

1. Consider the system of linear equations
- $$\begin{aligned} x_1 - 2x_2 + 3x_3 + x_4 &= 6 \\ 2x_1 - x_2 + 3x_3 - x_4 &= 3 \\ -x_1 + x_2 - 2x_3 &= -3 \end{aligned}$$

Write this system in vector form, find the rank of the coefficient matrix, and find the solution set of the system.

In vector form, the system is

$$\begin{bmatrix} 1 & -2 & 3 & 1 \\ 2 & -1 & 3 & -1 \\ -1 & 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix}.$$

The rank of the coefficient matrix is two.

The solution set of the system is  $\{(-s+t, s+t-3, s, t) : s, t \text{ real}\}$ .

2. Consider the matrix  $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 4 \\ 1 & -1 & 3 \end{bmatrix}$  and the constant vector  $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$ .

- (a) Use Gauss-Jordan elimination to find the inverse of  $A$ .  
 (b) Use your answer from part (a) to solve the system of equations  $A\vec{x} = \vec{b}$ .  
 (c) Is the transpose of  $A$  non-singular? Explain.

(a)  $A^{-1} = \begin{bmatrix} 5 & -1 & -2 \\ 2 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$  (b)  $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} 12 \\ 5 \\ -2 \end{bmatrix}$  (c) Yes,  $(A^T)^{-1} = (A^{-1})^T$ .

3. Consider the matrix  $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 4 & -2 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & 3 & -2 & 4 \end{bmatrix}$ .

- (a) Compute the determinant of  $A$ . Does the matrix  $A$  have an inverse? Explain.  
 (b) If  $B$  is a  $4 \times 4$  matrix with  $\det(B) = -7$ , what is the determinant of the product  $AB$ ? Explain.  
 (c) Can a system  $A\vec{x} = \vec{b}$  involving this matrix  $A$ , and any constant vector  $\vec{b}$ , be inconsistent? Explain.

- (a)  $\det A = -7$ . Since  $\det A \neq 0$ ,  $A$  is non-singular.  
 (b)  $\det AB = \det A \det B = 49$ .  
 (c) Since  $A$  is non-singular, any system of equations with coefficient matrix  $A$  has a unique solution (so is consistent).

4. Consider the homogeneous system of linear equations
- $$\begin{aligned} (1-\lambda)x_1 + 2x_2 + 6x_3 &= 0 \\ (2-\lambda)x_2 + 3x_3 &= 0 \\ x_2 + (4-\lambda)x_3 &= 0 \end{aligned}$$

- (a) Find all values of  $\lambda$  for which this system has infinitely many solutions.  
 (b) Find the solution set of this homogeneous system when  $\lambda = 1$ .

- (a) The system has infinitely many solutions if  $\lambda = 1$  or  $\lambda = 5$ .  
 (b) When  $\lambda = 1$ , the solution set is  $\{(s, -3t, t) : s, t \text{ real}\}$ .