1. (a) Solve the initial value problem $\dfrac{dy}{dx} = \dfrac{x(2y-1)}{x^2+1}$, $y(0) = 1$.

   $y = (x^2+1)/2$ solves the IVP

   (b) Briefly explain why this initial value problem has a unique solution.

   Use the existence and uniqueness theorem for first-order IVPs.

2. Find the general solution of each of the following differential equations. (Your solutions should be real-valued.)

   (a) $y' + \dfrac{2}{x}y = 8x$ $\quad y = 2x^2 + Cx^{-2}$

   (b) $y'' + 6y' + 10y = 0$ $\quad y = c_1 e^{-3x}\cos x + c_2 e^{-3x}\sin x$

   (c) $y'' - 4y' + 4y = 4x$ $\quad y = c_1 e^{2x} + c_2 x e^{2x} + x + 1$

3. (a) Solve the initial value problem $y'' + 2y' - 3y = 0$, $y(0) = 5$, $y'(0) = 1$.

   $y = e^{-3x} + 4e^{x}$

   (b) Find a particular solution of the differential equation $y'' + 2y' - 3y = 8e^{x}$.

   $y = 2xe^{x}$

4. Consider the differential equation $x^2y'' - xy' + y = 0$, $x > 0$.

   (a) Verify that the function $y_1(x) = x\ln(x)$ is a solution of this differential equation.

   If $y_1 = x\ln(x)$, then $y_1' = \ln(x) + 1$ and $y_1'' = x^{-1}$.
   So $x^2y_1'' - xy_1' + y_1 = x^2x^{-1} - x(\ln(x)+1) + x\ln(x) = 0$.

   (b) Find all values of $r$ for which the function $y(x) = x^r$ is a solution of this differential equation.

   If $y = x^r$, then $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$.
   So $x^2y'' - xy' + y = r(r-1)x^r - rx^r + x^r = (r^2 - 2r + 1)x^r$.
   So $y = x^r$ is a solution if $r = 1$.

   (c) Find the general solution of this differential equation on the interval $I = (0, \infty)$. Give a thorough explanation why the solution you give is the general solution.

   The general solution is $y = c_1 x\ln(x) + c_2 x$.
   The solutions $y_1 = x\ln(x)$ and $y_2 = x$ not proportional on $I$,
   so are linearly independent on $I$.
   This can also be established using the Wronskian: $W[x\ln(x), x] = -x \neq 0$ for $x > 0$

5. A certain antique coin increases in value at a rate proportional to the square root of the current value. The value of the coin was \$100 in the year 2000.
   Write down the initial value problem you would solve to determine the value of the coin $t$ years from 2000.

   If $V(t)$ is the value of the coin $t$ years from 2000, $\dfrac{dV}{dt} = k\sqrt{V}$, $V(0) = 100$

**Extra Credit**. If the value of the coin in problem 5 will be \$144 next year (in 2002), when will the value of the coin be \$400?

In the year 2010.