

1. Consider the consistent system of linear equations, with coefficient matrix A :

$$\begin{array}{rcl} x_1 & - & 2x_2 + x_3 - x_4 = 1 \\ 3x_1 & - & 6x_2 + x_3 + x_4 = 5 \end{array} \quad A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 3 & -6 & 1 & 1 \end{bmatrix}$$

- (a) Find the solution set of this system of linear equations.

The solution set is $\{(2s - t + 2, s, 2t - 1, t) : s, t \text{ any real numbers}\}$

- (b) For this matrix A , is there a 2×1 constant vector \vec{b} for which the system of linear equations $A\vec{x} = \vec{b}$ is inconsistent? Explain.

A is 2×4 and $\text{rank } A = 2$, so for any \vec{b} , $\text{rank } A = \text{rank } A^\# = 2$, and $A\vec{x} = \vec{b}$ is consistent.

2. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \\ 0 & 1 & 1 & 1 \\ 2 & 4 & 2 & 6 \end{bmatrix} \xrightarrow[R_2 \rightarrow R_2 - 2R_1]{R_4 \rightarrow R_4 - 2R_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix} \xrightarrow[R_4 \rightarrow R_4 - 2R_2]{R_3 \rightarrow R_3 - R_2} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$

- (a) Find a row echelon matrix that is row equivalent to A , and find the rank of A .

The row echelon matrix U given above is row equivalent to A . So $\text{rank } A = 2$.

Answer the following questions based (only) on your calculations in part (a).
Give a brief explanation with each answer.

- (b) Is the matrix A non-singular?

No, since A is 4×4 and $\text{rank } A = 2 < 4$, A is singular.

- (c) Is the determinant of A equal to zero, or different from zero?

$\det A = 0$. Since A and U are row equivalent, $\det A = k \det U$ for some non-zero scalar k .
 U is upper triangular, so $\det U = 1 \cdot 1 \cdot 0 \cdot 0 = 0$ is the product of the diagonal entries.

- (d) Does the homogeneous system of linear equations $A\vec{x} = \vec{0}$ have a unique solution?

$A\vec{x} = \vec{0}$ has infinitely many solutions. Since $\text{rank } A = 2 < 4$, there are two free variables.

3. [35 points] Let $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}$. The inverse of A is $A^{-1} = \begin{bmatrix} -1 & -2 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & -2 \end{bmatrix}$.

- (a) Use Gauss-Jordan elimination to calculate the inverse of the matrix A by hand.

The following sequence of row operations will reduce $[A:I]$ to $[I:A^{-1}]$:
 $R_1 \leftrightarrow R_3$, $R_2 \rightarrow R_2 - R_1$, $R_3 \rightarrow R_3 + 2R_2$, $R_1 \rightarrow R_1 - R_3$, $R_2 \rightarrow (-1)R_2$

- (b) Use the inverse of the matrix A to solve the system of linear equations $A\vec{x} = \vec{b}$.

$$\vec{x} = A^{-1}\vec{b} = [1 \ 2 \ 3]^T$$

- (c) Use cofactor expansion (on any row or column) to calculate the determinant of A .

$$\det A = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + (-1)^{3+1} \begin{vmatrix} 0 & 1 \\ -1 & 1 \end{vmatrix} = -(2-0) + (2+1) = 1 \quad (\text{expanding on column one})$$

4. [20 points] Let $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$.

In parts (a), (b), and (c), either carry out the indicated calculation, or state that it is not defined.

(a) BC is not defined

(b) $CB = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$

(c) $C^T + 3C = \begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix}$

- (d) A square matrix A is called orthogonal if the transpose of A is equal to the inverse of A , that is, $A^T = A^{-1}$. Is the 2×2 matrix C above orthogonal? Explain.

$$C^T C = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I, \text{ so } C^T \neq C^{-1} \text{ and } C \text{ is not orthogonal}$$

Extra Credit. If A is an orthogonal matrix, what is the determinant of A ?

If $A^T = A^{-1}$, then $A^T A = I$ so $\det A^T \det A = \det I = 1$. But $\det A^T = \det A$, so $\det A^T \det A = \det A \det A = (\det A)^2 = 1$ and $\det A = \pm 1$.