MATH 2090 **Answers to the Review Problems for Exam 1** **Fall 2001**

1. Solve the initial value problems $y' = 4x\sqrt{y-1},\ y(2) = 1$ and $y' = y^2 \sin x,\ y(0) = 1$.
   Can you be certain if either of these initial value problems has a unique solution?

   Using separation of variables, a solution of $y' = 4x\sqrt{y-1}$, $y(2) = 1$ is $y = 1 + (x^2 - 4)^2$.
   This solution is **not** unique. For instance, the function $y = 1$ is also a solution.
   The existence and uniqueness theorem does not apply to this IVP.
   Using separation of variables, a solution of $y' = y^2 \sin x$, $y(0) = 1$ is $y = \sec x$.
   The existence and uniqueness theorem does apply to this IVP, and this solution is unique.

2. Find the general solution of each of the following differential equations.

   (a) $y' + 4y = e^{-x}$ (c) $xy' - y = 4xy^2$ Hint: make the substitution $u = y^{-1}$

   (b) $x + xy^2 + e^{x^2} y \dfrac{dy}{dx} = 0$ (d) $y'' + 10y' + 25y = 0$ (e) $y'' - 2y' + 5y = 0$

   (a) $y = \dfrac{1}{3}e^{-x} + Ce^{-4x}$ (b) $\ln(1 + y^2) = e^{x^2} + C$ (c) $y = \dfrac{x}{C - 2x^2}$
   (d) $y = c_1 e^{-5x} + c_2 x e^{-5x}$ (e) $y = c_1 e^x \cos(2x) + c_2 e^x \sin(2x)$

3. (a) Solve the initial value problem $y'' - 4y = 0,\quad y(0) = 1,\quad y'(0) = 6$.
   (b) Find the general solution of each of the following non-homogeneous differential equations.
   (i) $y'' - 4y = 8e^{2x}$ (ii) $y'' - 4y = 5\sin x$ (iii) $y'' - 4y = 4e^{2x} - 5\sin x$
   (c) Explain how the results of parts (i) and (ii) above can be used to obtain the answer to part (iii).

   (a) $y = 2e^{2x} - e^{-2x}$ (b) (i) $y = c_1 e^{2x} + c_2 e^{-2x} + 2xe^{2x}$
   (b) (ii) $y = c_1 e^{2x} + c_2 e^{-2x} - \sin(x)$ (b) (iii) $y = c_1 e^{2x} + c_2 e^{-2x} + xe^{2x} + \sin(x)$
   (c) The "forcing function" $4e^{2x} - 5\sin x$ in (iii) is a linear combination of those from (i) and (ii).

4. The function $y_1(x) = x$ is solution of the differential equation $x^2 y'' - 2xy' + 2y = 0,\quad x > 0$.
   Use the method of reduction of order to find a second linearly independent solution $y_2(x)$ on the interval $I = (0, \infty)$. Explain why the solutions $y_1(x)$ and $y_2(x)$ are linearly independent on I, and find the general solution of this differential equation.

   If $y = xu$, $y' = u + xu'$, and $y'' = 2u' + xu''$. So $x^2 y'' - 2xy' + 2y = x^3 u''$. Since $x > 0$, the function $y = xu$ is a solution of the DE if $u'' = 0$. So $u = ax + b$ for constants a and b, and $y = ax^2 + bx$.
   Since $y_1 = x$ is a known solution, a "new" solution is $y_2 = x^2$. These solutions are linearly independent on I (e.g., $W[y_1, y_2] = W[x, x^2] = x^2 \neq 0$ on I), so $y = c_1 x + c_2 x^2$ is the general solution.

5. Newton's law of cooling states that the rate of change of the temperature of an object with respect to time t is proportional to difference between the temperature, $W(t)$, of the object at time t and the temperature, R, of the surrounding medium.
   Suppose that the temperature of the water in my cup is 45°F at the beginning of class, and is 50°F ten minutes later. Assume that room temperature $R = 72$°F is constant. To determine the temperature of my water at the end of class using Newton's law of cooling, you would have to solve a certain initial value problem. Write this initial value problem down. Then solve it.

   $\dfrac{dW}{dt} = k(W - 72)$, $W(0) = 45$, $W(10) = 50$ $\quad W(t) = 72 - 27\left(\dfrac{22}{27}\right)^{t/10}$

6. Boyle-Mariotte's law for ideal gases. For a gas at low pressure p (and constant temperature), the rate of change of the volume $V = V(p)$ is equal to $-V/p$. Determine the volume as a function of p.

   If $\dfrac{dV}{dp} = -\dfrac{V}{p}$, then $V(p) = \dfrac{k}{p}$, where k is a (positive) constant.