{A}}^* \to \mathrm{gr}_V^*(\mathcal{L})$ is an isomorphism. See Remark 5.3 for another equivalent reformulation of this open question.

After the completion of this work, we learned about the preprint [16], which contains similar results related to the construction of the universal $\mathbb{Z}$-invariant, for *pure* braids. The methods of [16] are more elementary; at the same time, their results are less general (see Theorem 3.1). Note also that the authors of [16] do not investigate the connection with finite-type link theory.

1. The construction

Our main result relates two filtered objects which are naturally connected with braids, like in the Vassiliev theory for knots and links. For braids, we will follow the standard notation and conventions from [4]. For singular links and braids, we will follow [6,5,13].

The first object is $\mathbb{Z}[B_n]$, the integral group ring of Artin's braid group on n strings, B_n. Its *Vassiliev-type filtration*, $\{VF_k\}_{k\geqslant 0}$, is defined geometrically, via *resolutions of singular n-braids*. For a singular braid β with r transverse double points, $R(\beta) \in \mathbb{Z}[B_n]$ denotes the usual signed sum of the 2^r possible resolutions of the double points into positive or negative crossings. Then VF_k is additively generated, by definition, by all resolutions $R(\beta)$, where β has at least k double points.

If $\beta = \alpha_0\tau_{i_1}\alpha_1\cdots\tau_{i_r}\alpha_r$, as in [5], where $\alpha_0,\ldots,\alpha_r \in B_n$ and τ_i denotes the elementary singular n-braid with one double point involving the strings i and $i+1$, for $i = 1,\ldots,n-1$, then

$$R(\beta) = \alpha_0\bigl(\sigma_{i_1}^{-1} - \sigma_{i_1}\bigr)\alpha_1\cdots\bigl(\sigma_{i_r}^{-1} - \sigma_{i_r}\bigr)\alpha_r, \tag{1.1}$$

where $\sigma_1,\ldots,\sigma_{n-1}$ are the standard generators of B_n, like in [4]. Denote by J the two-sided ideal of $\mathbb{Z}[B_n]$ generated by $\sigma_i - \sigma_i^{-1}$, for $i = 1,\ldots,n-1$. It follows that we may describe the Vassiliev filtration of $\mathbb{Z}[B_n]$ purely algebraically; namely it coincides with the *J-adic filtration*: $VF_k = J^k$, for all k.

The second object is defined combinatorially. To describe it, we start with $\mathbb{Z}[t]$, the graded algebra of noncommutative polynomials with integer coefficients generated by the indeterminates t_{ij}, where $1 \leqslant i \neq j \leqslant n$ and $t_{ij} = t_{ji}$. The elements of filtration k are by definition the linear combinations of monomials of degree at least k. Denote by U_n the quotient graded algebra obtained from $\mathbb{Z}[t]$ by imposing the relations

$$[t_{ij}, t_{rs}] = 0, \tag{1.2}$$

for all distinct indices $1 \leqslant i, j, r, s \leqslant n$, and

$$[t_{ij}, t_{ik} + t_{jk}] = 0, \tag{1.3}$$

for all distinct $1 \leqslant i, j, k \leqslant n$. (Here the bracket $[x, y]$ stands for the algebra commutator, $xy - yx$.) It is endowed with the canonically induced filtration, $F_kU_n =: U_n^{\geqslant k}$. (Here and in the sequel upper indices denote degree.)

The symmetric group Σ_n acts on the graded algebra $\mathbb{Z}[t]$, by permuting the indices of the indeterminates. Obviously this action preserves the defining relations (1.2) and (1.3);

therefore we may consider the semidirect product graded algebra, $\mathcal{B}_n =: U_n \rtimes \mathbb{Z}[\Sigma_n]$, filtered by

$$F_k\mathcal{B}_n = U_n^{\geqslant k} \otimes \mathbb{Z}[\Sigma_n]. \tag{1.4}$$

This combinatorial construction corresponds to the algebra of *closed chord diagrams* from the Vassiliev knot theory. In more detail, the degree k monomials of $\mathbb{Z}[t]$ may be thought of as diagrams of k *horizontal chords*, with endpoints on n vertical segments oriented upwards. The product d_1d_2 of two such diagrams is defined by stacking d_1 on the top of d_2. The algebra generator t_{ij} corresponds to the one chord diagram with endpoints on the strings i and j; then one associates to every degree k monomial the corresponding product of k elementary horizontal chord diagrams. The commutation relations (1.2) translate to horizontal isotopy of chord diagrams without collision between endpoints, while the infinitesimal Yang–Baxter relations (1.3) correspond to the four-term relations from Vassiliev link theory. A typical element $a \otimes x$ of $\mathcal{B}_n$, where a is a degree k monomial and $x \in \Sigma_n$, gives then rise, by Artin closure, to a diagram of k chords on a set of oriented circles corresponding to the cycles in the decomposition of the permutation x; see Section 5 for more details.

Finally we have to complete, like in the Kontsevich theory for knots and links. The completion of $\mathbb{Z}[t]$ will be denoted by $\mathbb{Z}[[t]]$. It is the complete algebra of noncommutative formal series with integer coefficients in the indeterminates t_{ij}. Its quotient by the (closed) ideal generated by (1.2) and (1.3) will be denoted by $\widehat{U_n}$, the completion of U_n. Set $\widehat{\mathcal{B}}_n =: \widehat{U_n} \otimes \mathbb{Z}[\Sigma_n]$, and filter it by

$$F_k\widehat{\mathcal{B}}_n = \widehat{U_n}^{\geqslant k} \otimes \mathbb{Z}[\Sigma_n]. \tag{1.5}$$

(Here $\widehat{U_n}^{\geqslant k}$ denotes the formal series of order k, that is without terms in degree strictly less than k.)

Our construction of the universal finite type invariant for braids involves two choices. Firstly, let s be a (set map) section of the natural group epimorphism $\pi : B_n \to \Sigma_n$. Secondly, let $S_{ij} \in \mathbb{Z}[[t]]$ be an arbitrary formal series of the form

$$S_{ij} = 1 + t_{ij} + \text{ higher terms}, \tag{1.6}$$

for $1 \leqslant i < j \leqslant n$.

One knows that the pure braid group $P_n =: \ker \pi$ is an iterated semidirect product of free groups. More precisely, let $\{a_{ij} \mid 1 \leqslant i < j \leqslant n\}$ be the standard generators of P_n, see [4, Ch. 1]. Denote by G_j the subgroup generated by $\{a_{ij} \mid i < j\}$, for $j = 2, \ldots, n$. It is known that these subgroups are free, and $P_n = G_n \rtimes \cdots \rtimes G_2$.

