1. Consider the homogeneous system of linear equations, with coefficient matrix A:

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 0 \\ -x_1 + 2x_2 + 2x_4 &= 0 \\ x_1 - 2x_2 + 4x_3 + 2x_4 &= 0 \end{aligned} \qquad A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ -1 & 2 & 0 & 2 \\ 1 & -2 & 4 & 2 \end{bmatrix}$$

(a) Find a row echelon matrix that is row equivalent to A, and find the rank of A.

> The matrix A is row equivalent to $\begin{bmatrix} 1 & -2 & 1 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, so rank $A = 2$.

(b) Find the solution set of this system of linear equations.

> The solution set is $\{(2s + 2t, s, -t, t) \mid s, t \in \mathbb{R}\}$.

(c) For this matrix A and any 3×1 constant vector $\vec{b}$, can you be certain that the non-homogeneous system of linear equations $A\vec{x} = \vec{b}$ is consistent? Explain.

> You cannot be certain.
> The system $A\vec{x} = \vec{b}$ is consistent if rank A = rank $A^\#$, where $A^\# = [A \mid \vec{b}]$.
> This need not be the case for every $\vec{b}$.
> For instance, the system $A\vec{x} = \vec{b}$ is inconsistent if $\vec{b} = [1\ 1\ 8]^T$.

2. Let $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & -2 \\ 1 & 0 & -2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$. The inverse of A is $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -1 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to calculate the inverse of the matrix A by hand.

> Row reduce $[A \mid I]$ until you reach $[I \mid A^{-1}]$.

(b) Solve the system of linear equations $A\vec{x} = \vec{b}$.

> The solution is $\vec{x} = A^{-1}\vec{b} = [10\ \ -4\ \ \ 3]^T$.

(c) Does the homogeneous system $A^T\vec{x} = \vec{0}$ have infinitely many solutions? Explain.

> Since A is nonsingular, A^T is also nonsingular. So $A^T\vec{x} = \vec{0}$ has only the trivial solution.

3. Consider the system of linear equations

$$\begin{aligned} x_1 - x_2 &= 1, \\ x_1 - x_3 &= 2, \\ x_2 - x_3 &= k^2, \end{aligned}$$

where k is a constant.

(a) Find all values of k for which this system has infinitely many solutions.

> The system has infintely many solutions if $k = 1$ or $k = -1$.

(b) Find all values of k for which this system has no solution.

> The system has no solution k is any number other than 1 or -1.

(c) Are there any values of k for which this system has a unique solution? Explain.

> There are not. For instance, the rank of the coefficient matrix is 2, which is less than the number of variables.

4. Consider the matrix $A = \begin{bmatrix} 3 & 2 & 0 & 1 \\ 2 & 4 & 2 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$.

(a) Calculate the determinant of the matrix A.

> The determinant of this matrix is -18.

(b) Can the answer to part (a) be used to determine if the matrix A is nonsingular? Explain.

> It can: $\det A \neq 0$ if and only if A is nonsingular.