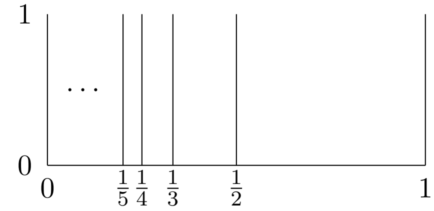
1. Let C be the topologist's comb, given on the right.



$C = I \cup \big(\bigcup_{k=0}^{\infty} J_k\big)$, where

$I = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1 \text{ and } y = 0\}$,

$J_0 = \{(x,y) \in \mathbb{R}^2 \mid x = 0 \text{ and } 0 \le y \le 1\}$, and

$J_k = \{(x,y) \in \mathbb{R}^2 \mid x = \dfrac{1}{k} \text{ and } 0 \le y \le 1\}$, for each positive integer k.

(a) State the definition of a limit point, and find all limit points of C.

x is a limit point of a set A if every open set containing x meets A in a point other than x

C contains all of its limit points. Recall that x is a limit point of $C \subset \mathbb{R}^2$ if and only if there is a sequence of points (x_i) in C such that $x_i \to x$ and $x_i \neq x \forall i$ (Proposition 1.1.6). From the definition of C, one can check that if (x_i) is a sequence in C which converges, it must converge to a point in C.

(b) Is C path connected? Explain.

Yes, C is path connected. This can be proved, for instance, by induction, using Exercise 1.3.12. First note that I and J_0 are clearly path connected, and that $I \cap J_0 = \{(0,0)\}$. Then, $I \cup J_0$ is path connected. Since J_1 is path connected and $(I \cup J_0) \cap J_1 = \{(1,0)\}$, $I \cup J_0 \cup J_1$ is path connected. An inductive argument then shows that $C = I \cup \big(\bigcup_{k=0}^{\infty} J_k\big)$ is path connected.

(c) Is C compact? Explain.

Yes, C is compact. From part (a), C contains all of its limit points, so C is closed. Since C is contained in a disk centered at the origin (say, of radius 2), C is also bounded. So, by the Heine-Borel Theorem, C is compact.

2. Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be topological spaces.

(a) State the definition in terms of open sets, a function $f\colon X \to Y$ is *continuous* if ...

...for every open set $V \subset Y$, inverse image $f^{-1}(V) = \{x \in X \mid f(x) \in V\}$ is open in X.

(b) Prove that the composition of continuous functions is continuous:
If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, prove that $g \circ f\colon X \to Z$ is continuous.

Let $h = g \circ f$, and let W be an open set in Z. We must show that $h^{-1}(W)$ is open in X. Since g is continuous, $V = g^{-1}(W)$ is open in Y. Since f is continuous, $U = f^{-1}(V)$ is open in X. Since $U = f^{-1}(g^{-1}(W)) = h^{-1}(W)$ is open, $h = g \circ f$ is continuous.

(c) Let $(a,b) = \{x \in \mathbb{R}^1 \mid a < x < b\} \subset \mathbb{R}^1$. Are $X = (0,1)$ and $Y = (4,9)$ homeomorphic? Explain.

Yes, $(0,1)$ and $(4,9)$ are homeomorphic. For instance, one can check that $f(x) = 5x + 4$ is a one-to-one continuous function from $(0,1)$ to $(4,9)$, with continuous inverse function $g(x) = (x-4)/5$, i.e., a homeomorphism.

3. Let A be a subspace of $\mathbb{R}^n$, and let B be a subset of A.

(a) Prove that if B is closed in A, then $B = A \cap D$ for some closed set $D \subset \mathbb{R}^n$.

Since B is closed in A, the complement $U = A \setminus B$ is open in A. Since A has the subspace topology, this means that $U = A \cap V$, where V is open in $\mathbb{R}^n$. Let $D = \mathbb{R}^n \setminus V$. Observe that D is closed in $\mathbb{R}^n$, and that $B = A \setminus U = A \setminus A \cap V = A \cap (\mathbb{R}^n \setminus V) = A \cap D$.

(b) Given an example of a subspace A of $\mathbb{R}^2$ and a subset B of A for which B is closed in A, but B is not closed in $\mathbb{R}^2$.

One example: $A = \{(x,y) \in \mathbb{R}^2 \mid |x| < 1, |y| < 1\}$, an open square, and $B = \{(x,y) \in A \mid x \ge 0\}$
$B = A \cap \{(x,y) \in \mathbb{R}^2 \mid x \ge 0\}$ is closed in A, but B is not closed in $\mathbb{R}^2$ since, for instance, $(1,0)$ is a limit point of B in $\mathbb{R}^2$ which is not in B. (Note that B is not open in $\mathbb{R}^2$ either.)