We are going to view the (images of the) elements S_{ij} as units in the algebra $\widehat{U_n}$. We thus have, for each $j = 2, \ldots, n$, a group representation $\mu_j : G_j \to \widehat{U_n}$, defined on the free generators by

$$\mu_j(a_{ij}) = S_{ij}, \quad \text{for } i < j. \tag{1.7}$$

These give $\mu : P_n \to \widehat{U_n}$, by setting

$$\mu(p) = \mu_n(p_n)\cdots\mu_2(p_2), \quad \text{where } p = p_n\cdots p_2, \text{ with } p_j \in G_j,\ j = 2,\ldots,n. \tag{1.8}$$

(Here $p = p_n \cdots p_2$ is the *normal form* of the pure braid $p \in P_n$.)

Finally, every braid $b \in B_n$ may be uniquely written in the form

$$b = p \cdot s(x), \quad \text{with } p \in P_n \text{ and } x = \pi(b). \tag{1.9}$$

Set then $M(b) = \mu(p) \otimes x \in \widehat{\mathcal{B}}_n$, and extend linearly to $M : \mathbb{Z}[B_n] \to \widehat{\mathcal{B}}_n$. Our main result is the following.

Theorem 1.1. *The above M respects the filtrations and induces a multiplicative isomorphism at the associated graded level.*

The proof will be in two steps. A key role is played by the *I-adic filtration*, $\{I^k(G)\}_{k\geqslant 0}$. Given any group G, the *augmentation ideal* $I(G) \subset \mathbb{Z}[G]$ is additively generated by the elements of the form $g - 1$, with $g \in G$. Set $I = I(P_n)$. The first step is the following.

Proposition 1.2. *The map $\Phi : \mathbb{Z}[B_n] \to \mathbb{Z}[P_n] \otimes \mathbb{Z}[\Sigma_n]$, defined for $b \in B_n$ by $\Phi(b) = p \otimes x$, where $p \in P_n$ and $x \in \Sigma_n$ are as in* (1.9), *and then extended by linearity, is an isomorphism which identifies the filtrations $\{J^k\}$ of $\mathbb{Z}[B_n]$ and $\{I^k \otimes \mathbb{Z}[\Sigma_n]\}$ of $\mathbb{Z}[P_n] \otimes \mathbb{Z}[\Sigma_n]$.*

The second step of the proof of the theorem is as follows. (A similar result was obtained in [16, Theorem 5]; see Theorem 3.1 for a comparison between the methods of proof.)

Proposition 1.3. *The linear extension $\mu : \mathbb{Z}[P_n] \to \widehat{U_n}$ sends I^k into $\widehat{U}_n^{\geqslant k}$, for all $k \geqslant 0$, and induces an isomorphism at the associated graded level.*

Corollary 1.4. *Set $\mathrm{gr}_J^k\, \mathbb{Z}[B_n] = J^k/J^{k+1}$, for $k \geqslant 0$.*

(i) (Lin [13]) *The groups $\mathrm{gr}_J^k\, \mathbb{Z}[B_n]$ are torsion free, for all k.*

(ii) $\bigcap_k J^k = 0$, *in particular M is faithful on B_n.*

Proof. (i) Theorem 1.1 says that $\mathrm{gr}_J^k\, \mathbb{Z}[B_n]$ is isomorphic to $U_n^k \otimes \mathbb{Z}[\Sigma_n]$. On the other hand it is known [3, Ch. 3B] that all the groups U_n^k are torsion free and finitely generated. Explicit bases are described in [13, Theorem 2.3].

(ii) By Proposition 1.2 it is enough to prove that $\bigcap_k I^k = 0$, that is to verify that the pure braid group P_n is residually torsion free nilpotent, see [7, Proposition 2.2.1]. This fact in turn is well-known; it follows from the corresponding property of free groups [15, Ch. 5] and from the semidirect product structure of P_n, via the main result from [9, §3]. □

Set $\mathrm{gr}_J^*\, \mathbb{Z}[B_n] = \bigoplus_{k\geqslant 0} \mathrm{gr}_J^k\, \mathbb{Z}[B_n]$; it is a graded algebra, with multiplication induced from $\mathbb{Z}[B_n]$. Likewise, set $\mathrm{gr}_F^* \widehat{\mathcal{B}}_n = \bigoplus_{k\geqslant 0}(F_k\widehat{\mathcal{B}}_n/F_{k+1}\widehat{\mathcal{B}}_n)$; as a graded algebra, this equals the semidirect product $U_n^* \rtimes \mathbb{Z}[\Sigma_n]$, see (1.5). Finally, the multiplicative part in the statement of our theorem is clarified as below.

Proposition 1.5. *The induced map,* $\mathrm{gr}^*(M):\mathrm{gr}_J^*\mathbb{Z}[B_n] \to U_n^* \rtimes \mathbb{Z}[\Sigma_n]$, *is a graded algebra isomorphism.*

Remark 1.6. Over $\mathbb{Q}$, it is known that one may actually construct a *representation* of $\mathbb{Q}[B_n]$ in $\widehat{\mathcal{B}_n} \otimes \mathbb{Q}$, having all the properties stated in Theorem 1.1 above. See [8, Proposition 5.1] and [18]. The key fact is the existence of a Drinfel'd associator over $\mathbb{Q}$ [8]; nothing is known about the case of $\mathbb{Z}$ or mod p coefficients. The existence of a *multiplicative* universal invariant for braids, over $\mathbb{Z}$, is an open problem, too, even for pure braids.

We are going to give the proof of Proposition 1.2 in the next section, and to develop the proof of Proposition 1.3 in Sections 3 and 4. At the end of Section 4 we will derive Proposition 1.5.

2. The reduction to the case of pure braids

This is the easy step in the proof of our main result. It uses the next simple lemma, which is inspired by [20, Section 1].

Lemma 2.1. *Set* $B = \mathbb{Z}[B_n]$ *and* $I = I(P_n)$. *Then* $J^k = BI^kB = BI^k = I^kB$, *for all* k.

Proof. Obviously BI^k and I^kB are included in BI^kB. To see that we actually have equality, start with a typical element $a(p_1 - 1)\cdots(p_k - 1)b \in BI^kB$, where $a, b \in B_n$ and $p_1, \ldots, p_k \in P_n$. It can be rewritten in the form $({}^a p_1 - 1)\cdots({}^a p_k - 1)ab$ or $ab({}^{b^{-1}}p_1 - 1)\cdots({}^{b^{-1}}p_k - 1)$, which clearly belong to I^kB and BI^k respectively, since P_n is a normal subgroup of B_n. (Here we have denoted by ${}^x y =: xyx^{-1}$ the conjugation in a group.) Knowing that $BI^kB = BI^k = I^kB$, for all k, it will thus suffice to verify that $J = BIB$, in order to complete the proof of the lemma. In one direction, note that the generator $\sigma_i - \sigma_i^{-1}$ of J equals $\sigma_i^{-1}(\sigma_i^2 - 1) \in BIB$, since $\sigma_i^2 \in P_n$, and therefore $J \subset BIB$. The converse inclusion was noted in [20, Lemma 1.2]. □

2.2. Proof of Proposition 1.2

Define a map $\Psi : \mathbb{Z}[P_n] \otimes \mathbb{Z}[\Sigma_n] \to \mathbb{Z}[B_n]$ on basis elements, by $\Psi(p \otimes x) = p \cdot s(x)$, for $p \in P_n$ and $x \in \Sigma_n$. Plainly Φ and Ψ are inverse isomorphisms, see (1.9). We will check next that $\Phi(I^kB) \subset I^k \otimes \mathbb{Z}[\Sigma_n]$ and $\Psi(I^k \otimes \mathbb{Z}[\Sigma_n]) \subset I^kB$. Thanks to the equality $J^k = I^kB$ provided by the previous lemma, this will complete the proof of the proposition.

