1. Recall that $\mathbb{R}^3$ denotes the vector space of all ordered 3-tuples of real numbers.
Consider the vectors $\vec{v}_1 = (1, 2, 1)$, $\vec{v}_2 = (2, 7, 3)$, $\vec{v}_3 = (-1, 0, 1)$ in $\mathbb{R}^3$.

(a) Can the vector $\vec{b} = (3, -4, 1)$ be expressed as a linear combination of $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$?

(b) Does the set of vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ form a basis for $\mathbb{R}^3$? Explain.

> (a) Yes, solve $A\vec{x} = \vec{b}$, where $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$, to see that $\vec{b} = (12)\vec{v}_1 + (-4)\vec{v}_2 + (1)\vec{v}_3$.
> (b) Yes, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly independent, spanning set, so is a basis.
> Both of these properties follow from the fact that the matrix $A = [\vec{v}_1\ \vec{v}_2\ \vec{v}_3]$ has rank three.

2. Let S be the set of vectors $\vec{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$ satisfying the condition $x_1 - 3x_2 + x_3 = 0$.

(a) Show that S is a subspace of $\mathbb{R}^3$. (b) Find a basis for S, and determine the dimension of S.

> (a) Note that S is non-empty. For instance, $\vec{0} = (0, 0, 0)$ is in S.
> S is closed under vector addition:
> If $\vec{u} = (u_1, u_2, u_3)$ and $\vec{v} = (v_1, v_2, v_3)$ are in S, then $u_1 - 3u_2 + u_3 = 0$ and $v_1 - 3v_2 + v_3 = 0$.
> It follows that $\vec{u} + \vec{v} = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$ is also in S, since
> $(u_1 + v_1) - 3(u_2 + v_2) + (u_3 + v_3) = u_1 - 3u_2 + u_3 + v_1 - 3v_2 + v_3 = 0 + 0 = 0$
> S is closed under scalar multiplication:
> If $\vec{u}$ is in S as above, and k is a scalar, then $k\vec{u} = (ku_1, ku_2, ku_3)$ is in S, since
> $ku_1 - 3ku_2 + ku_3 = k(u_1 - 3u_2 + u_3) = k(0) = 0$
> (b) $\{(3, 1, 0), (-1, 0, 1)\}$ is a basis for S. $\dim S = 2$

3. Recall that $M_2(\mathbb{R})$ denotes the vector space of all 2×2 matrices with real entries.
Consider the matrices $A_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, $A_2 = \begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 3 & 5 \\ 0 & 6 \end{bmatrix}$ in $M_2(\mathbb{R})$.

(a) Determine if the set $\{A_1, A_2, A_3\}$ is linearly independent. (b) Explain why $\{A_1, A_2, A_3\}$ does **not** span $M_2(\mathbb{R})$.

> (a) This set **is** linearly independent. Check that the system of equations $c_1A_1 + c_2A_2 + c_3A_3 = 0$ has only the trivial solution $c_1 = c_2 = c_3 = 0$.
> (b) For instance, the matrix $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ can not be expressed as a linear combination of A_1, A_2, and A_3.

4. In parts (a) and (b), determine if the mapping $T : \mathbb{R}^2 \to \mathbb{R}^2$ defines a linear transformation.

(a) $T(x_1, x_2) = (x_1 + 2x_2, 3x_1)$ (b) $T(x_1, x_2) = (x_1 + 2x_2, 3 + x_1)$

(c) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation satisfying $T(1, 0) = (0, 3)$ and $T(0, 1) = (1, -2)$.
Find $T(-1, 3)$, and determine the matrix of T.

> (a) This mapping **is** a linear transformation. Check that $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(k\vec{x}) = kT(\vec{x})$.
> (b) This **is not** a linear transformation. Neither of the two conditions above holds.
> (c) $T(-1, 3) = (3, -9)$. The matrix of T is $\begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$.

5. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$, where $A = \begin{bmatrix} 2 & -2 & 4 \\ 1 & -1 & 2 \end{bmatrix}$.

(a) Find a basis for $\mathrm{Rng}(T)$. (b) Find the dimension of $\mathrm{Ker}(T)$. (c) Do the columns of A span $\mathbb{R}^2$? Explain.

> (a) $\{(2, 1)\}$ is a basis for $\mathrm{Rng}(T)$.
> (b) $\dim \mathrm{Ker}(T) = 2$.
> (c) The columns of A **do not** span $\mathbb{R}^2$. $A\vec{x} = \vec{b}$ is consistent only when $\vec{b} = (b_1, b_2)$ satisfies $b_1 = 2b_2$.