(c) If A is open in $\mathbb{R}^2$ and B is open in A, is B open in $\mathbb{R}^2$? Explain.
Yes. If B is open in A, then $B = A \cap V$ for some open V in $\mathbb{R}^2$. The intersection of two open sets in $\mathbb{R}^2$ is open in $\mathbb{R}^2$.

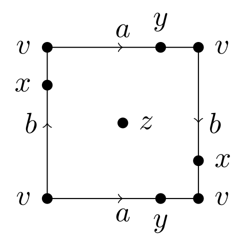
4. Explain why...

(a) ...the unit interval $I = [0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ and the circle $\mathbb{S}^1$ are not homeomorphic.
For instance, I has the fixed point property (by the intermediate value theorem), while $\mathbb{S}^1$ does not (e.g., rotation by $\pi/2$ radians has no fixed point in $\mathbb{S}^1$).

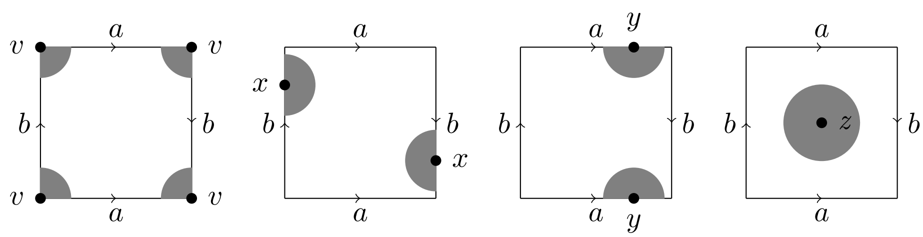
(b) ...$\mathbb{R}^2$ and $D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ are not homeomorphic.
For instance, $\mathbb{R}^2$ is not compact, while D^2 is compact.

(c) ...the 2-sphere $\mathbb{S}^2$ and the torus $\mathbb{T}$ are not homeomorphic.
For instance, the complement of any simple closed curve $\mathcal{C}$ on $\mathbb{S}^2$ is not path connected, but there are simple closed curves on $\mathbb{T}$ for which the complement is path connected. A homeomorphism $h\colon \mathbb{S}^2 \to \mathbb{T}$ would restrict to a homeomorphism $\mathbb{S}^2 \smallsetminus \mathcal{C} \to \mathbb{T} \smallsetminus h(\mathcal{C})$. But $\mathbb{S}^2 \smallsetminus \mathcal{C}$ is not path connected, while $\mathbb{T} \smallsetminus h(\mathcal{C})$ may be. Since path connectivity is a topological property, there cannot be such a homeomorphism.

5. Consider the plane model for the Klein bottle $\mathbb{K}$ given on the right, with a, b indicating the identifications, and v, x, y, z representing points in $\mathbb{K}$.

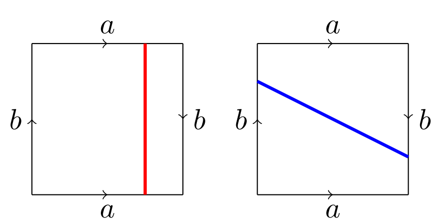


(a) Exhibit Euclidean 2-disk neighborhoods of each of the points v, x, y, z in separate copies of the plane model for $\mathbb{K}$, and explain why $\mathbb{K}$ is Hausdorff.



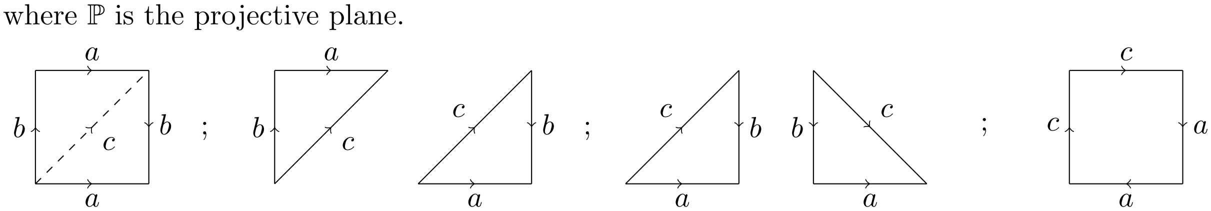
If u and v are distinct points in $\mathbb{K}$, one can produce neighborhoods U of u and V of v as above which are disjoint.