Pick any additive generator $(p_1 - 1)\cdots(p_k - 1)b \in I^kB$, with $p_1, \ldots, p_k \in P_n$ and $b = p \cdot s(x)$ as in (1.9). It is sent by Φ to $(p_1 - 1)\cdots(p_k - 1)p \otimes x$, which plainly belongs to $I^k \otimes \mathbb{Z}[\Sigma_n]$. Conversely, start with a typical element $(p_1 - 1)\cdots(p_k - 1) \otimes x$ from

$I^k \otimes \mathbb{Z}[\Sigma_n]$, where $p_1, \ldots, p_k \in P_n$ and $x \in \Sigma_n$. Its Ψ-image is $(p_1 - 1) \cdots (p_k - 1) \cdot s(x) \in I^k B$. $\square$

3. The I-adic filtration of an almost-direct product

We are now beginning the proof of Proposition 1.3. Recall from Section 1 that P_n is an iterated semidirect product of free subgroups, $P_n = G_n \rtimes \cdots \rtimes G_2$. We will identify in the sequel $\mathbb{Z}[P_n]$ and $\mathbb{Z}[G_n] \otimes \cdots \otimes \mathbb{Z}[G_2]$, by means of the additive isomorphism

$$\mathbb{Z}[P_n] \xrightarrow{\sim} \mathbb{Z}[G_n] \otimes \cdots \otimes \mathbb{Z}[G_2], \tag{3.1}$$

which sends $p \in P_n$ to $p_n \otimes \cdots \otimes p_2$; see (1.8). (Note, however, that this is not a *ring* isomorphism.) Firstly, we will show that

$$I^k(P_n) = \sum_{k_n + \cdots + k_2 = k} I^{k_n}(G_n) \otimes \cdots \otimes I^{k_2}(G_2), \tag{3.2}$$

for all k.

We will do this by induction on n, exploiting the semidirect product decomposition, $P_n = G_n \rtimes P_{n-1}$, see [4, Ch. 1]. Identify $\mathbb{Z}[P_n]$ and $\mathbb{Z}[G_n] \otimes \mathbb{Z}[P_{n-1}]$, as in (3.1). We will show that one has

$$I^k(P_n) = \sum_{l+h=k} I^l(G_n) \otimes I^h(P_{n-1}), \tag{3.3}$$

for all k. This will imply the decomposition formula (3.2) above, by induction.

Recall now the notion of *almost-direct product*. A semidirect product of groups, $G \rtimes B$, is called an almost-direct product if

$$bgb^{-1} \equiv g \text{ modulo } (G, G), \tag{3.4}$$

for every $b \in B$ and $g \in G$. (Here and in the sequel we denote by (H, K), where H and K are subgroups of a group G, the subgroup generated by all *group commutators*, $(h, k) =: hkh^{-1}k^{-1}$, with $h \in H$ and $k \in K$.) One knows that $P_n = G_n \rtimes P_{n-1}$ is an almost-direct product [4, Ch. 1].

The next result is the key ingredient for the proof of Proposition 1.3 (like in the approach from [16]). It is established in [16, Section 2], for the particular case of $P_n = G_n \rtimes P_{n-1}$, by ad-hoc methods. At our level of generality, it implies the decomposition formula (3.2), for fundamental groups of complements of arbitrary *fiber-type* complex hyperplane arrangements. (The structure of iterated almost-direct product of such groups is provided by [9, Section 2].)

Theorem 3.1. *If $G \rtimes B$ is an almost-direct product, then*

$$I^k(G \rtimes B) = \sum_{l+h=k} I^l(G) \otimes I^h(B), \quad \textit{for all } k.$$

Proof. Denote by F_k the right-hand side of the above equality. Plainly $\{F_k\}$ is a descending filtration on $\mathbb{Z}[G] \otimes \mathbb{Z}[B]$, which is identified with $\mathbb{Z}[G \rtimes B]$ as usual, see (3.1). With this identification, it is immediate to see that $F_k \subset I^k(G \rtimes B)$, for all k.

To verify the other inclusion, we have to check on additive generators that $(g_1 b_1 - 1) \cdots (g_k b_k - 1) \in F_k$, for every $g_1, \dots, g_k \in G$ and $b_1, \dots, b_k \in B$. We start by remarking that it will suffice to verify this for *special* elements, of the form $e = (e_1 - 1) \cdots (e_k - 1)$, where either $e_i \in G$ or $e_i \in B$, for $i = 1, \dots, k$. Indeed, we may write $gb - 1 = (g - 1)(b - 1) + (g - 1) + (b - 1)$, for any $g \in G$ and $b \in B$. Therefore we may expand the product $(g_1 b_1 - 1) \cdots (g_k b_k - 1)$ as a sum of special elements, belonging to $F_k + F_{k+1} + \cdots + F_{2k} \subset F_k$, and we are done.

We will associate to each special element e as above a vector with k components, which are either 0 or 1. Set $\mathrm{type}(e) = (c_1, \dots, c_k)$, where $c_i = 0$ if $e_i \in G$ and $c_i = 1$ if $e_i \in B$, for $i = 1, \dots, k$. We will use the lexicographic order from the left on types of special elements to prove that $e \in F_k$ by induction on $\mathrm{type}(e)$.

Let us say that the special element e is *standard* if

$$\mathrm{type}(e) = (\overbrace{0, \dots, 0}^{l}, \overbrace{1, \dots, 1}^{h}).$$

In this case, plainly $e \in I^l(G) \otimes I^h(B) \subset F_k$, and we are done. If e is not standard, then it must be of the form

$$\begin{aligned} e = {} & (g_1 - 1) \cdots (g_r - 1)(b_1 - 1) \cdots (b_s - 1)(b - 1) \\ & \times (g - 1)(e_1 - 1) \cdots (e_t - 1), \end{aligned} \tag{3.5}$$

where $g_1, \dots, g_r, g \in G$, $b_1, \dots, b_s, b \in B$, $(e_1 - 1) \cdots (e_t - 1)$ is special and $r + s + t + 2 = k$.

