1. [15 points] Consider the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 7 \\ 1 & 1 & -3 \end{bmatrix}$.

(a) Find the eigenvalues of the matrix A.

(b) Can the answer to part (a) be used to determine if the matrix A is defective? Explain.

(a) $\det(A - \lambda I) = (1-\lambda)(\lambda^2 - 16)$ (e.g., expand on row 1) eigenvalues: $\lambda_1 = 1$, $\lambda_2 = 4$, and $\lambda_3 = -4$

(b) Since A has distinct eigenvalues, A has three linearly independent eigenvectors, so A is non-defective.

2. [15 points] The matrix $A = \begin{bmatrix} 1 & -3 & 1 \\ -1 & -1 & 1 \\ -1 & -3 & 3 \end{bmatrix}$ is diagonalizable. The characteristic polynomial of A is $p(\lambda) = \det(A - \lambda I) = -\lambda^3 + 3\lambda^2 - 4 = -(\lambda-2)^2(\lambda+1)$.

Find a diagonal matrix D and a nonsingular matrix S so that $S^{-1}AS = D$.

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\quad S = \begin{bmatrix} -3 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ (one choice of D and S)

3. [12 points] One eigenvalue of the matrix $A = \begin{bmatrix} 0 & 10 \\ -1 & 2 \end{bmatrix}$ is $\lambda_1 = 1 + 3\,\mathrm{i}$.

Find all eigenvalues and corresponding eigenvectors for this matrix.

$\lambda_1 = 1 + 3\,\mathrm{i}$, $\vec{v}_1 = \begin{bmatrix} 1 - 3\,\mathrm{i} \\ 1 \end{bmatrix}$ $\quad \lambda_2 = 1 - 3\,\mathrm{i}$, $\vec{v}_2 = \begin{bmatrix} 1 + 3\,\mathrm{i} \\ 1 \end{bmatrix}$ (one choice of complex conjugate eigenvectors)

4. [22 points] Consider the differential equation $(D+7)(D^2 - 4D + 5)y = e^{2x} + 3x^2$.

(a) Find the general solution of the associated homogeneous differential equation, $(D+7)(D^2 - 4D + 5)y = 0$.

(b) Find the annihilator of the function $F(x) = e^{2x} + 3x^2$.

(c) Determine the form of a particular solution of the non-homogeneous differential equation given above.

(a) $y(x) = c_1 e^{-7x} + c_2 e^{2x}\cos(x) + c_3 e^{2x}\sin(x)$ $\quad$ (b) $(D-2)D^3$ $\quad$ (c) $y_p(x) = Ae^{2x} + Bx^2 + Cx + D$

5. [18 points] Let $A = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix}$, and let $\vec{x}_1(t) = \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix}$ and $\vec{x}_2(t) = \begin{bmatrix} 3e^{-t} \\ e^{-t} \end{bmatrix}$.

(a) Verify that the function $\vec{x}_1(t)$ is a solution of the system of differential equations $\vec{x}\,' = A\,\vec{x}$.

(b) The function $\vec{x}_2(t)$ is also a solution of the system $\vec{x}\,' = A\,\vec{x}$. (Do not verify this.)
Use the Wronskian to show that the set $\{\vec{x}_1(t), \vec{x}_2(t)\}$ is linearly independent on the interval $I = (-\infty, \infty)$.

(c) Solve the initial value problem $\vec{x}\,' = A\,\vec{x}$, $\vec{x}(0) = \begin{bmatrix} 2 \\ 9 \end{bmatrix}$.

(a) $A\,\vec{x}_1 = \begin{bmatrix} -2 & 3 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} e^{4t} \\ 2e^{4t} \end{bmatrix} = \begin{bmatrix} 4e^{4t} \\ 8e^{4t} \end{bmatrix} = \vec{x}_1'$ $\quad$ (b) $W[\vec{x}_1, \vec{x}_2] = \begin{vmatrix} e^{4t} & 3e^{-t} \\ 2e^{4t} & e^{-t} \end{vmatrix} = -5e^{3t} \neq 0$ for all t

(c) $\vec{x}(t) = 5\vec{x}_1(t) - \vec{x}_2(t)$ solves the IVP

6. [18 points] In each part below, find a fundamental set of real-valued solutions of the system of differential equations $\vec{x}\,' = A\,\vec{x}$. (You do not have to verify that it is a fundamental set.)

(a) $A = \begin{bmatrix} 1 & -8 \\ 2 & -7 \end{bmatrix}$ is defective; eigenvalues: $\lambda_1 = \lambda_2 = -3$; associated eigenvector: $\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

(b) A is a certain 2×2 matrix with real entries;
One eigenvalue of A is $\lambda_1 = 2 + \mathrm{i}$, and a corresponding eigenvector is $\vec{v}_1 = \begin{bmatrix} 3+\mathrm{i} \\ 2 \end{bmatrix}$.

(a) $\left\{ \vec{x}_1(t) = e^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \vec{x}_2(t) = te^{-3t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + e^{-3t} \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \right\}$ is one fundamental set

(b) $\left\{ \vec{x}_1(t) = e^{2t} \cos(t) \begin{bmatrix} 3 \\ 2 \end{bmatrix} + e^{2t} \sin(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{x}_2(t) = e^{2t} \sin(t) \begin{bmatrix} 3 \\ 2 \end{bmatrix} + e^{2t} \cos(t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is one fundamental set

Extra Credit. Find the matrix A from problem **6**, part (b).

The eigenvalues $\lambda_1 = 2 + \mathrm{i}$ and $\lambda_2 = 2 - \mathrm{i}$ of A are distinct, so the corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2 = \overline{(\vec{v}_1)}$ (the complex conjugate) are linearly independent. So the matrix $S = [\vec{v}_1 \vec{v}_2]$ is nonsingular, and we have $AS = SD$, or $A = SDS^{-1}$, where D is a diagonal matrix with diagonal entries λ_1 and λ_2.

Thus $A = SDS^{-1} = \begin{bmatrix} 3+\mathrm{i} & 3-\mathrm{i} \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2+\mathrm{i} & 0 \\ 0 & 2-\mathrm{i} \end{bmatrix} \begin{bmatrix} 2/(4\,\mathrm{i}) & (-3+\mathrm{i})/(4\,\mathrm{i}) \\ -2/(4\,\mathrm{i}) & (3+\mathrm{i})/(4\,\mathrm{i}) \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ 2 & -1 \end{bmatrix}$