(b) Exhibit a simple closed curve $\mathcal{C}$ on $\mathbb{K}$ for which $\mathbb{K} \smallsetminus \mathcal{C}$ is path connected.
Draw $\mathcal{C}$ on the space model for $\mathbb{K}$ given on the back of this sheet.
Here are a couple examples (there are many others):



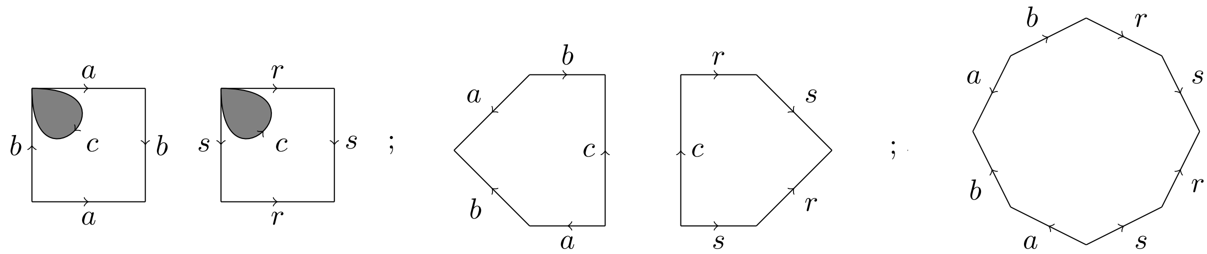
The complement of the red curve is a cylinder.
The complement of the blue curve is a Möbius band.
The red curve is drawn on the space model below.

(c) Give a "cut-and-paste" argument explaining why $\mathbb{K}$ is homeomorphic to $\mathbb{P}\#\mathbb{P}$, where $\mathbb{P}$ is the projective plane.



6. Let $\mathbb{K}$ be the Klein bottle, and $\mathbb{T}$ the torus.

(a) Sketch a plane model and write down a word that represents the connected sum $\mathbb{K}\#\mathbb{T}$.



one word representing $\mathbb{K}\#\mathbb{T}$ is $aba^{-1}brsr^{-1}s^{-1}$

(b) Is the surface $\mathbb{K}\#\mathbb{T}$ orientable? Explain.

No, if S_1 and S_2 are surfaces, with at least one nonorientable, then $S_1\#S_2$ is nonorientable.
Or, the presence of $\cdots b \cdots b \cdots$ in the word representing $\mathbb{K}\#\mathbb{T}$ indicates that this surface contains a Möbius band, and hence an orientation reversing loop.

(c) State the *Classification Theorem for Surfaces.*
What surface in this theorem is $\mathbb{K}\#\mathbb{T}$ homeomorphic to? Explain.

Any compact, path connected surface is homeomorphic to a sphere, a connected sum of tori, or a connected sum of projective planes.
Since $\mathbb{K} \cong \mathbb{P}\#\mathbb{P}$, and $\mathbb{P}\#\mathbb{T} \cong 3\mathbb{P}$, we have $\mathbb{K}\#\mathbb{T} \cong \mathbb{P}\#\mathbb{P}\#\mathbb{T} \cong \mathbb{P}\#3\mathbb{P} \cong 4\mathbb{P}$.

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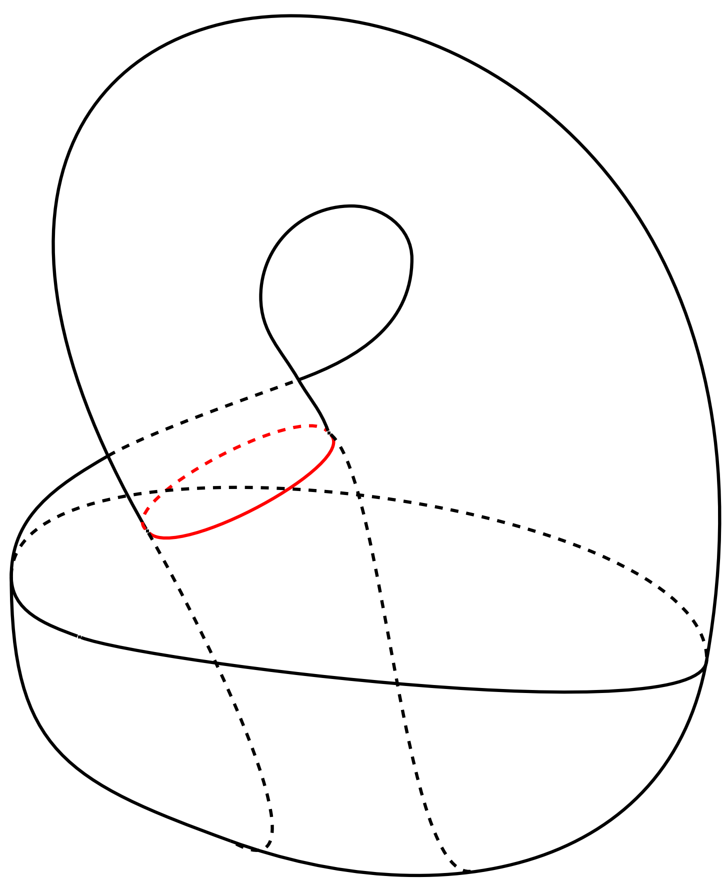
5. A space model for the Klein bottle $\mathbb{K}$, with simple closed curve $\mathcal{C}$ for which $\mathbb{K} \setminus \mathcal{C}$ is path connected.



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