At this point we are going to use the hypothesis, namely the fact that $G \rtimes B$ is an almost-direct product. To be precise, we claim that under this assumption one has commutation relations in $\mathbb{Z}[G \rtimes B]$ expressing the difference $(b - 1)(g - 1) - (g - 1)(b - 1)$ as a linear combination of terms of the form

$$\bigl(g' - 1\bigr)\bigl(g'' - 1\bigr)b, \quad \text{with } g', g'' \in G, \tag{3.6}$$

for any $b \in B$ and $g \in G$. Indeed,

$$(b - 1)(g - 1) - (g - 1)(b - 1) = \bigl({}^b g - g\bigr)b = (f - 1)gb, \tag{3.7}$$

for some $f \in (G, G)$; the last equality in (3.7) comes from the fact that ${}^b g = fg$, where $f \in (G, G)$, see (3.4). On the other hand, one knows that $f - 1$ is a linear combination of terms of the form

$$\bigl(f' - 1\bigr)\bigl(f'' - 1\bigr), \quad \text{with } f', f'' \in G, \tag{3.8}$$

if $f \in (G, G)$, see, e.g., [7, p. 194]. Substitute then (3.8) in (3.7), and rewrite $(f'' - 1)g$ as $(f'' g - 1) - (g - 1)$, to get (3.6), as claimed.

Coming back to (3.5), we may use (3.6) to express e as a sum, whose first term is the special element

$$e' = (g_1 - 1)\cdots(g_r - 1)(b_1 - 1)\cdots(b_s - 1)(g - 1) \cdot (b - 1)(e_1 - 1)\cdots(e_t - 1), \tag{3.9}$$

with $\mathrm{type}(e') < \mathrm{type}(e)$, and whose second term is a linear combination of elements of the form $e'' \cdot b$, where

$$e'' = (g_1 - 1)\cdots(g_r - 1)(b_1 - 1)\cdots(b_s - 1)\bigl(g' - 1\bigr) \cdot \bigl(g'' - 1\bigr)\bigl({}^b e_1 - 1\bigr)\cdots\bigl({}^b e_t - 1\bigr), \tag{3.10}$$

with $\mathrm{type}(e'') < \mathrm{type}(e)$, again. Use induction to infer that e' and e'' in (3.9) and (3.10) above belong to F_k. Finally, it is easily seen that $F_k \cdot b \subset F_k$, for any $b \in B$, hence $e \in F_k$ and our proof is complete. □

Corollary 3.2. *The map $\mu : \mathbb{Z}[P_n] \to \widehat{U_n}$ in the statement of Proposition* 1.3 *sends $I^k(P_n)$ into $\widehat{U_n}^{\geqslant k}$, for all k.*

Proof. We will perform several reductions, by using the remark that both filtrations are multiplicative (that is, $I^r \cdot I^s \subset I^{r+s}$ and $\widehat{U_n}^{\geqslant r} \cdot \widehat{U_n}^{\geqslant s} \subset \widehat{U_n}^{\geqslant r+s}$, for all r, s), which follows immediately from their construction. Firstly, the decomposition (3.2) says that it is enough to verify the property stated in the corollary for each map $\mu_j : \mathbb{Z}[G_j] \to \widehat{U_n}$, $j = 2, \ldots, n$, see the construction of μ (1.8). Secondly, it will plainly suffice to check that $\mu_j(I(G_j)) \subset \widehat{U_n}^{\geqslant 1}$. In other words, we have to verify that $\mu_j(g) \equiv 1$ modulo $\widehat{U_n}^{\geqslant 1}$, for every $g \in G_j$. Since $\widehat{U_n}^{\geqslant 1}$ is an ideal of $\widehat{U_n}$ and μ_j is a representation, we are left with checking this on the group generators a_{ij}. That is, all we have to see is that $\mu_j(a_{ij}) \equiv 1$ modulo $\widehat{U_n}^{\geqslant 1}$, for $1 \leqslant i < j \leqslant n$, which is obvious from the construction, see (1.6) and (1.7). □

4. The role of the lower central series filtration

To finish the proof of Proposition 1.3, we will need one more ingredient. This will be provided by a powerful result of Quillen, which establishes a close relation between the I-adic filtration of an arbitrary group ring, $\mathbb{Z}[G]$, and the *lower central series filtration* of G, $\{\Gamma_k G\}_{k \geqslant 1}$.

We recall that $\Gamma_k G$ is defined inductively, by setting $\Gamma_1 G = G$, and then $\Gamma_{k+1} G = (G, \Gamma_k G)$, for $k \geqslant 1$. This gives a natural descending series of normal subgroups, with abelian quotients denoted by $\mathrm{gr}^k_\Gamma(G) =: \Gamma_k G / \Gamma_{k+1} G$, for $k \geqslant 1$. The *associated graded Lie algebra* of G is defined by $\mathrm{gr}^*_\Gamma(G) =: \bigoplus_{k \geqslant 1} \mathrm{gr}^k_\Gamma(G)$; it is a graded Lie algebra with Lie bracket induced by the group commutator. Denote by $U\mathrm{gr}^*_\Gamma(G)$ its enveloping algebra. The (multiplicative) I-adic filtration gives rise to another graded algebra,

$$\mathrm{gr}^*_I \mathbb{Z}[G] =: \bigoplus_{k \geqslant 0} \mathrm{gr}^k_I \mathbb{Z}[G],$$

where $\mathrm{gr}_I^k \mathbb{Z}[G] = I^k(G)/I^{k+1}(G)$, for $k \geqslant 0$. One knows that $g - 1 \in I^k(G)$, if $g \in \Gamma_k G$; see for example [7, p. 194]. Denote by

$$\chi : \Gamma_k G \to I^k(G) \tag{4.1}$$

the map defined by $\chi(g) = g - 1$.

Theorem 4.1 (Quillen [19]). *The above map χ induces a graded algebra surjection, $\chi : U\mathrm{gr}_\Gamma^*(G) \to \mathrm{gr}_I^*\mathbb{Z}[G]$, with the property that $\chi \otimes \mathbb{Q}$ is an isomorphism, for any group G.*

On the other hand, denote by $\mathbb{L}(t)$ the graded $\mathbb{Z}$-Lie algebra freely generated by indeterminates t_{ij}, where $1 \leqslant i \neq j \leqslant n$ and $t_{ij} = t_{ji}$, like in Section 1; the grading is given by bracket length. Define E_n to be the graded Lie algebra quotient of $\mathbb{L}(t)$ modulo the (Lie) relations (1.2) and (1.3). One knows that there is a graded Lie algebra isomorphism between E_n^* and $\mathrm{gr}_\Gamma^*(P_n)$, which sends the algebraic generator $t_{ij} \in E_n^1$ to the class modulo $\Gamma_2 P_n$ of the standard geometric generator $a_{ij} \in \Gamma_1 P_n$; see for example [3, Ch. 3B]. Passing to universal enveloping algebras, this will enable us to identify in the sequel the graded algebras $UE_n = U_n$ and $U\mathrm{gr}_\Gamma(P_n)$, by an isomorphism

$$U_n^* \xrightarrow{\sim} U\,\mathrm{gr}_\Gamma^*(P_n), \tag{4.2}$$

which sends t_{ij} to the class $\overline{a_{ij}} \in \mathrm{gr}_\Gamma^1(P_n)$, for $1 \leqslant i < j \leqslant n$.

