MATH 2090 **Review Problems for Exam 3** **Fall 2000**

1. Consider the vectors $\vec{v}_1 = (1, 3, 4)$, $\vec{v}_2 = (0, 1, 1)$, $\vec{v}_3 = (-1, 0, -1)$, $\vec{v}_4 = (5, 3, -2)$ in $\mathbb{R}^3$.

   (a) Determine if the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ spans $\mathbb{R}^3$.

   (b) Determine if this set of vectors is linearly independent.

   (c) Does this set of vectors form a basis for $\mathbb{R}^3$? Explain.

2. Recall that the trace of a square matrix is the sum of the diagonal entries. Let S be the set of all 2×2 matrices with real entries and zero trace: $S = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;:\; a, b, c, d \text{ real, and } a + d = 0 \right\}$.

   (a) Show that S is a subspace of $\mathbb{M}_2(\mathbb{R})$, the vector space of all 2×2 matrices with real entries.

   (b) Verify that $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for S.

   (c) Find the components of the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$ with respect to the basis in part (b).
   In other words, express the matrix A as a linear combination of the matrices given in part (b).

3. (a) Consider the mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2, x_3, k)$, where k is a scalar. Find all values of k for which T is a linear transformation, and find the matrix of T for these values of k.

   (b) If $T : \mathbb{R}^3 \to \mathbb{R}$ is a linear transformation and $T(1, 0, 0) = 1$, $T(1, 1, 0) = 3$, $T(1, 1, 1) = 5$, find $T(2, 4, 6)$.

4. Consider the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 2 & 3 & 1 \\ -4 & 0 & 2 \end{bmatrix}$ and the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$.

   (a) Find a basis for $\text{Ker}(T)$, the kernel of the linear transformation T. What is the dimension of $\text{Ker}(T)$?

   (b) Find a basis for $\text{Rng}(T)$, the range of the linear transformation T. What is the dimension of $\text{Rng}(T)$?

   (c) Do the rows of the matrix A form a basis for $\mathbb{R}^3$? Explain.

   (d) Determine all eigenvalues and corresponding eigenvectors of the matrix A.