1. Consider the vectors $\vec{v}_1 = (1, -1, 0, 2)$, $\vec{v}_2 = (-1, 2, 1, -2)$, $\vec{v}_3 = (0, 1, 3, 0)$ in $\mathbb{R}^4$.

   (a) Can the vector $\vec{b} = (2, 3, 9, 4)$ be expressed as a linear combination of $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? In other words, is $\vec{b}$ in the subspace span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$?

   > The system $c_1(1-1,0,2) + c_2(-1,2,1,-2) + c_3(0,1,3,0) = (2,3,9,4)$ is consistent. So $\vec{b} = (2,3,9,4) = a(1-1,0,2) + b(-1,2,1,-2) + c(0,1,3,0)$ is in span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

   (b) Determine if the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent in $\mathbb{R}^4$.

   > The system $c_1(1-1,0,2) + c_2(-1,2,1,-2) + c_3(0,1,3,0) = (0,0,0,0)$ has only the trivial solution. So this set of vectors **is** linearly independent.

   (c) Explain why the set $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ does **not** form a basis for $\mathbb{R}^4$.

   > A basis for $\mathbb{R}^4$ must consist of 4 vectors since $\dim \mathbb{R}^4 = 4$.

2. Let P_3 be the vector space of all polynomials (in the variable x) of degree less than 3 with real coefficients, $P_3 = \{ax^2 + bx + c \,:\, a, b, c \text{ real}\}$. Consider the polynomials $p_1 = 1 + x$, $p_2 = 1 + x^2$, $p_3 = x + x^2$ in P_3. Determine if the set $\{p_1, p_2, p_3\}$ forms a basis for P_3.

   > This set **is** a basis for P_3. Check that $c_1(1+x) + c_2(1+x^2) + c_3(x+x^2) = ax^2 + bx + c$ has a unique solution $\{c_1, c_2, c_3\}$ for any a, b, c by comparing coefficients of $1 = x^0, x^1, x^2$.

3. In parts (a) and (b), determine if the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defines a linear transformation.

   (a) $T(x_1, x_2) = (x_2 - x_1, x_1)$

   > This mapping **is** a linear transformation. Check that $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(c\vec{x}) = cT(\vec{x})$ for any $\vec{x}$, $\vec{y}$, and c.

   (b) $T(x_1, x_2) = (x_2 - 1, x_1)$

   > This mapping **is not** a linear transformation. For instance, $T(\vec{x} + \vec{y}) = (x_2 + y_2 - 1, x_1 + y_1) \neq (x_2 + y_2 - 2, x_1 + y_1) = T(\vec{x}) + T(\vec{y})$.

   (c) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation and $T(1, 0) = (2, 3)$ and $T(0, 1) = (-1, 1)$, what is $T(4, 2)$?

   > $T(4,2) = T(4(1,0) + 2(0,1)) = 4T(1,0) + 2T(0,1) = 4(2,3) + 2(-1,1) = (6,14)$

4. Consider the matrix $A = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 9 & -6 \end{bmatrix}$ and the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$.

   (a) Find a basis for Rng(T), the range of the linear transformation T. What is the dimension of Rng(T)?

   > $\text{Rng}(T) = \{\vec{b} \text{ in } \mathbb{R}^2 \,:\, A\vec{x} = \vec{b} \text{ is consistent}\} = \{\vec{b} \text{ in } \mathbb{R}^2 \,:\, b_2 + 3b_1 = 0\}$
   > $= \{(b_1, -3b_1)\} = \text{span}\{(1, -3)\}$ $\dim \text{Rng}(T) = 1$

   (b) Find the dimension of Ker(T), the kernel of the linear transformation T.

   > $\text{Ker}(T) = \{\vec{x} \text{ in } \mathbb{R}^3 \,:\, A\vec{x} = \vec{0}\} = \{\vec{x} \text{ in } \mathbb{R}^3 \,:\, x_1 - 3x_2 + 2x_3 = 0\}$
   > $= \{(3x_2 - 2x_3, x_2, x_3)\} = \text{span}\{(3,1,0), (-2,0,1)\}$ $\dim \text{Ker}(T) = 2$

   (c) Do the columns of the matrix A span $\mathbb{R}^2$? Explain.

   > They do not. Only vectors on the line span$\{(1, -3)\}$ can be expressed as linear combinations of the columns of A.

5. Consider the matrix $A = \begin{bmatrix} 3 & -2 & 1 \\ 0 & 2 & 0 \\ -2 & 4 & 0 \end{bmatrix}$.

   (a) One eigenvalue of A is $\lambda = 2$. Find the eigenvectors corresponding to this eigenvalue.

   > The eigenvectors corresponding to $\lambda = 2$ are the non-zero solutions of $(A - 2I)\,\vec{v} = \vec{0}$. These vectors are of the form $\vec{v} = (2s - t, s, t)$, where s and t are not both zero.

   (b) Find all of the eigenvalues of the matrix A.

   > The eigenvalues of A are the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$. Expand along row two to get $p(\lambda) = (2 - \lambda)(\lambda^2 - 3\lambda + 2) = (2 - \lambda)(\lambda - 2)(\lambda - 1)$. So the eigenvalues of A are $\lambda_1 = \lambda_2 = 2$ and $\lambda_3 = 1$.

   (c) Can the answer to part (b) be used to determine if the columns of the matrix A are linearly independent? Explain.

   > It can. The columns of A are linearly **dependent** if and only if $A\vec{x} = \vec{0}$ has a non-trivial solution. In other words, A has dependent columns if and only if there is a non-zero vector $\vec{x}$ so that $A\vec{x} = 0\vec{x}$. So A has dependent columns if and only if 0 is an eigenvalue of A.