Corollary 4.2. *With the above identification, the Quillen map* (4.1) *gives a graded algebra isomorphism*

$$\chi : U_n^* \xrightarrow{\sim} \mathrm{gr}_I^*\mathbb{Z}[P_n],$$

which associates to $t_{ij} \in U_n^1$ the class of $a_{ij} - 1$ in $\mathrm{gr}_I^1\mathbb{Z}[P_n]$, for $1 \leqslant i < j \leqslant n$.

Proof. Given Quillen's result (Theorem 4.1) and the isomorphism (4.2), it is enough to recall that U_n^* is a torsion free graded abelian group; see the proof of Corollary 1.4(i). □

We have shown in Corollary 3.2 that the map μ from Proposition 1.3 induces $\mathrm{gr}^k(\mu) : \mathrm{gr}_I^k \mathbb{Z}[P_n] \to U_n^k$, for all k. We are going to finish the proof of Proposition 1.3 by showing a little more than stated, namely that $\mathrm{gr}^k(\mu)$ is actually the inverse of Quillen's isomorphism from Corollary 4.2.

Lemma 4.3. *The composition*

$$\mathrm{gr}^k(\mu) \circ \chi : U_n^k \to U_n^k$$

equals the identity, for all k.

Proof. We begin by recalling from the previous section that $I^k(P_n)$ is additively generated by the elements of the form

$$(g_{n1} - 1)\cdots(g_{n,k_n} - 1)\cdots(g_{21} - 1)\cdots(g_{2,k_2} - 1), \tag{4.3}$$

where $g_{ji} \in G_j$, for $j = 2, \ldots, n$ and $i = 1, \ldots, k_j$, and $k_2 + \cdots + k_n = k$; see (3.2). We also know from Theorem 4.1 that there is a natural surjection, $\chi : G/\Gamma_2 G \to I(G)/I^2(G)$, for any group G. This enables us to replace in (4.3) all elements g_{ji} by generators of the group G_j, for $j = 2, \ldots, n$, to get a convenient system of additive generators for $I^k(P_n)$ modulo $I^{k+1}(P_n)$,

$$a = (a_{i_1,n} - 1) \cdots (a_{i_{k_n},n} - 1) \cdots (a_{i_1,2} - 1) \cdots (a_{i_{k_2},2} - 1), \tag{4.4}$$

where $\{a_{ij} \mid 1 \leqslant i < j \leqslant n\}$ are the standard generators of P_n. By Corollary 4.2, the elements of the form

$$\tau = (t_{i_1,n} \cdots t_{i_{k_n},n}) \cdots (t_{i_1,2} \cdots t_{i_{k_2},2}) \tag{4.5}$$

will then provide a generating system for U_n^k, since plainly $\chi(\tau) \equiv a$, modulo $I^{k+1}(P_n)$. We may now finish the proof of the lemma, by showing that $\mu(a) \equiv \tau$ modulo $\widehat{U_n}^{\geqslant k+1}$. This in turn follows immediately from the construction of μ; see (1.7)–(1.8) and the normalization requirement (1.6). $\square$

At the end, we will complete the proof of our main result, Theorem 1.1.

4.4. Proof of Proposition 1.5

We will break again the proof in two steps. As an intermediate object we will use the graded algebra $\mathrm{gr}_I^* \mathbb{Z}[P_n] \rtimes \mathbb{Z}[\Sigma_n]$. The semidirect product structure comes from the natural action of B_n on the graded algebra $\mathrm{gr}_I^* \mathbb{Z}[P_n]$, given by conjugation. To see that this factors through Σ_n, it plainly suffices to show that the conjugation action of any $p \in P_n$ is trivial on the algebra generators, $\mathrm{gr}_I^1 \mathbb{Z}[P_n]$. This fact follows in turn from the naturality property of the surjection $\chi : G/\Gamma_2 G \to I(G)/I^2(G)$, together with the remark that the conjugation action of any $g \in G$ is trivial modulo $\Gamma_2 G$. We know from Propositions 1.2 and 1.3 that $\mathrm{gr}^*(M)$ equals the composition of two isomorphisms, $\mathrm{gr}^*(\Phi) : \mathrm{gr}_J^* \mathbb{Z}[B_n] \xrightarrow{\sim} \mathrm{gr}_I^* \mathbb{Z}[P_n] \rtimes \mathbb{Z}[\Sigma_n]$ and $\mathrm{gr}^*(\mu) \otimes \mathrm{id} : \mathrm{gr}_I^* \mathbb{Z}[P_n] \rtimes \mathbb{Z}[\Sigma_n] \xrightarrow{\sim} U_n^* \rtimes \mathbb{Z}[\Sigma_n]$; see (1.9). We are going to prove Proposition 1.5 by showing that both inverse isomorphisms, $\mathrm{gr}^*(\Psi) : \mathrm{gr}_I^* \mathbb{Z}[P_n] \rtimes \mathbb{Z}[\Sigma_n] \xrightarrow{\sim} \mathrm{gr}_J^* \mathbb{Z}[B_n]$ (see Section 2.2), and $\chi \otimes \mathrm{id} : U_n^* \rtimes \mathbb{Z}[\Sigma_n] \xrightarrow{\sim} \mathrm{gr}_I^* \mathbb{Z}[P_n] \rtimes \mathbb{Z}[\Sigma_n]$ (see Lemma 4.3), are multiplicative.

The multiplicative property of $\mathrm{gr}^*(\Psi)$ follows from a direct straightforward computation, as soon as one knows that $s(xy) \equiv s(x) \cdot s(y)$ modulo J, for any $x, y \in \Sigma_n$. (Recall that s is an arbitrary section of the canonical epimorphism $\pi : B_n \to \Sigma_n$.) This in turn is a direct consequence of the fact that J equals $\ker\{\mathbb{Z}[\pi] : \mathbb{Z}[B_n] \to \mathbb{Z}[\Sigma_n]\}$. Lastly, this equality is easily derived from the well-known fact that Σ_n is the quotient of B_n modulo the relations $\sigma_i^2 = 1$, for $i = 1, \ldots, n-1$; see [4, Ch. 1].

