1. Consider the consistent system of linear equations, with coefficient matrix A:

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & - & x_4 & = & 1 \\ 3x_1 & - & 6x_2 & + & x_3 & + & x_4 & = & 5 \end{array} \qquad A = \begin{bmatrix} 1 & -2 & 1 & -1 \\ 3 & -6 & 1 & 1 \end{bmatrix}$$

(a) Find the solution set of this system of linear equations.

The solution set is $\{(2s - t + 2, s, 2t - 1, t) : s, t$ any real numbers$\}$

(b) For this matrix A, is there a 2×1 constant vector $\vec{b}$ for which the system of linear equations $A\vec{x} = \vec{b}$ is inconsistent? Explain.

A is 2×4 and rank $A = 2$, so for any $\vec{b}$, rank $A =$ rank $A^\# = 2$, and $A\vec{x} = \vec{b}$ is consistent.

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 3 & 1 & 5 \\ 0 & 1 & 1 & 1 \\ 2 & 4 & 2 & 6 \end{bmatrix} \xrightarrow[\substack{R2\to R2-2R1 \\ R4\to R4-2R1}]{} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{bmatrix} \xrightarrow[\substack{R3\to R3-R2 \\ R4\to R4-2R2}]{} \boxed{\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U}$$

(a) Find a row echelon matrix that is row equivalent to A, and find the rank of A.

The row echelon matrix U given above is row equivalent to A. So rank $A = 2$.

Answer the following questions based (only) on your calculations in part (a).
Give a brief explanation with each answer.

(b) Is the matrix A non-singular?

No, since A is 4×4 and rank $A = 2 < 4$, A is singular.

(c) Is the determinant of A equal to zero, or different from zero?

$\det A = 0$. Since A and U are row equivalent, $\det A = k \det U$ for some non-zero scalar k.
U is upper triangular, so $\det U = 1 \cdot 1 \cdot 0 \cdot 0 = 0$ is the product of the diagonal entries.

(d) Does the homogeneous system of linear equations $A\vec{x} = \vec{0}$ have a unique solution?

$A\vec{x} = \vec{0}$ has infinitely many solutions. Since rank $A = 2 < 4$, there are two free variables.

3. [35 points] Let $A = \begin{bmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 7 \\ 2 \\ 4 \end{bmatrix}$. The inverse of A is $A^{-1} = \begin{bmatrix} -1 & -2 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & -2 \end{bmatrix}$.

(a) Use Gauss-Jordan elimination to calculate the inverse of the matrix A by hand.

The following sequence of row operations will reduce $[A \vdots I]$ to $[I \vdots A^{-1}]$:
$R1 \leftrightarrow R3$, $\quad R2 \to R2 - R1$, $\quad R3 \to R3 + 2R2$, $\quad R1 \to R1 - R3$, $\quad R2 \to (-1)R2$

(b) Use the inverse of the matrix A to solve the system of linear equations $A\vec{x} = \vec{b}$.

$\vec{x} = A^{-1}\vec{b} = [1\ 2\ 3]^T$

(c) Use cofactor expansion (on any row or column) to calculate the determinant of A.

$\det A = (-1)^{2+1} \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} + (-1)^{3+1} \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = -(2 - 0) + (2 + 1) = 1$ (expanding on column one)

4. [20 points] Let $B = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$.

In parts (a), (b), and (c), either carry out the indicated calculation, or state that it is not defined.

(a) BC is not defined (b) $CB = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 1 & 1 \end{bmatrix}$ (c) $C^T + 3C = \begin{bmatrix} 0 & 2 \\ -2 & 4 \end{bmatrix}$

(d) A square matrix A is called orthogonal if the transpose of A is equal to the inverse of A, that is, $A^T = A^{-1}$. Is the 2×2 matrix C above orthogonal? Explain.

$C^TC = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$, so $C^T \neq C^{-1}$ and C is not orthogonal

Extra Credit. If A is an orthogonal matrix, what is the determinant of A?

If $A^T = A^{-1}$, then $A^TA = I$ so $\det A^T \det A = \det I = 1$. But $\det A^T = \det A$,
so $\det A^T \det A = \det A \det A = (\det A)^2 = 1$ and $\det A = \pm 1$.