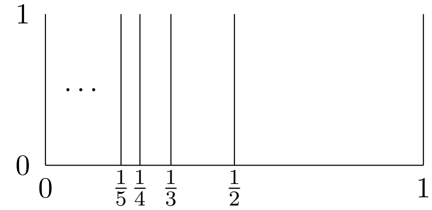
Return this sheet with your exam. Each problem is worth 17 points. NAME: ____________________
Give detailed proofs when asked to. Otherwise, (briefly justified) short answers are sufficient.

1. Let C be the topologist's comb, given on the right.
(The formal definition of C is given on the back of this sheet.)



(a) State the definition of a limit point, and find all limit points of C.
(b) Is C path connected? Explain.
(c) Is C compact? Explain.

2. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be topological spaces.

(a) State the definition in terms of open sets, a function $f\colon X \to Y$ is *continuous* if ...
(b) Prove that the composition of continuous functions is continuous:
If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, prove that $g \circ f\colon X \to Z$ is continuous.
(c) Let $(a,b) = \{x \in \mathbb{R}^1 \mid a < x < b\} \subset \mathbb{R}^1$, with the subspace topology.
Are $X = (0,1)$ and $Y = (4,9)$ homeomorphic? Explain.

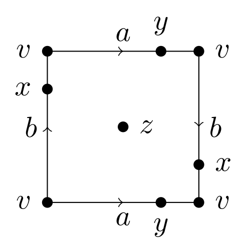
3. Let A be a subspace of $\mathbb{R}^n$, and let B be a subset of A.

(a) Prove that if B is closed in A, then $B = A \cap D$ for some closed set $D \subset \mathbb{R}^n$.
(b) Given an example of a subspace A of $\mathbb{R}^2$ and a subset B of A for which B is closed in A, but B is not closed in $\mathbb{R}^2$.
(c) If A is open in $\mathbb{R}^2$ and B is open in A, is B open in $\mathbb{R}^2$? Explain.

4. Give a brief explanation why...

(a) ...the unit interval $I = [0,1] = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$ and the circle $\mathbb{S}^1$ are not homeomorphic.
(b) ...the plane $\mathbb{R}^2$ and the disk $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \le 1\}$ are not homeomorphic.
(c) ...the 2-sphere $\mathbb{S}^2$ and the torus $\mathbb{T}$ are not homeomorphic.

5. Consider the plane model for the Klein bottle $\mathbb{K}$ given on the right, with a, b indicating the identifications, and v, x, y, z representing points in $\mathbb{K}$.



(a) Exhibit Euclidean 2-disk neighborhoods of each of the points v, x, y, z in separate copies of the plane model for $\mathbb{K}$, and explain why $\mathbb{K}$ is Hausdorff.
(b) Exhibit a simple closed curve $\mathcal{C}$ on $\mathbb{K}$ for which $\mathbb{K} \setminus \mathcal{C}$ is path connected.
Draw $\mathcal{C}$ on the space model for $\mathbb{K}$ given on the back of this sheet.
(c) Give a "cut-and-paste" argument explaining why $\mathbb{K}$ is homeomorphic to $\mathbb{P}\#\mathbb{P}$, where $\mathbb{P}$ is the projective plane.

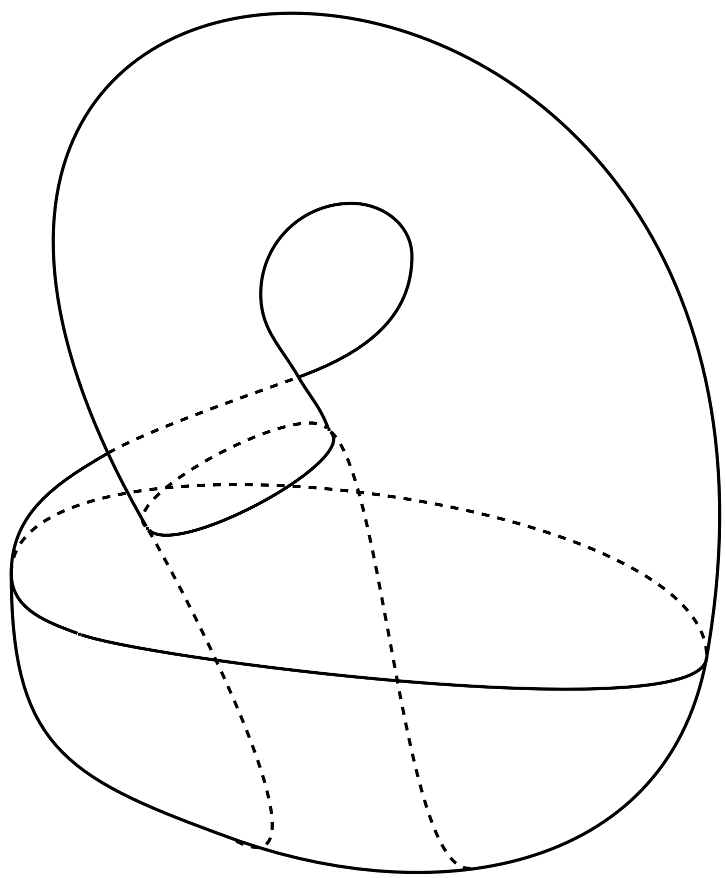
6. Let $\mathbb{K}$ be the Klein bottle, and $\mathbb{T}$ the torus,

(a) Sketch a plane model and write down a word that represents the connected sum $\mathbb{K}\#\mathbb{T}$.
(b) Is the surface $\mathbb{K}\#\mathbb{T}$ orientable? Explain.
(c) State the *Classification Theorem for Surfaces*.
What surface in this theorem is $\mathbb{K}\#\mathbb{T}$ homeomorphic to? Explain.

1. The topologist's comb C is given by $C = I \cup \left(\bigcup_{k=0}^{\infty} J_k\right)$, where

$$\begin{aligned} I &= \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \text{ and } y = 0\}, \\ J_0 &= \{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ and } 0 \le y \le 1\}, \text{ and} \\ J_k &= \{(x,y) \in \mathbb{R}^2 \mid x = \frac{1}{k} \text{ and } 0 \le y \le 1\}, \text{ for each positive integer } k. \end{aligned}$$

5. A space model for the Klein bottle $\mathbb{K}$.



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