The last step of the proof is to show that the graded algebra isomorphism χ from Corollary 4.2 is in addition equivariant with respect to the Σ_n-actions. (This implies at once that $\chi \otimes \mathrm{id}$ is multiplicative, as needed to conclude.) Thus we have to check, on the algebra generators of U_n^*, that

$$a_{x(i),x(j)} - 1 \equiv {}^{b}(a_{ij} - 1) \text{ modulo } I^2(P_n), \tag{4.6}$$

for every $x \in \Sigma_n$ and $1 \leqslant i \neq j \leqslant n$. (Here $b \in B_n$ is such that $\pi(b) = x$, and we are using the usual convention, $a_{kl} = a_{lk}$, for $k \neq l$.) By the naturality property of $\chi : G/\Gamma_2 G \to I(G)/I^2(G)$, (4.6) follows from

$$^{b}a_{ij} \equiv a_{x(i),x(j)} \text{ modulo } \Gamma_2 P_n. \tag{4.7}$$

Finally, it is not difficult to verify (4.7) above; see [17, Lemma 5.2(i)]. □

Remark 4.5. Note that the graded algebra isomorphism $\mathrm{gr}^*(M) : \mathrm{gr}_J^* \mathbb{Z}[B_n] \xrightarrow{\sim} U_n^* \rtimes \mathbb{Z}[\Sigma_n]$ is independent of the choices made for the construction of M (namely the section s and the formal series $\{S_{ij}\}$). This is implicitly contained in the above proof. To make it explicit, it suffices to add that not only $\chi \otimes \mathrm{id}$ but also $\mathrm{gr}^*(\Psi)$ is independent of our choices. This is easy to deduce from the construction of Ψ. More precisely, one has that $\mathrm{gr}^*(\Psi)(v \otimes \xi) = \widetilde{v} \cdot \pi^{-1}(\xi)$, for $v \in \mathrm{gr}_I^* \mathbb{Z}[P_n]$ and $\xi \in \mathbb{Z}[\Sigma_n]$. Here $\widetilde{v}$ is the image of v, via the canonical graded algebra map $\mathrm{gr}_I^* \mathbb{Z}[P_n] \to \mathrm{gr}_J^* \mathbb{Z}[B_n]$ (see Lemma 2.1), and $\pi^{-1} : \mathbb{Z}[\Sigma_n] \xrightarrow{\sim} \mathbb{Z}[B_n]/J = \mathrm{gr}_J^0 \mathbb{Z}[B_n]$.

Remark 4.6. Lin derived in [13] the additive isomorphism between $\mathrm{gr}_J^* \mathbb{Z}[B_n]$ and $U_n^* \otimes \mathbb{Z}[\Sigma_n]$ by lengthy direct computations. This result follows at once, by combining Proposition 1.2 and Corollary 4.2.

5. The connection with braid closure

In this section our goal is to show that, at the associated graded level, the universal finite-type invariant for braids M is compatible with the basic geometric construction in the finite-type theory for *oriented links*, which relates closed chord diagrams to the Vassiliev filtration of links. The comparison between (finite-type) braid theory and link theory will be done via the classical *Artin closure* of braids, see [4, Ch. 2].

We start by reviewing the relevant standard material on singular links and braids, and on closed and horizontal chord diagrams, following [5,6,13,21].

The *Vassiliev filtration of links*, $\{VF_k(\mathcal{L})\}_{k\geqslant 0}$ is defined via *resolutions of singular links*, $R(L)$, in the same way the Vassiliev-type filtration of n-braids, $\{VF_k(B_n)\}_{k\geqslant 0}$, is constructed using resolutions of singular n-braids, $R(\beta)$; see the beginning of Section 1. We will denote the associated graded objects by $\mathrm{gr}_V^*(\mathcal{L})$, and $\mathrm{gr}_V^*(B_n)$, respectively. The Artin closure of (singular) braids induces a degree zero additive map, denoted by

$$\widehat{(\cdot)} : \mathrm{gr}_V^*(B_n) \to \mathrm{gr}_V^*(\mathcal{L}).$$

Denote by $\mathbb{Z}[\mathcal{D}^k]$ the free abelian group generated by all diagrams $d \in \mathcal{D}^k$, consisting of k chords with $2k$ distinct endpoints on a set of oriented circles, considered up to orientation-preserving homeomorphisms of the supporting circles. The quotient of $\mathbb{Z}[\mathcal{D}^k]$ by the so-called *four-term relations* will be denoted by $\mathcal{A}^k$; see [21]. Define the graded algebra H_n^* to be the quotient of $\mathbb{Z}[t]$ modulo the relations (1.2); its elements may be identified with linear combinations of horizontal chord diagrams on n oriented vertical segments modulo horizontal isotopy, as explained in Section 1.

If m is a degree k monomial from $\mathbb{Z}[t]$ and $x \in \Sigma_n$, define the Artin closure $\widehat{m \otimes x} \in \mathcal{D}^k$ by identifying the endpoint of the ith vertical segment with the initial point of the $x(i)$th vertical segment, for $i = 1, \ldots, n$. This construction gives rise to another degree zero additive map, $\{\widehat{(\cdot)} : H_n^k \otimes \mathbb{Z}[\Sigma_n] \to \mathbb{Z}[\mathcal{D}^k]\}_{k \geqslant 0}$, which factors to

$$\widehat{(\cdot)} : U_n^* \otimes \mathbb{Z}[\Sigma_n] \to \mathcal{A}^*.$$

The diagram $D(L) \in \mathcal{D}^k$ of a singular link L with k double points is defined by associating to each double point a chord having the preimages of the double point as extremities. Similarly, the diagram of a singular n-braid β with k double points, $D(\beta) \in H_n^k \otimes \mathbb{Z}[\Sigma_n]$, is constructed as follows. Define $D(\beta)$ to be $d \otimes x$. Here $x = \pi(\beta)$ (the permutation associated to β). The horizontal k-chord diagram d is obtained by placing on the ith vertical position the strand of β whose endpoint is on the ith position, for $i = 1, \ldots, n$, and then splitting each double point into a horizontal chord, as before.

It is easy to see that one has

$$D(\widehat{\beta}) = \widehat{D(\beta)}, \tag{5.1}$$

for any singular n-braid β. It will be equally useful to recall from Section 1 the natural semidirect product multiplicative structure of $H_n^* \otimes \mathbb{Z}[\Sigma_n]$, to be denoted by $H_n \rtimes \mathbb{Z}[\Sigma_n]$. It is not difficult to verify that

$$D(\alpha\beta) = D(\alpha) \cdot D(\beta), \tag{5.2}$$

for any singular braids α and β.

Finally, recall the fundamental geometric construction from finite-type link theory. One has a map $f : \mathbb{Z}[\mathcal{D}^k] \to \mathrm{gr}_V^k(\mathcal{L})$, which associates to $d \in \mathcal{D}^k$ the class modulo $VF_{k+1}(\mathcal{L})$ of $R(L)$, where L is any singular link with the property that

$$D(L) = d. \tag{5.3}$$

In this way one gets an induced surjective map $f : \mathcal{A}^k \to \mathrm{gr}_V^k(\mathcal{L})$; see [21].

