1. Determine if the matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}$ is defective or non-defective.

$\det(A - \lambda I) = (3 - \lambda)^3$, so $\lambda = 3$ is an eigenvalue with algebraic multiplicity 3.
$\ker(A - 3I) = \text{span}\{(1,0,1),(0,1,0)\}$, so the geometric multiplicity of $\lambda = 3$ is $2 < 3$, and A is defective.

2. Find all eigenvalues and associated eigenvectors of the matrix $A = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$.

$\det(A - \lambda I) = \lambda^2 - 2\lambda + 5$, so $\lambda_1 = 1 + 2\,\mathrm{i}$ and $\lambda_2 = 1 - 2\,\mathrm{i}$.
Associated eigenvectors are $\vec{v}_1 = (-2\,\mathrm{i}, 1)$ and $\vec{v}_2 = (2\,\mathrm{i}, 1)$

3. Consider the differential equation $(D-2)(D^2 - 2D + 5)y = 3e^{2x} + 7x$.

(a) Find the general solution of the associated homogeneous differential equation, $(D-2)(D^2 - 2D + 5)y = 0$.

$y(x) = c_1 e^{2x} + c_2 e^{x}\cos(2x) + c_3 \sin(2x)$

(b) Find the annihilator of the function $F(x) = 3e^{2x} + 7x$.

$D^2(D-2)$ annihilates $F(x) = 3e^{2x} + 7x$.

(c) Find the form of a particular solution of the non-homogeneous differential equation given above.

$y_p(x) = Axe^{2x} + Bx + C$

4. (a) Use the definition to compute the Laplace transform of the function $f(t) = e^{6t}$.

$$L[e^{6t}] = \int_0^\infty e^{-st}e^{6t}\,dt = \int_0^\infty e^{(6-s)t}\,dt = \lim_{N\to\infty}\left[\frac{e^{(6-s)t}}{6-s}\Big|_0^N\right]$$
$$= \lim_{N\to\infty}\left[\frac{e^{(6-s)N}}{6-s} - \frac{1}{6-s}\right] = -\frac{1}{6-s} = \frac{1}{s-6} \text{ if } 6 - s < 0$$

(b) Find the Laplace transform of the function $f(t) = 4t^3 - 2\sin 3t$.

$L[f(t)] = \dfrac{24}{s^4} - \dfrac{6}{s^2+9}$

(c) Find the inverse Laplace transform of the function $F(s) = \dfrac{4}{s^3} + \dfrac{3}{s+4} + \dfrac{2s}{s^2+4}$.

$L^{-1}[F(s)] = 2t^2 + 3e^{-4t} + 2\cos(2t)$

5. In each part below, find a fundamental set of solutions for the system of differential equations $\vec{x}\,' = A\vec{x}$. Your answers should be real-valued.

(a) $A = \begin{bmatrix} 3 & -5 \\ 2 & 1 \end{bmatrix}$ an eigenvalue/eigenvector pair is $\lambda_1 = 2 + 3\,\mathrm{i}$, $\vec{v}_1 = \begin{bmatrix} 1+3\,\mathrm{i} \\ 2 \end{bmatrix}$.

$$\vec{x}_1(t) = e^{2t}\cos(3t)\begin{bmatrix}1\\2\end{bmatrix} - e^{2t}\sin(3t)\begin{bmatrix}3\\0\end{bmatrix}, \quad \vec{x}_2(t) = e^{2t}\sin(3t)\begin{bmatrix}1\\2\end{bmatrix} + e^{2t}\cos(3t)\begin{bmatrix}3\\0\end{bmatrix}$$

(b) $A = \begin{bmatrix} 13 & -9 \\ 4 & 1 \end{bmatrix}$ (defective); eigenvalues: $\lambda_1 = \lambda_2 = 7$; associated eigenvector: $\vec{v} = \begin{bmatrix}3\\2\end{bmatrix}$.

$\vec{x}_1(t) = e^{7t}\begin{bmatrix}3\\2\end{bmatrix}$, $\vec{x}_2(t) = te^{7t}\begin{bmatrix}3\\2\end{bmatrix} + e^{7t}\begin{bmatrix}2\\1\end{bmatrix}$ is one fundamental set

6. The eigenvalues of certain 2×2 matrix A are $\lambda_1 = 3$ and $\lambda_2 = 2$.

Eigenvectors corresponding to λ_1 and λ_2 are $\vec{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix}1\\3\end{bmatrix}$.

(a) Solve the initial value problem $\vec{x}\,' = A\vec{x}$, $\vec{x}(0) = \begin{bmatrix}2\\7\end{bmatrix}$.

$\vec{x}(t) = -e^{3t}\begin{bmatrix}1\\2\end{bmatrix} + 3e^{2t}\begin{bmatrix}1\\3\end{bmatrix}$

(b) Find the matrix A.

$$A = SDS^{-1} = \begin{bmatrix}1 & 1\\2 & 3\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}3 & -1\\-2 & 1\end{bmatrix} = \begin{bmatrix}5 & -1\\6 & 0\end{bmatrix}$$