1. [35 points] Consider the system of linear equations $A\vec{x} = \vec{b}$:

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 4x_2 & + & 3x_3 & = & 3 \\ x_1 & + & 3x_2 & + & x_3 & = & 5 \end{array} \quad \text{where} \quad A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 1 & 3 & 1 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

(a) Solve this (consistent) non-homogeneous system of linear equations.

(b) Is the matrix A nonsingular? Explain.

(c) Do the columns of the matrix A form a basis for $\mathbb{R}^3$? Explain.

(a) $$A^\sharp = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 3 & 1 & 5 \end{bmatrix} \underset{\substack{R2-2R1 \\ R3-R1}}{\longrightarrow} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 3 \end{bmatrix} \underset{R2 \leftrightarrow R3}{\longrightarrow} \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \operatorname{rank} A = 3$$

The solution of $A\vec{x} = \vec{b}$ is $\vec{x} = (x_1, x_2, x_3) = (-3, 3, -1)$

(b) Since $\operatorname{rank} A = 3$, the matrix A is nonsingular.

(c) Since $\operatorname{rank} A = 3$, the system $A\vec{x} = \vec{0}$ has only the trivial solution, and the system $A\vec{x} = \vec{b}$ has a (unique) solution for any $\vec{b}$. Thus the columns of A are linearly independent, and span $\mathbb{R}^3$. So they do form a basis for $\mathbb{R}^3$.

2. [25 points] Consider the mapping $T : \mathbb{R}^3 \to \mathbb{R}$ defined by $T(x_1, x_2, x_3) = x_1 - x_3$.

(a) Verify that the mapping T is a linear transformation.

(b) Find a basis for Ker(T), the kernel of T, and find the dimension of Ker(T).

(a) $$\begin{aligned} T(\vec{u} + \vec{v}) &= T(u_1 + v_1, u_2 + v_2, u_3 + v_3) = u_1 + v_1 - (u_3 + v_3) = u_1 - u_3 + v_1 - v_3 \\ &= T(u_1, u_2, u_3) + T(v_1, v_2, v_3) = T(\vec{u}) + T(\vec{v}) \end{aligned}$$
$$T(k\,\vec{u}) = T(ku_1, ku_2, ku_3) = ku_1 - ku_3 = k(u_1 - u_3) = k\,T(u_1, u_2, u_3) = k\,T(\vec{u}).$$

(b) $\operatorname{Ker}(T) = \{\vec{x} \text{ in } \mathbb{R}^3 \mid T(\vec{x}) = 0\} = \{\vec{x} \text{ in } \mathbb{R}^3 \mid x_1 - x_3 = 0\} = \operatorname{span}\{(1,0,1), (0,1,0)\}$
basis: $\{(1,0,1), (0,1,0)\}$ $\quad \dim \operatorname{Ker}(T) = 2$

3. [40 points] Find the (general) solution of each of the following differential equations. Your answers should be real-valued.

(a) $\dfrac{dy}{dx} + \dfrac{2}{x}\,y = \dfrac{6}{x}$ (b) $D(D+1)^2(D^2+9)y = 0$ (c) $y'' - 2y' - 3y = 6e^{2x}$

(a) $y = 3 + Cx^{-2}$

(b) $y = c_1 + c_2e^{-x} + c_3xe^{-x} + c_4\cos(3x) + c_5\sin(3x)$

(c) $y = c_1e^{3x} + c_2e^{-x} - 2e^{2x}$

4. [20 points] A table of Laplace transforms was included.

(a) Find the Laplace transform of the function $f(t) = e^{2t}\cos(3t) - 3e^{-2t}$.

(b) Find the inverse Laplace transform of the function $F(s) = \dfrac{6}{s^3} + \dfrac{s+6}{s^2+9}$.

(a) $L[e^{2t}\cos(3t) - 3e^{-2t}] = \dfrac{s-2}{(s-2)^2+9} - \dfrac{3}{s+2}$ (b) $L^{-1}\left[\dfrac{6}{s^3} + \dfrac{s+6}{s^2+9}\right] = 3t^2 + \cos(3t) + 2\sin(3t)$

5. [20 points] Solve the initial value problem $y'' - 2y' - 3y = 0$, $y(0) = 0$, $y'(0) = 8$ *two* ways:

(a) using the Laplace transform (b) *without* using the Laplace transform $\boxed{y = 2e^{3t} - 2e^{-t}}$

6. [30 points] Consider the matrix $A = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 2 & -2 \\ 1 & 0 & 0 \end{bmatrix}$.

(a) Verify that the characteristic polynomial of the matrix A is $p(\lambda) = (1-\lambda)^2(2-\lambda)$.

(b) Find all eigenvalues and eigenvectors of A, and determine if A is defective or diagonalizable.

(a) $p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 0 & -1 \\ 1 & 2-\lambda & -2 \\ 1 & 0 & 0-\lambda \end{vmatrix} = (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = (2-\lambda)(\lambda^2 - 2\lambda + 1)$

$= (2-\lambda)(1-\lambda)^2$

(b) eigenvalues/eigenvectors: $\lambda_1 = 2$, $\vec{v}_1 = (0,1,0)$, $\lambda_2 = \lambda_3 = 1$, $\vec{v}_2 = (1,1,1)$

The matrix A is defective. $\lambda = 1$ has algebraic multiplicity two and geometric multiplicity one.

7. [30 points] The eigenvalues of a certain diagonalizable 2×2 matrix A are $\lambda_1 = 1$ and $\lambda_2 = -1$. Eigenvectors corresponding to λ_1 and λ_2 are $\vec{v}_1 = (2,1)$ and $\vec{v}_2 = (1,1)$.

(a) Find a fundamental set of solutions for the system of differential equations $\vec{x}\,' = A\,\vec{x}$. Verify that the solutions you find form a fundamental set.

(b) Find a diagonal matrix D and a nonsingular matrix S so that $S^{-1}AS = D$.

(c) Find the matrix A.

(a) $\left\{\vec{x}_1(t) = e^t\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \vec{x}_2(t) = e^{-t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ is a fundamental set of solutions $W[\vec{x}_1, \vec{x}_2] = \begin{vmatrix} 2e^t & e^{-t} \\ e^t & e^{-t} \end{vmatrix} = 1 \neq 0$

(b) $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $S = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ (c) $A = SDS^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$

Extra Credit. For the matrix A from problem **7**, compute A^{2001}.

$A^{2001} = SD^{2001}S^{-1} = SDS^{-1} = A$