We may now spell out our main result in this section.

Theorem 5.1. *There is a commutative diagram*

$$\begin{array}{ccc}
\mathrm{gr}_V^k(B_n) & \xrightarrow[\sim]{\mathrm{gr}^k(M)} & U_n^k \otimes \mathbb{Z}[\Sigma_n] \\
{\scriptstyle \widehat{(\cdot)}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{(\cdot)}} \\
\mathrm{gr}_V^k(\mathcal{L}) & \xleftarrow[f]{} & \mathcal{A}^k
\end{array}$$

up to the sign $(-1)^k$, *for all* k.

Proof. We are going to check the assertion of the theorem on a convenient system of additive generators of $\mathrm{gr}_V^k(B_n)$, as in the proof of Lemma 4.3; see also the proof of Proposition 1.2 from Section 2.

Let us then pick an element $e \in VF_k(B_n)$ of the form

$$e = (a_{i_1 j_1} - 1) \cdots (a_{i_k j_k} - 1) \cdot s(x), \tag{5.4}$$

where $(a_{i_1 j_1} - 1) \cdots (a_{i_k j_k} - 1) \in I^k(P_n)$ is in normal form, as in (4.4), and $x \in \Sigma_n$. We start by computing $\widehat{e}$. One knows that

$$a_{ij} - 1 = a\big(\sigma_i - \sigma_i^{-1}\big)b, \quad \text{with } a = \sigma_{j-1} \cdots \sigma_{i+1}\sigma_i \text{ and } b = \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}, \tag{5.5}$$

for $1 \leqslant i < j \leqslant n$; see [4, Ch. 1]. Denote by a_r and b_r the braids a and b appearing in the above decomposition (5.5) of $a_{i_r j_r} - 1$, for $r = 1, \ldots, k$. Define the singular n-braid β by

$$\beta = a_1 \tau_{i_1} b_1 \cdots a_k \tau_{i_k} b_k \cdot s(x), \tag{5.6}$$

where τ_i stands for the elementary singular braid with one double point, for $i = 1, \ldots, n-1$, as in [5]. We infer from (5.4)–(5.6) that $R(\beta) = (-1)^k e$; see (1.1). Consequently

$$\widehat{e} = (-1)^k R\big(\widehat{\beta}\big). \tag{5.7}$$

Set $u = t_{i_1 j_1} \cdots t_{i_k j_k} \otimes x \in U_n^k \otimes \mathbb{Z}[\Sigma_n]$. Use (1.9) and the definition of M, and then argue exactly as in the proof of Lemma 4.3 to deduce that

$$\mathrm{gr}^k(M)(\bar{e}) = u, \tag{5.8}$$

where $\bar{e}$ denotes the class of e modulo $VF_{k+1}(B_n)$.

By comparing (5.7) and (5.8), we see that all we need to finish our proof is to show that $f(\widehat{u})$ equals the class of $R(\widehat{\beta})$ modulo $VF_{k+1}(\mathcal{L})$. (Here $\widehat{u} \in \mathcal{D}^k$, as explained before.) In other words, it will be enough to verify that $D(\widehat{\beta}) = \widehat{u}$; see (5.3). This in turn will follow from

$$D\big(a_1 \tau_{i_1} b_1 \cdots a_k \tau_{i_k} b_k \cdot s(x)\big) = t_{i_1 j_1} \cdots t_{i_k j_k} \otimes x \tag{5.9}$$

(equality in $H_n^* \rtimes \mathbb{Z}[\Sigma_n]$); see (5.1).

Finally, we will check (5.9) by resorting to the multiplicative property (5.2). It follows directly from the definitions that $D(a) = 1 \otimes \pi(a)$, for $a \in B_n$, and $D(\tau_i) = t_{i,i+1} \otimes \pi(\sigma_i)$, for $i = 1, \ldots, n-1$. This may be used, together with the definition of the semidirect product multiplicative structure of $H_n^* \rtimes \mathbb{Z}[\Sigma_n]$, to check that

$$D(a_r) \cdot D(\tau_{i_r}) \cdot D(b_r) = t_{i_r j_r} \otimes 1, \tag{5.10}$$

for $r = 1, \ldots, k$. We infer from (5.10) that $D(\beta) = (t_{i_1 j_1} \cdots t_{i_k j_k} \otimes 1) \cdot (1 \otimes x) = t_{i_1 j_1} \cdots t_{i_k j_k} \otimes x$, which verifies (5.9) and thus completes our proof. □

Set now $\mathrm{gr}_V^*(B) =: \bigoplus_{n \geqslant 1} \mathrm{gr}_V^*(B_n)$. The singular Alexander theorem (see [5, Section 6]) implies that braid closure induces a degree zero surjection, $\widehat{(\cdot)} : \mathrm{gr}_V^*(B) \to \mathrm{gr}_V^*(\mathcal{L})$. Similarly, set $U^* \otimes \mathbb{Z}[\Sigma] =: \bigoplus_{n \geqslant 1}(U_n^* \otimes \mathbb{Z}[\Sigma_n])$, and denote by $\mathrm{gr}^*(M)$ the direct sum isomorphism, $\mathrm{gr}^*(M) : \mathrm{gr}_V^*(B) \xrightarrow{\sim} U^* \otimes \mathbb{Z}[\Sigma]$. We deduce from the singular Alexander theorem that the Artin closure of diagrams induces another degree zero surjection, $\widehat{(\cdot)} : U^* \otimes \mathbb{Z}[\Sigma] \to \mathcal{A}^*$; use (5.1).

Recall the fundamental object in closed chord diagram theory, $\widetilde{\mathcal{A}}^*$. It is the graded quotient of $\mathcal{A}^*$ obtained by killing those diagrams which contain an *isolated chord*. (This is a chord with both endpoints on some circle C, with the property that there is no chord endpoint on one of the two arcs determined by this chord on C.) It is known that f factors to give a surjection $f : \widetilde{\mathcal{A}}^* \to \mathrm{gr}_V^*(\mathcal{L})$; see [21].

Let $\widehat{M}^k$ be the (surjective) composition $\mathrm{gr}_V^k(B) \overset{\mathrm{gr}^k(M)}{\underset{\sim}{\longrightarrow}} U^k \otimes \mathbb{Z}[\Sigma] \overset{\widehat{(\cdot)}}{\longrightarrow} \widetilde{\mathcal{A}}^k$, multiplied by $(-1)^k$. Theorem 5.1 provides then a commutative diagram:

$$\begin{array}{ccc} \mathrm{gr}_V^k(B) & & \\ \downarrow{\scriptstyle\widehat{(\cdot)}} & \searrow^{\widehat{M}^k} & \\ \mathrm{gr}_V^k(\mathcal{L}) & \underset{f}{\longleftarrow} & \widetilde{\mathcal{A}}^k \end{array} \tag{5.11}$$

for all k.

Our last theorem is a partial factorization result for $\widehat{M}^k$ through braid closure, $\widehat{(\cdot)}$. More precisely, consider the following families of Markov-type elements of $\mathrm{gr}_V^k(B)$ (with the convention that, from now on, $\overline{(\cdot)}$ will stand for the class modulo VF_{r+1} of an element belonging to VF_r):

$$\overline{R(\alpha\beta)} - \overline{R(\beta\alpha)} \in \mathrm{gr}_V^k(B_n), \tag{I}$$

where α is a singular n-braid with l double points and β is a singular n-braid with h double points, $l + h = k$;

$$\overline{R(\alpha)} - \overline{R(\alpha)\cdot\sigma_n^{\pm1}} \in \mathrm{gr}_V^k(B_n) \oplus \mathrm{gr}_V^k(B_{n+1}), \tag{II}$$

where α is a singular n-braid with k double points, and $\mathbb{Z}[B_n] \subset \mathbb{Z}[B_{n+1}]$ is the usual inclusion (see [4, Ch. 1]);

$$\overline{R(\alpha)\cdot(\sigma_n - \sigma_n^{-1})} \in \mathrm{gr}_V^k(B_{n+1}), \tag{III}$$

where α is a singular n-braid with $k-1$ double points. Clearly all these elements are sent to zero in $\mathrm{gr}_V^k(\mathcal{L})$, by braid closure. Actually, we can say more.

Theorem 5.2. *All three types of elements described above belong to* $\ker(\widehat{M}^k)$.

Proof. Type (I) elements. It is easy to see that one has the equality $\overline{R(\alpha\beta)} - \overline{R(\beta\alpha)} = \overline{R(\alpha)} \cdot \overline{R(\beta)} - \overline{R(\beta)} \cdot \overline{R(\alpha)}$, in the graded algebra $\mathrm{gr}_V^*(B_n)$; use (1.1). Given the multiplicative property of $\mathrm{gr}^*(M)$ (see Proposition 1.5), it will be enough in this case to show that $\widehat{uv} = \widehat{vu}$ (equality in $\mathcal{A}^*$), for $u, v \in U_n^* \rtimes \mathbb{Z}[\Sigma_n]$. This in turn is verified without difficulty, starting from the definitions.

For the remaining two cases, we make a preliminary remark. One has stabilization graded algebra maps,

$$U_n^* \rtimes \mathbb{Z}[\Sigma_n] \overset{S}{\to} U_{n+1}^* \rtimes \mathbb{Z}[\Sigma_{n+1}] \quad \text{and} \quad \mathrm{gr}_V^*(B_n) \overset{S}{\to} \mathrm{gr}_V^*(B_{n+1}).$$

The first one is induced by the obvious graded algebra map $U_n^* \to U_{n+1}^*$ which is defined by the canonical inclusion, on algebra generators, and by the usual inclusion [4, Ch. 1], $\Sigma_n \subset \Sigma_{n+1}$; the other one is induced by the usual inclusion, $\mathbb{Z}[B_n] \subset \mathbb{Z}[B_{n+1}]$. It is easy to see that $\mathrm{gr}^*(M)$ commutes with stabilization. One may check this on a convenient system of additive generators for $\mathrm{gr}_V^*(B_n)$, as in the proof of Theorem 5.1; see (5.4) and (5.8).

Type (II) elements. We may resort to the multiplicative and stabilization properties of $\mathrm{gr}^*(M)$ to compute that

$$\begin{aligned}\mathrm{gr}^k(M)\big(\overline{R(\alpha)\cdot\sigma_n^{\pm 1}}\big) &= S\mathrm{gr}^k(M)\big(\overline{R(\alpha)}\big)\cdot\mathrm{gr}^0(M)\big(\overline{\sigma_n^{\pm 1}}\big)\\ &= S\,\mathrm{gr}^k(M)\big(\overline{R(\alpha)}\big)\cdot(1\otimes x_n).\end{aligned}$$

(We have denoted $\pi(\sigma_n^{\pm 1})$ by x_n.) We thus see in this case that it will be enough to show that the Artin closure of $(m\otimes x)\cdot(1\otimes x_n)=m\otimes xx_n$ equals the Artin closure of $m\otimes x$ (as diagrams), for any monomial m in the generators t_{ij} with $i,j\leqslant n$ and for any $x\in\Sigma_n$. To check this, start by remarking that if $(i_1,\ldots,i_{r-1},n)$ is the cycle of x which contains n, then $(i_1,\ldots,i_{r-1},n,n+1)$ will be a cycle of xx_n; the other cycles of x and xx_n are the same. Since m contains no chord endpoint on the last vertical segment, we infer at once that $\widehat{m\otimes xx_n}=\widehat{m\otimes x}$, as desired.

Type (III) elements. As before, we have

$$\begin{aligned}\mathrm{gr}^k(M)\big(\overline{R(\alpha)\cdot(\sigma_n-\sigma_n^{-1})}\big) &= S\mathrm{gr}^{k-1}(M)\big(\overline{R(\alpha)}\big)\cdot\mathrm{gr}^1(M)\big(\overline{(a_{n,n+1}-1)\sigma_n^{-1}}\big)\\ &= S\mathrm{gr}^{k-1}(M)\big(\overline{R(\alpha)}\big)\cdot(t_{n,n+1}\otimes x_n).\end{aligned}$$

This time we will conclude by showing that, if m and x are as in the previous case, then the Artin closure of $(m\otimes x)\cdot(t_{n,n+1}\otimes x_n)=m\cdot t_{i_1,n+1}\otimes xx_n$ contains an isolated chord (and therefore represents zero in $\widetilde{\mathcal{A}}^*$). The chord corresponding to $t_{i_1,n+1}$, with endpoints on the circle corresponding to the cycle $(i_1,\ldots,i_{r-1},n,n+1)$, is indeed isolated, since there are no chord endpoints of m situated on the last vertical segment. □

Remark 5.3. It is immediate to see that $\widehat{M}^*$ factors in (5.11) if and only if $f:\widetilde{\mathcal{A}}^*\to\mathrm{gr}_V^*(\mathcal{L})$ is an isomorphism. On the other hand, there is the surjection given by diagram Artin closure, $U^*\otimes\mathbb{Z}[\Sigma]\to\widetilde{\mathcal{A}}^*$. Using the singular Markov theorem [10], we can show that its kernel is generated by the $\mathrm{gr}^*(M)$-images of the above types (I)–(III) of elements (for any coefficients). It follows that f is an isomorphism if and only if the kernel of braid Artin closure, $\mathrm{gr}_V^*(B)\to\mathrm{gr}_V^*(\mathcal{L})$, is generated by the Markov families (I)–(III).

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